Field of an Arbitrarily Accelerating Point Mass

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A metric of the Kerr-Schild type is derived which contains four arbitrary functions of time. It is a generalization of Vaidya's shining-star metric, and permits arbitrary acceleration of the source.

I. INTRODUCTION

Since 1924, it has been known that the Schwarzschild vacuum solution can be written in the particularly simple form[1]

$$ds^2=du^2+2du\,dr-r^2(d\theta^2+\sin^2\theta\,d\phi^2)-(2m/r)du^2. \quad (1)$$

The coordinate u is retarded time and is obtained from the usual Schwarzschild time t by the transformation

$$u=t-r-2m\ln(r-2m). \quad (2)$$

This coordinate system is intrinsically related to one of the two principal null congruences of the Riemann tensor. The vector field

$$l_\mu=u_{,\mu}$$
$$=(1,0,0,0)$$

is a principal null vector, and r is an affine parameter along l_μ:

$$l^\mu=dx^\mu/dr.$$

Furthermore, the Schwarzschild metric may now be written as

$$g_{\mu\nu}=\eta_{\mu\nu}-(2m/r)l_\mu l_\nu, \quad (3)$$

where $\eta_{\mu\nu}$ is a flat-space metric. Kerr and Schild[2] have found all vacuum solutions of the form

$$g_{\mu\nu}=\eta_{\mu\nu}+Hl_\mu l_\nu, \quad (4)$$

with

$$\eta^{\mu\nu}l_\mu l_\nu=0, \quad l^\mu{}_{;\mu}\neq 0;$$

and Debney, Kerr, and Schild[3] have generalized these results to include Einstein-Maxwell fields.

In this paper we wish to exhibit metrics of the same form as Eq. (4) that are neither vacuum nor electromagnetic, but still possess physical interest.

II. WORLD-LINE GEOMETRY

Consider an arbitrary smooth world line L in Minkowski space that is everywhere timelike (see Fig. 1). Let u be the proper time along the curve. Let $\lambda^\mu(u)$ denote the unit tangent vector at any point, directed toward the future. For each point x in space-time, the past null cone at x intersects L exactly once.[4] Hence there exists a unique retarded null vector connecting x with the curve. Let the null vector be $\sigma^\mu(x)$ and let y be the point of contact. Now the definitions of u and $\lambda^\mu(u)$ may be extended off the world line by setting

$$u(x)=u(y), \quad \lambda^\mu(x)=\lambda^\mu(y).$$

Thus the fields u, σ^μ, and λ^μ are well defined everywhere and may be differentiated. To obtain expressions for their derivatives we first note that, even off the world line, u measures proper distance in the λ^μ direction. Thus we have

$$\lambda^\mu u_{,\mu}=1.$$

Because σ_μ lies in the null surface $u=\text{const.}$, σ_μ and $u_{,\mu}$ must be proportional:

$$u_{,\mu}=r^{-1}\sigma_\mu, \quad (5)$$

where

$$r=\lambda^\mu\sigma_\mu. \quad (6)$$

The vector λ^μ depends only on u, and applying the chain rule to it gives

$$\lambda^\mu{}_{;\nu}=(\delta\lambda^\mu/\delta u)u_{,\nu}$$
$$=r^{-1}\dot{\lambda}^\mu\sigma_\nu, \quad (7)$$

where the dot denotes a u derivative. Finally, the derivative of σ^μ is obtained by differentiating the relation

$$\sigma^\mu=x^\mu-y^\mu$$

and using the chain rule on y. We get

$$\sigma^\mu{}_{;\nu}=\delta^\mu{}_\nu-r^{-1}\lambda^\mu\sigma_\nu. \quad (8)$$

III. CURVED-SPACE METRIC

The metric we wish to consider is

$$g_{\mu\nu}=\eta_{\mu\nu}-2mr^{-3}\sigma_\mu\sigma_\nu, \quad (9)$$

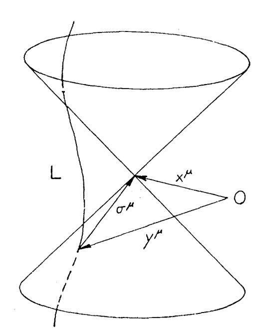

Fig. 1. Unique retarded null vector σ^μ connecting an arbitary point x to the timelike world line L.

[1] A. S. Eddington, Nature **113**, 192 (1924).
[2] R. Kerr and A. Schild (unpublished report).
[3] G. Debney, R. Kerr, and A. Schild (to be published).
[4] The single exception, which we exclude, is when the world line is asymptotically null at $u=-\infty$.

where $m=m(u)$, and the other quantities are defined as in Sec. II. This metric reduces to the Schwarzschild metric, Eq. (3), if L is a straight line and $m=\text{const}$. To avoid confusion we will require that σ^μ, λ^μ, and $\dot{\lambda}^\mu$ be given initially as contravariant vectors and that indices will be raised and lowered using only $\eta_{\mu\nu}$. Appearances of $g_{\mu\nu}$ will be explicitly written out.

We may now calculate the Christoffel symbols and Riemann tensor using Eqs. (5)–(8). The results are

$$\begin{aligned}\Gamma_{\mu\nu}{}^{\tau}=2mr^{-3}\eta_{\mu\nu}\sigma^{\tau}&\\ +mr^{-4}(3\lambda_\mu\sigma_\nu\sigma^\tau+3\lambda_\nu\sigma_\mu\sigma^\tau-\sigma_\mu\sigma_\nu\lambda^\tau)&\\ +3mr^{-5}(1-\dot{\lambda}^\alpha\sigma_\alpha)\sigma_\mu\sigma_\nu\sigma^\tau+2m^2r^{-6}\sigma_\mu\sigma_\nu\sigma^\tau&\\ +\dot{m}r^{-4}\sigma_\mu\sigma_\nu\sigma^\tau,&\quad(10)\end{aligned}$$

$$\begin{aligned}R_{\mu\nu\sigma\tau}=4mr^{-3}\eta_{[\mu[\sigma}\eta_{\tau]\nu]}&\\ +12mr^{-4}\eta_{[\mu[\sigma}\sigma_{\tau]}\lambda_{\nu]}+12mr^{-4}\eta_{[\mu[\sigma}\lambda_{\tau]}\sigma_{\nu]}&\\ -24mr^{-5}\sigma_{[\mu}\lambda_{\nu]}\sigma_{[\sigma}\lambda_{\tau]}+12mr^{-5}\eta_{[\mu[\sigma}\sigma_{\tau]}\sigma_{\nu]}&\\ +8m^2r^{-6}\eta_{[\mu[\sigma}\sigma_{\tau]}\sigma_{\nu]}+4\dot{m}r^{-4}\eta_{[\mu[\sigma}\sigma_{\tau]}\sigma_{\nu]}&\\ -12mr^{-5}(\dot{\lambda}^\alpha\sigma_\alpha)\eta_{[\mu[\sigma}\sigma_{\tau]}\sigma_{\nu]},&\quad(11)\end{aligned}$$

$$R_{\mu\nu}=2\dot{m}r^{-4}\sigma_\mu\sigma_\nu-6mr^{-5}(\dot{\lambda}^\alpha\sigma_\alpha)\sigma_\mu\sigma_\nu. \quad (12)$$

In general, the metric is not a vacuum metric, but has a Ricci tensor proportional to $\sigma_\mu\sigma_\nu$.[5] A Ricci tensor of this form could be produced, for example, by an incoherent cloud of zero-mass particles streaming out from the world line. Their distribution is not necessarily isotropic, but generally of the form $A(u)+B(u)\cos\theta(u)$.

The source at the center undergoes a net loss of mass and of linear momentum. The recoil it suffers is manifest in the acceleration of the world line. By integrating the outflow of $T_{\mu\nu}$ at infinity, one may verify directly that the central source at any instant carries momentum $P^\mu(u)=m(u)\lambda^\mu(u)$, in agreement with the principle of equivalence. In the particular case $\dot{\lambda}^\alpha\sigma_\alpha=0$, we recover Vaidya's metric for a shining star.[6] Note that our metric has four arbitrary functions of time, namely, $m(u)$ and the three independent components of the acceleration $\dot{\lambda}^\mu(u)$.

[5] The Weyl tensor turns out to be the same as Eq. (10), but without the last two terms. Hence it is "quasistatic," identical in form to the Weyl tensor for a Schwarzschild field, and therefore Petrov type D.

IV. COORDINATES

For some purposes it may be more convenient to have the metric written in a particular set of coordinates rather than the vector form given above. Quite general coordinate systems based on an accelerating world line have been described by Newman and Unti.[7] One such system, which is suitable for our purposes, results in the metric

$$\begin{aligned}&g_{uu}=1-2ar\cos\theta-r^2(f^2+g^2\sin^2\theta)-2m(u)r^{-1},\\ &g_{ur}=1,\quad g_{u\theta}=r^2f,\quad g_{u\phi}=r^2g\sin^2\theta,\\ &g_{\theta\theta}=-r^2,\quad g_{\phi\phi}=-r^2\sin^2\theta,\end{aligned}\quad(13)$$

where

$$\begin{aligned}&f=-a(u)\sin\theta+b(u)\sin\phi+c(u)\cos\phi,\\ &g=b(u)\cot\theta\cos\phi-c(u)\cot\theta\sin\phi,\end{aligned}\quad(14)$$

and a, b, c, and m are all arbitrary functions of u. It may be seen that θ and ϕ are spherical coordinates which rotate so as to keep the north pole $\theta=0$ pointed toward the direction of acceleration at all times. The quantity $a(u)$ is the magnitude of the acceleration, while b and c describe the rate of change of its direction. For uniform acceleration, $a=\text{const.}$, and $b=c=0$.

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[6] P. Vaidya, Indian Acad. Sci. **A33**, 264 (1951).

[7] E. Newman and T. Unti, J. Math. Phys. **4**, 1467 (1963).