Absolute Zero of Time*

CHARLES W. MISNER
Department of Physics and Astronomy, University of Maryland, College Park, Maryland 20742
(Received 8 July 1969)

A singularity involving infinite densities at a finite proper time in the past is strongly suggested for the beginnings of the Universe by Einstein's general relativity theory, and is consistent with the few relevant observational data. There is no reasonable point at which to anticipate a failure of the theory, especially since a simplified quantum calculation in the accompanying paper predicts that quantum effects do not change the nature of the singularity. Therefore, we suggest that the singularity be treated as an essential element of cosmological theory, and indicate how this can be made more palatable by refining our concepts of time.

THE hot-big-bang theory of cosmology has received strong support in recent years. The microwave background radiation[1] fits so naturally into this theory, as a relic of earlier hotter epochs, that it was predicted[2] long before it was observed.[3] Not only does this 3°K radiation offer difficulties to the steady-state theory of cosmology, but if the quasars should prove to be cosmologically distant objects, their redshift distribution conflicts with the basic steady-state hypothesis.[4] At the same time that observational indications have been accumulating in favor of the existence of very different, hotter, and denser, earlier epochs of the Universe, the theoretical implications of Einstein's general relativity concerning the origins of the universe have been greatly clarified. The feature of these theoretical models we shall be concerned with is the initial singularity. An initial singularity already appeared in Friedman's relativistic cosmologies[5] which predated Hubble's discovery[6] of the expansion of the Universe. It persists in the entire Robertson-Walker class of homogeneous isotropic cosmological models for all plausible equations of state. Nor is the singularity suppressed by introducing a cosmological constant[1] provided the value of this constant is chosen to allow the Universe to have expanded from a state of sufficiently high density ($z\gtrsim 7$) to have thermalized the microwave radiation, nor is it eliminated by including Dicke's scalar field.[1]

There are two principal views regarding this initial singularity: one, that Einstein 's equations do not really imply a singularity; the other, that because Einstein's equations do demand a singularity the equations must fail to describe nature at some point. I will suggest still a third viewpoint.

The first viewpoint ("Einstein avoids a singularity") is the most plausible first reaction to the Robertson-Walker singularities. The mathematically perfect symmetry which these models postulate may show the expansion beginning from a mathematically ideal point, but a more realistic and more irregular model should show the singularity replaced by an exceedingly complex, but regular, high-density phase. Khalatnikov and Lifshitz[7] and their collaborators are the strongest exponents of this view. They have studied increasingly complex singular solutions of the Einstein equations in order to show that the typical or generic "singularity" is nonsingular, being merely a failure of the coordinate system chosen for sorting out the different solutions rather than any physically significant infinity. Unfortunately for their viewpoint, almost all the solutions they found had true (infinite curvature) singularities. The one exception is the metric $ds_0^2=-dt^2+t^2dz^2+dx^2+dy^2$ which gives, for instance, the local behavior of the singularity ($\sqrt{-g}\rightarrow 0$) in Taub's closed universe.[8] Although there are no intrinsic infinities in this singularity (the metric ds_0^2 is even flat: set $\zeta=t\sinh z$, $\eta=t\cosh z$), most cosmological perturbations do lead to true infinites,[8] including the introduction of arbitrarily small (or large) amounts of matter moving along the lines of constant xyz in these empty universes or the introduction of any small asymmetry between the x and y axes. Thus all known examples suggest that infinites do occur.[9]

The second viewpoint ("Nature avoids Einstein's singularities") is a reaction to the singularity theorems[1,10] of Penrose, Hawking, and Geroch. In these theorems "singularity" no longer means necessarily an infinity, but may be indicated by any time-like geodesic which cannot be continued beyond some finite proper time. Although the singularity is not described in full detail, the methods are rigorous rather than persuasive, and there is no question that the conclusions follow from the hypotheses. Further, one can argue persuasively that the hypotheses are actually satisfied in our universe.[1]

* Supported in part by NSF Grant No. GP 8560 and NASA Grant No. NSG-21-002-010.

[1] S. W. Hawking and G. F. R. Ellis, Astrophys. J. **152**, 25 (1968). A brief review and references to earlier work can be found here. For a more extensive review see R. B. Partridge, Am. Scientist **57**, 37 (1969).

[2] G. Gamow, Nature **162**, 680 (1948); R. A. Alpher and R. C. Herman, *ibid.* **162**, 774 (1948); Rev. Mod. Phys. **22**, 153 (1950).

[3] A. A. Penzias and R. W. Wilson, Astrophys. J. **142**, 419 (1965).

[4] M. J. Rees and D. W. Sciama, Nature **211**, 1283 (1966).

[5] A. A. Friedman, Z. Physik **10**, 377 (1922); **21**, 326 (1924).

[6] E. Hubble, Proc. Natl. Acad. Sci. U. S. **15**, 168 (1929).

[7] E. M. Lifshitz and I. M. Khalatnikov, Advan. Phys. **12**, 185 (1963).

[8] C. W. Misner and A. H. Taub, Zh. Eksperim. i Teor. Fiz. **55**, 233 (1968) [English transl.: Soviet Phys.—JETP **28**, 122 (1969)].

[9] Lifshitz *et al.* draw a different conclusion, as is discussed in the Appendix.

[10] R. Penrose, in *Battelle Rencontres 1967*, edited by C. DeWitt and J. Wheeler (W. A. Benjamin, Inc., New York, 1968), Chap. 7.

Other theories in physics have also rigorously predicted singularities: In Euler's hydrodynamic equations most solutions develop discontinuities, and in Lorentz's electrodynamics a classical Bohr-like atom radiates infinite energy in a finite time. In these cases a more refined and accurate physical theory (transport processes, quantum theory) showed no true singularity, so to accept Einstein's theory would be to accept something never before known in physics, the concept of a true singularity actually achieved in nature. Therefore most authors[11] presume that refinements will be required in Einstein's theory under extreme conditions, and that a true singularity in nature is not to be expected. The significance of the singularity theorems, then, is that they tell us the Universe has evolved from an earlier state in which conditions were so extreme that the presently known laws of physics were inadequate.

I prefer a more optimistic viewpoint ("Nature and Einstein are subtle but tolerant") which views the initial singularity in cosmological theory not as proof of our ignorance, but as a source from which we can derive much valuable understanding of cosmology. Thus, while I presume that relativity, like other physical theories, will be improved from time to time, I do not see that these changes need bear directly on the present problem of the cosmological singularity. Therefore I propose to discuss the singularity within Einstein's theory. This means accepting all known theorems and examples. As a working hypothesis I presume that the inevitable singularity required by the theorems is of the worst possible type, typical of the examples, where all matter experiences infinite density at a finite proper time in the past. One then sets about tolerating this assumed consequence of the mathematics.

Progress in physics can proceed both from tolerance and from intolerance. One could be intolerant of classical models of the atom because atoms were observed not to suffer catastrophic radiative decay. But Einstein[12] and Bohr[13] were tolerant of Planck's theory of radiation (in spite of its singular discontinuities) which violated no observation. Relativistic cosmology has had reasonable success (the expansion of the Universe, the microwave background) tending in the direction of support for its most dramatic theoretical novelty (the initial singularity), and there are no observational indications of, say, an era of contraction preceding the present expansion. Since the objections to a singularity are conceptual, rather than observational, then, I judge it is a situation in which tolerance is indicated. We should stretch our minds, find some more acceptable set of words to describe the mathematical situation now identified as "singular," and then proceed to incorporate this singularity into our physical thinking until observational difficulties force revisions on us.

The concept of a true initial singularity (as distinct from an indescribable early era at extravagant but finite high densities and temperatures) can be a positive and useful element in cosmological theory. For instance, I have proposed[14] that neutrino viscosity above 10^{10}°K could lead to a specific finite limit on the 12-h anisotropy of the microwave background radiation. Whether this calculation based on a viscosity approximation can be validated by kinetic-theory computations remains to be seen.[15,16] Stewart[16] has suggested a more basic objection to the argument, however. He points out that the equations which govern the problem are regular, well-posed differential equations, so that the simple continuity of the solutions as functions of the initial conditions shows that no finite limit on the present anisotropy can result if arbitrary anisotropy is admitted at some finite initial epoch, whether that be 10^{14}°K, or even higher. The continuity requires that the differential equations be regular on a finite interval, and serves to point up the essential contribution which a singularity brings. For equations which are singular at the initial time (or which set the initial conditions in the infinite past), an infinite range of initial conditions could evolve into a finite range of possible present conditions. Thus any argument that some features of the present universe are independent of most parameters specifying the initial conditions[17] could only succeed if initial conditions are specified at a true singularity, or in the infinite past, but not at any finite and regular past era.

Some possibilities for understanding the large-scale homogeneity of the Universe may also be opened up by treating the initial singularity as an acceptable element in the theory. There is a model of a homogeneous closed universe which I have called the "mixmaster universe"[18] and which has a very interesting initial singularity. It is an *anisotropic* modification of the closed Robertson-Walker model, and could evolve into a closed *isotropic* present-day universe if neutrino viscosity, or some other anisotropy limiting process, is effective. In this model the anisotropy is measured by the ratios of the circumference of the Universe in the three orthogonal space

---

[11] J. A. Wheeler, in *Relativity, Groups, and Topology—Les Houches 1963*, edited by C. DeWitt and B. DeWitt (Gordon and Breach, Science Publishers, Inc., New York, 1964); *Battelle Rencontres*, edited by C. Dewitt and J. Wheeler (W. A. Benjamin, Inc., New York, 1968) Chap. IX; R. Penrose, Ref. 10; S. Hawking and G. Ellis, Ref. 1; K. S. Thorne, in *High Energy Astrophysics—Les Houches 1966*, edited by C. DeWitt, E. Schatzman, and P. Veron (Gordon and Breach, Science Publishers, Inc., New York, 1967), Vol. III; I. D. Novikov, Astron. Zh. **43**, 911 (1966), [English transl.: Soviet Astron.—AJ **10**, 731 (1967)]; R. H. Dicke, Phys. Today **20**, 55 (1967).

[12] A. Einstein, Ann. Physik **17**, 132 (1905); **22**, 180 (1907).

[13] N. Bohr, Phil. Mag. **26**, 1 (1913).

[14] C. W. Misner, Phys. Rev. Letters **19**, 533 (1967); Astrophys. J. **151**, 431 (1968).

[15] A. G. Doroshkevich, Ya. B. Zel'dovich, and I. D. Novikov, Zh. Eksperim. i Teor. Fiz. (to be published); **53**, 644 (1967) [English transl.: Soviet Phys.—JETP **26**, 408 (1968)]. R. F. Carswell, Monthly Notices Roy. Astron. Soc. **144**, 279 (1969).

[16] J. M. Stewart (unpublished.)

[17] C. W. Misner, in *Battelle Rencontres 1967*, edited by C. DeWitt and J. Wheeler (W. A. Benjamin, Inc., New York, 1968), Chap. VI.

[18] C. W. Misner, Phys. Rev. Letters **22**, 1071 (1969).

directions. If we follow the development of this Universe backwards toward the singularity, we discover that first one axis is the longest, then another. The relative expansion rates change in an ergodic way, first maintaining one ratio for several e-foldings of the volume, then changing to another expansion rate ratio, etc. When two axes become very nearly equal, the solution approximates what is known as the Taub universe.[8] The approximation is a stable perturbation of the more highly symmetric Taub space, and therefore holds for very many decades of decrease in volume (as one approaches the singularity at zero volume), but small nonlinear terms eventually destroy the near symmetry. It is this close and long maintained approximation to the Taub universe which suggests that the establishment of homegeneity in the Universe could be understood with the help of this model. For in the Taub universe, light rays can travel many times around the Universe near the singularity, with each circumnavigation requiring a change in the volume of space by a factor $e^{4\pi}$. With causal interactions now capable of spanning the length of the Universe, the way is open to look for physical processes which could equalize gross discrepancies between the properties of different parts of the Universe. In the Taub universe, however, causal interactions propagate around the Universe in only one direction along the symmetry axis. But in the mixmaster universe, a prolonged Taub-like period (involving a volume change of some high power of $e^{4\pi}$) with near symmetry about one axis, is followed by a period of asymmetry in which first one axis, then another, expands most rapidly, until the ergodic nature of changes in expansion rate ratios again leads to a Taub-like epoch where two axes are nearly equivalent. This time the approximate symmetry axis need not be the same as before, however, and since each step in this process requires only a finite number of e-foldings of the volume, every possibility will be tried infinitely many times before the singularity is reached. Thus one anticipates that causal interactions spanning the entire Universe in all directions are possible near the singularity in the mixmaster universe. This approach to an understanding of the large scale homogeneity of the Universe makes little sense, however, if one is not prepared to take the singularity seriously. If high anisotropy only occurs at temperatures above 10^{10}°K, and if one was not prepared to discuss any physics above 10^{20}°K, this would leave only about ten decades of expansion in which to invoke the mechanisms of the mixmaster universe. With each step in the ergodic search for near symmetry already involving several decades of expansion, and each Taub-like era of long-range interaction lasting many decades of expansion, little if any mixing could be achieved in these few decades. But if the computations accept the singularity and therefore contemplate infinitely many decades of expansion, we have here a promising approach to an understanding of large-scale homogeneity.

Two questions remain to be discussed: First, if nothing else modifies the singularity out of Einstein's theory a more sensible epoch, must we not at least presume that the quantization of space-time curvature will make classical singularity ideas break down when the radius of the Universe is less than $(G\hbar/c^3)^{1/2}=10^{-33}$ cm? Second, how can we rephrase the language of physics to make the singularity appear more reasonable once we decide to accept it?

The surprising answer to the first question is no; quantizing gravitation does not appear to have any major effect on the singularity. As is shown in the accompanying paper, the quantum theory of gravity[19] can be applied to models of the universe as complex as those that can be treated classically. The two main results of quantum cosmology are: (1) The degree of freedom corresponding to the expansion of the Universe is not directly quantized, but like longitudinal electromagnetic fields, is constrained to be a specified function of the independent metric and matter degrees of freedom; and (2) the independent metric degrees of freedom, like electromagnetic and other quanta, are compressed adiabatically as one approaches the singularity, increasing their energies but not their quantum numbers. Thus modes which are classical now (high quantum number) remain classical at the singularity. Briefly stated, the quantum theory of gravity gives no indication that it will significantly change the nature of the initial singularity.

On to the final question then, and the title of this essay: Can we make a lower limit on time (at the singularity) some finite proper time in the past seem as harmless as a lower limit on temperatures some finite number of degrees below normal body temperature? I think so. We must distinguish two senses of time, a philosophical or psychological sense of time, and a physical definition of time which is proper time. Now proper time is a very precise and therefore limited concept, while the other concept of time is a very untidy but fertile concept which can evolve as we carry it to richer intellectual grounds. It has already evolved. Who now would argue that a pendulum keeps good time by referring to a heartbeat as the standard? And we recognize that the second, the day, and the year are not just different units for the same thing (like the centimeter and the meter), but genuinely different kinds of time tied to atomic constants, the earth's rotation, and its orbit period, respectively. When men settle on other planets they will no doubt adopt a different day and year to live by, but do physics with the same second. It is clear that the age of the Universe is not to be measured by counting earth days or years, then, since the earth's motions change or disappear as we push back into history. But ammonia molecules also disappear as we delve back toward the orgins of the universe, so

[19] C. W. Misner, Phys. Rev. **186**, 1319 (1969), accompanying paper.

an atomic clock is not qualitatively superior to the earth's motion. Nuclear transitions are available at temperatures below 10^9°K, but we might prefer to use the muon lifetime as a unit near 10^{12}°K (100 MeV) when nuclei could not exist but thermal muon pairs were plentiful. At 10^{16}°K (10^3 GeV) something unfamiliar, no doubt, is the natural prevalent standard of proper time. With a good knowledge of physics, all these time standards can be interrelated, but none of them really appeals to us as being the ultimate heart of the Universe, whose beats give us a standard of time more enduring than man, or the earth, or the atom.

In this discussion we have recited some history of the Universe in an increasingly standard way. Epochs are labeled by temperature, which is roughly the same as redshift, $T=(3°\mathrm{K})(1+z)$. The useful label is really the exponent, $\ln(1+z)$ or $\log T$. In theoretical studies of expanding universe models, including the closed mixmaster model[18] described previously, a convenient epoch label in Einstein's equations has turned out to be equivalent, namely $\Omega=-\ln(V^{1/3})$, where V is the volume of space at the given epoch. I find this Ω time to be very attractive as a primary standard;[20] the enduring measure of evolution throughout the history of the Universe is its own expansion. We see that significant cosmological epochs (galaxy formation, nucleosynthesis, hadron era, etc.) are spaced at reasonable intervals of Ω or $\log T$. The most general example yet studied of the original singularity[18,21] (which is by definition at $\Omega\rightarrow\infty$, an infinite Ω time in the past) is the mixmaster universe, where expansion rates in the different directions maintain essentially constant ratios, for finite intervals $\Delta\Omega$ of Ω time, then change to new values which again hold for another finite $\Delta\Omega$ interval. Successive Ω intervals of constant expansion rate ratios are, on the average, longer than the preceding one by a small factor, and are occasionally interspersed by a very long, but still finite, $\Delta\Omega$ interval in which two axes have nearly equal (very small) expansion rates. Thus, as we approach the singularity $\Omega\rightarrow\infty$, we see the universe ticking away in Ω time quite actively. *The Universe is meaningfully infinitely old because infinitely many things have happened since the beginning.*

Since objections to a finite age for the Universe are philosophical and not observational, one must use a philosophical concept of time in such discussions. In my view, then, one is satisfied by an infinite age reflecting an infinite succession of noticeable events, as in the history of the Universe apparently suggested by Einstein's general relativity. When one considers the times required at different densities for the constituents of the Universe to reach thermal equilibrium, one can even justify the summary: In the first second of its existence the Universe evolved slowly through infinite epochs, gradually speeding up toward an explosive expansion during the ten billion years of the most recent epochs. For it was in the hot early stages that the expansion was slow enough to allow thermal equilibrium, and only recently has matter expanded too fast to thermalize radiation. The cosmologically significant rates are measured by $d/d\Omega$: How many reactions occur per e-folding of the size of the Universe? A proper time rate, how many reactions per revolution of the earth, or per π^0 lifetime, can seem very pointless or parochial in the wrong epoch. Only by such parochial standards does the cosmological singularity present itself as an object of concern. By a better standard, the approach to the singularity shows a Universe which beats at a rather regular but gradually slowing rate in the infinite past, $\Omega\rightarrow\infty$.

APPENDIX

Khalatnikov, Lifshitz *et al.*[7,20,22] do not draw from the examples the same conclusion which I have at the conclusion of the third paragraph of the above essay. They assume, contrary to the experience from the examples, that most cosmological solutions have no singularity. I discern two main arguments for this conclusion in their work. The first argument is that their examples have been produced specifically by studying singularities and will include essentially all conceivable types of singularities. Then, since the more general undiscovered solutions are not of these catalogued singular types, they must be intrinsically nonsingular. The flaw in this agrument is that the typical cosmological solution may have a singularity of a presently inconceivable type. There is even an example to illustrate this flaw, for subsequent to the paper[22] which first suggested that a sufficiently complete survey of possible singularities had been made, a subsequent paper[21] (and independently my own work[18]) exhibited a true singularity of a previously inconceivable type.

The second argument of Lifshitz and Khalatnikov and the one to which they appear to give more weight, concerns not a singularity in a solution of Einstein's equations, but in a coordinate representation of some portion of a solution. (By "solution" I mean an abstract Reimannian manifold of Lorentz signature whose Ricci tensor, in the case of empty space, vanishes. A "singularity of a solution" means, in this paper, that some curvature scalar becomes infinite.) Now a singularity in a coordinate representation of a solution may occur either because the solution is singular in, or on the boundary of, the region where the coordinates are de-

[20] Professor Chandrasekhar has called to my attention that a similar time, $\tau\propto\log t$ has been advocated by E. A. Milne, *Kinematic Relativity* (Oxford University Press, New York, 1948).

[21] V. A. Belinsky and I. M. Khalatnikov, Zh. Eksperim. i Teor. Fiz. **56**, 1700 (1969) [English transl.: Soviet Phys.—JETP (to be published)].

[22] E. M. Lifshitz, V. V. Sudakov, and I. M. Khalatnikov, Zh. Eksperim. i Teor. Fiz. **40**, 1847 (1961) [English transl.: Soviet Phys.—JETP **13**, 1298 (1961)].

fined (true singularity), or because the coordinates themselves are singular at points where the solution is regular (coordinate singularity). As Lifshitz and Khalatnikov show, coordinate representations satisfying $g_{0\mu}=-\delta_\mu{}^0$, called synchronous coordinates, have singularities of the type $(-g)^{1/2}\rightarrow 0$, but this singularity they show is, in general, a coordinate singularity or "fictitious singularity." They then conclude[7] "Any grounds for the existence of another type of singularity, which would be real and at the same time peculiar to the general solutions, therefore essentially disappear." I am unable to follow this last quoted step in the argument.

Synchronous coordinate systems can be set up by simple constructions,[7] and one can easily arrange that any regular point of a solution lie on the $\sqrt{(-g)}=0$ singular set bounding an appropriately chosen synchronous coordinate system. But there is no demonstration that synchronous coordinate systems can be chosen in such a way that they reach the singular points of the solution. Thus there is no reason to believe that the singularities of solutions (conceived of in a coordinate-independent way) can be discussed on the basis of representations in synchronous coordinates. Another way to phrase this objection is to note that not only does the construction of a synchronous coordinate system (by choosing an arbitrary initial hypersurface and erecting geodesics normal to it[7]) lead almost always to a fictitious singularity in flat space,[7] but also in a Robertson-Walker cosmology, or in any other singular solutions. *Thus the nature of the singularity in the generic synchronous coordinate representation of a solution is unaffected by whether or not the underlying solution is singular.*

A synchronous coordinate system is essentially a hypersurface-orthogonal congruence of time-like geodesics. The coordinate system fails (becomes singular) when the geodesics intersect one another.[7] Let us estimate when this will occur. If the initial hypersurface has a radius of curvature ρ (second fundamental form $\sim 1/\rho$), then even in flat space the failure will occur within distance (or proper time) $\sim\rho$ from the hypersurface. In curved space-time, however, the Einstein equations show that the curvature always has a sign tending to make the failure occur earlier.[7] Thus if the congruence of geodesics passes through a region of space-time whose curvature is $\sim 1/\rho^2$, some geodesics can be expected to intersect a time ρ later. In particular, if the initial hypersurface for the synchronous coordinates includes our Galaxy at the present epoch, then it includes the sun. A congruence (continuous space-filling family) of geodesics covering the sun will fail in about one hour, since the radius of curvature of space-time near the sun or near any other object of comparable average density is about 1 h. (Think of the geodesics as paths of test particles; for particles starting from rest near the solar surface, the hour is the free-fall time to collisions at the center, or at moderate velocities it is the orbit period.) Thus synchronous coordinate systems constructed now in the real Universe last at most an hour, not the 10^{10} years required if they were to lead to any information about the initial singularity of cosmological theory. If the Universe now contains a neutron star (or anything of comparable density, e.g., a nucleus), the lifetime of a synchronous coordinate system is less than a millisecond, or if a galaxy is the smallest scale irregularity one is prepared to notice, the life of a synchronous coordinate system could be extended to 10^7 years, but it remains equally irrelevant to the study of the initial singularity. It also does not help to choose the initial hypersurface of a synchronous coordinate system closer to the singularity, since the generic solution which resembles our Universe now will, if extrapolated back toward the singularity, be even more irregular near the singularity,[7] so synchronous coordinates will then extend an even smaller fraction of the required distance to the singularity than now.

Note added in proof. In a recent report[23] Belinski and Khalatnikov have provided the final steps in the analysis of a general solution of the Einstein equations whose singularity is qualitatively similar to that of the mixmaster universe.[18,19,21] The aim of the singularity studies of Khalatnikov and Lifshitz, from the beginning, has been the construction of a general solution; they summarize the significance of this recent result in Ref. 24. One most important result is that the knowledge gained by these investigations[24] is now seen to be comfortably consistent with the singularity theorems.[1,10] But it furthermore complements the theorems in an extremely useful way by shedding the light of a very powerful and general example on the problem of describing the nature of the singularity. In this Khalatnikov-Lifshitz example,[24] the singularity corresponds to infinite curvature at every point of space. It is therefore of just the type which the viewpoint I have formulated in this paper is designed to accept. The generality of the example is expressed in the statement that the nature of the singularity is stable under arbitrary small changes in the Cauchy data defining it. It has not yet been established whether there exist stable singularities of any quite different character, nor does one know of any precise way to state that the nonsingular character of empty flat Minkowski space is stable. [The difficulty here is that, although most perturbations of flat space are believed to be stable, and all are in linear order, a singularity can arise from certain small perturbations. Let any small limit $m>0$ be placed on the total energy of the perturbation, and let $|g_{\mu\nu}-\eta_{\mu\nu}|<\epsilon$ be required at $t=0$. Then no matter how small the positive numbers m and ϵ are, a carefully focused pulse of incoming gravitational radiation can be found which (a) will satisfy these conditions

[23] V. A. Belinski and I. M. Khalatnikov, Zh. Eksperim. i Teor. Fiz. (to be published).

[24] I. M. Khalatnikov and E. M. Lifshitz, Phys. Rev. Letters (to be published).

and (b) will a finite time later localize the energy in a region of dimensions that are small compared to the Schwarzschild length m, and thus initiate a gravitational collapse to a singularity.[25] It seems to me, however, that the statement "Flat space-time is unstable..." would be a misleading summary of this situation.] Our understanding of the nature of the Einstein singularity is therefore quite meager, but the Khalatnikov-Lifshitz examples provide the bulk of it.

[25] R. Ruffini and J. A. Wheeler (unpublished).

PHYSICAL REVIEW VOLUME 186, NUMBER 5 25 OCTOBER 1969

Gravitational Field of a Sphere Composed of Concentric Shells

A. L. Mehra
Department of Mathematics, Jodhpur Polytechnic, Jodhpur, India

AND

P. C. Vaidya
Department of Mathematics, Gujarat University, Ahmedabad, India

AND

R. S. Kushwaha
Department of Mathematics, Jodhpur University, Jodhpur, India

(Received 30 April 1969)

A general solution of the field equations of general relativity theory has been obtained for a composite sphere having a number of shells, one above the other, of different densities.

I. INTRODUCTION

At present, we have few solutions[1] to describe the interior gravitational field of spherically symmetric bodies having variable densities. In general relativity it is very difficult to get analytic solutions of the field equations for any kind of variable density. In order to consider a fluid distribution with a variable density, we may use the simple device of considering the distribution as made up of different strata of uniform densities. Accordingly, the body may be considered as a composite sphere having a number of concentric shells, one above the other, of different densities. The number of shells and their densities may be assigned according to the distribution of matter in the body. The present investigation represents an attempt to provide a basis for a relativistic theory of stellar interiors.

II. FIELD EQUATIONS AND BOUNDARY CONDITIONS

Let us take the line element as

$$ds^2 = -e^{\lambda}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2 + e^{\nu}dt^2\,, \qquad (2.1)$$

where λ and ν are the functions of r only. Since the matter consists of a perfect fluid at rest, the components of the energy-momentum tensor $T_j{}^i$ satisfy the relations

$$T_1{}^1 = T_2{}^2 = T_3{}^3 = -p,\quad T_4{}^4 = \rho,\quad T_j{}^i = 0 \text{ for } i \neq j. \qquad (2.2)$$

The field equations for the line element (2.1) are given by Tolman[2] as

$$8\pi p = e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} \qquad (2.3)$$

$$= e^{-\lambda}\left(\frac{\nu''}{2} - \frac{\nu'\lambda'}{4} + \frac{\nu'^2}{4} + \frac{\nu' - \lambda'}{2r}\right) \qquad (2.4)$$

$$= e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2}\,, \qquad (2.5)$$

where the prime denotes differentiation with respect to r.

The solution of the field equations must satisfy the following conditions across any surface of discontinuity of density: (i) The gravitational potentials e^{λ} and e^{ν} must be continuous; (ii) the first differential coefficient of e^{ν} with respect to r must be continuous; (iii) the pressure must be positive and finite everywhere inside the sphere, and zero at the outer surface of the sphere. Condition (ii) ensures the continuity of pressure across any surface of discontinuity of density.

[1] H. R. Buchdahl, Astrophys. J. **140**, 1512 (1964); R. F. Tooper, *ibid.* **140**, 434 (1964); **142**, 1541 (1965); A. L. Mehra, J. Australian Math. Soc. **6**, 153 (1966).

[2] R. C. Tolman, *Relativity, Thermodynamics, and Cosmology* (Oxford University Press, London, 1962), p. 244.