Quantum Cosmology. I*†

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The Hamiltonian methods of Arnowitt, Deser, and Misner can be applied to homogeneous cosmological models, and prove to be an efficient way both of constructing the Einstein equations and of studying their solutions. By using an appropriate form for the metric, one finds that the constraint equations for these models can be solved explicitly, and the resulting problem in Hamiltonian mechanics resembles that of a particle in a potential well. The most unusual feature of the Hamiltonian is that it is explicitly time dependent. There is an easy and attractive choice of factor orderings which allows one to pass on to a quantum theory (by imposing canonical commutation relations on the independent canonical variables) while maintaining the signature of the quantized metric. For the closed-space cosmological model (Bianchi type IX) which is studied in most detail, a classical (high-quantum-number) state remains classical as the wave function is followed back in time toward the initial singularity. There is no tendency for significant contributions from states of low quantum number to develop even when the radius of the universe is much less than $(G\hbar/c^3)^{\frac{1}{2}}=10^{-33}$ cm.

I. INTRODUCTION

THE methods which Arnowitt, Deser, and Misner[1] (ADM) developed with the aim of quantizing Einstein's theory of gravity can be applied, as this paper will describe, to models of the Universe as complicated as any which have been studied in classical general relativity. The result is a quantized model of the Universe in which our main interest is directed toward quantum effects on the singularity at the beginning of time, which was discussed in the preceding paper.[2] It appears from this calculation that quantum effects do not significantly modify the nature of the initial singularity in relativistic cosmology. In particular, I find no suggestion of anything which would allow a contracting closed universe to pass through a quantum phase and emerge as an expanding universe.

The quantized model universe presented here has somewhat the relationship to a full quantum theory of gravity which the harmonic oscillator has to the quantum theory of the electromagnetic field. One may Fourier-analyze the free electromagnetic field, find that the amplitude of a single mode satisfies a harmonic-oscillator equation, and thus learn something of the quantum properties of the electromagnetic field by solving the Schrödinger equation for a harmonic oscillator. For the gravitational case where we wish to retain nonlinear effects near the cosmological singularity, Fourier analysis is not the appropriate tool, but something similar is achieved by imposing a definite space dependence on the metric (corresponding to a high degree of symmetry) and then letting quantum theory govern the time dependence of the remaining amplitudes. The method has, in common with Fourier analysis of free fields, the property that the single mode is an exact solution in the classical theory. It differs from Fourier analysis of free fields in the treatment of nonlinearities. For the modes we do include (two degrees of freedom in the present example) the mutual and self-interactions are treated completely. For the modes we omit (infinitely many), the interactions with the modes under study are ignored.

The two degrees of freedom in the gravitational field which our model includes may usefully be described[3,4] as the two polarization states of a gravitational wave mode whose wavelength is the maximum possible (lowest wave number) in this closed universe. The omitted modes are gravitational waves with any higher wave numbers. The classical exact solution[5,6] is obtained by simultaneously setting to zero both the amplitudes ("coordinates") and momenta of all these higher modes. This is, of course, not possible in quantum theory, so the model of a quantum theory of gravity presented here is just that, a model. It is not an exact description of a subspace or quotient space of the full theory of a quantized metric field. There is, however, some reason to expect that this model may approximate behaviors which would arise in the full theory. Most of the modes which have been neglected are those at high wave number. But precisely in the limit of high wave number, good approximations[7–9] are available in the classical theory which show (a) that different modes of high wave number do not interact with each other in the first or second order of approximation, and (b) that these short-wavelength gravitational waves influence the "expansion of the universe" modes (which our model retains) in just the same way as do electromagnetic

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† Dedicated to the memory of Eva Deser, 1958–1968.

[1] R. Arnowitt, S. Deser, and C. W. Misner, in *Gravitation: An Introduction to Current Research*, edited by L. Witten (Wiley-Interscience, Inc., New York, 1962), Chap. 7.

[2] C. W. Misner, Phys. Rev. **186**, 1328 (1969), accompanying paper.

[3] D. R. Brill, Nuovo Cimento Suppl. **2**, 3 (1964).

[4] J. A. Wheeler in *Conférence Internationale sur les Théories Relativistes de la Gravitation*, edited by L. Infeld (Gauthier-Villars, Paris, 1964), pp. 223–268.

[5] C. W. Misner, Phys. Rev. Letters **22**, 1071 (1969).

[6] C. W. Misner (unpublished).

[7] D. R. Brill and J. B. Hartle, Phys. Rev. **135**, B271 (1964).

[8] R. A. Isaacson, Phys. Rev. **166**, 1263 (1968); **166**, 1272 (1968).

[9] Y. Choquet-Bruhat (unpublished).

waves. But expanding-universe models containing electromagnetic radiation have been studied extensively[10] by many authors, and one finds that while the energy density of radiation increases as $(\text{volume})^{-4/3}$ as one approaches the initial singularity, $(\text{volume})\to 0$, the energy density of the lowest (anisotropy) modes increases more nearly as $(\text{volume})^{-2}$. These lowest modes, which our quantized cosmology retains, are therefore expected to be the dominant influence near the initial singularity. Thus, near the singularity, the lowest modes are effectively decoupled from the high-wave-number modes and the results of quantizing just the lowest modes should be the same (i.e., parallel to Fourier analysis of free fields) as if the highest modes had also been included. The major uncertainty in the significance of this model calculation, then, concerns the possible effect of including modes of intermediate wavelengths, where the size of the universe would be just a few wavelengths.

The model presented here is significant for two other reasons, beyond its implications for the nature of the cosmological singularity, namely as an introduction to ADM[1] Hamiltonian techniques in classical cosmological theory, and as a testing ground for methods, concepts, and interpretations in the theory of quantized geometry. The model is attractive for these purposes primarily because of the relative simplicity of the Hamiltonian, which reads

$$H=\{p_+^2+p_-^2+e^{-4\Omega}[V(\beta_+,\beta_-)-1]\}^{1/2}. \tag{1.1}$$

Here $\beta_\pm$ are the amplitudes of the two independent modes, $p_\pm$ are their conjugate momenta, and Ω is the time coordinate so that Hamilton's equations read $d\beta_+/d\Omega=\partial H/\partial p_+$, $dp_+/d\Omega=-\partial H/\partial\beta_+$, etc., and $dH/d\Omega=\partial H/\partial\Omega$. The function $V(\beta_+,\beta_-)$ is a certain positive definite combination of exponential functions which is defined and described in detail later. This Hamiltonian presents, then, a textbook-type problem in classical mechanics or quantum mechanics. Because it does not reduce to a one-dimensional problem, it is more difficult than the classical central-force problem, and is perhaps comparable in difficulty to the study of charged-particle orbits in an axially symmetric but nonuniform magnetic field (magnetic-mirror machine). Consequently, a qualitative description of the solutions can be given, and reliable approximations can be used to compute any quantitative information that is desired.

With a Hamiltonian as simple as Eq. (1.1), perplexing questions in the quantum theory of space-time geometry can be modelled in a very concrete form. For instance, the theory is given here with a particular choice of a time coordinate Ω chosen so that the volume of space at a time Ω is proportional to $e^{-3\Omega}$. The usual cosmic time t, which is proper time for the standard class of observers, can be related to Ω by

$$dt=-H^{-1}e^{-3\Omega}(2/3\pi)^{1/2}d\Omega. \tag{1.2}$$

But the Hamiltonian H is an operator in the quantum theory, so the coordinate time Ω and the cosmic time t cannot both be c-numbers. At least one of them must be an operator. Equation (1.2) thus models the problem of *quantum covariance*. In the present paper we will not make use of cosmic time t, and it seems natural and straightforward to treat the coordinate time Ω as a c-number. But we are led to ask if there is another formulation of the theory in which t could be treated as a c-number. Are the quantum theories based on t time and Ω time equivalent? Is there some evenhanded way to treat both t and Ω simultaneously as operators, with the present treatment corresponding simply to a choice of representation where Ω is diagonal? It would seem that further study and extension of the model theory presented here promises to shed some light on these questions. Other questions which could be simplified, made concrete and calculable, and illustrated working from this model are the relationships between the various techniques and emphases of Dirac,[11] ADM,[1] and Wheeler[12] in their approaches to quantizing general relativity. It is also possible that the classical analogy between the cosmological singularity and the one which occurs at $r=0$ in the Schwarzschild metric (i.e., as the endpoint of Oppenheimer-Snyder stellar collapse) can be extended to the quantum domain and will suggest some other relatively simple quantum theory models like that presented here, but with particular relevance to stellar collapse.

It may appear anomolous that as a first example of a quantized cosmology I choose a rather complicated cosmological model,[5,6] a closed space with different expansion rates along the three different axes. The reason for this is that the ADM Hamiltonian is so powerful that simpler models look implausible. For a radiation-dominated Robertson-Walker (RW) universe with flat space sections, for instance, which is the standard model of the early hot big-bang universe, the Hamiltonian is $H=\Gamma^{1/2}e^{-\Omega}$, where Γ is a constant. This Hamiltonian is an explicit function of coordinate time Ω, but does not depend on any canonical coordinate or momentum since the system has no unconstrained gravitational degrees of freedom. The general RW cosmology containing pressureless matter and fluid

[10] See, for example, Refs. 28 and 6.

[11] P. A. M. Dirac, Phys. Rev. **114**, 924 (1959); in *Recent Developments in General Relativity* (Pergamon Publishing Corp., New York, 1962), p. 191; in *Fluides et Champ Gravitationnel en Relativité Générale* (Centre National de la Recherche Scientifique, Paris, 1969), p. 13. For a recent development in the Dirac and ADM type of quantization, and for further references, see A. Peres, Phys. Rev. **171**, 1335 (1968).

[12] J. A. Wheeler in *Battelles Recontres 1967*, edited by C. M. DeWitt and J. A. Wheeler (W. A. Benjamin, Inc., New York, 1968), pp. 242–307; in *Relativity Groups and Topology, Les Houches 1963*, edited by C. DeWitt and B. deWitt (Gordon and Breach, Science Publishers, Inc., New York, 1964), pp. 317–520.

radiation is similar. The Hamiltonian is

$$H=(\Gamma e^{-2\Omega}+\mu e^{-3\Omega}-ke^{-4\Omega})^{1/2}, \tag{1.3}$$

where again there are no canonical variables and no Hamilton equations apart from $dH/d\Omega=\partial H/\partial\Omega$. The constants Γ and μ are determined from the total number of radiation quanta and massive particles, respectively, in the universe, while $k=0$, ± 1 gives the space curvature. Thus these RW models contain exclusively the unfamiliar features of quantum gravity, namely, the treatment of constraints, but none of the familiar features of ordinary mechanics or quantum mechanics. These RW models will probably prove useful in further studies of the unfamiliar parts of quantum cosmology, and the $k=+1$ model has already been subject to such a study by DeWitt.[13] It should, however, be helpful to see something familiar before approaching deeper problems, so we move on to models which do contain some independent dynamical variables. A simple model for this purpose is an empty expanding universe with flat space sections which have different expansion rates in different directions (the Bianchi type-I anisotropic homogeneous cosmology) which gives the Kasner[14] metrics in classical cosmology. In this case the Hamiltonian becomes

$$H=(p_+^2+p_-^2)^{1/2}, \tag{1.4}$$

with $p_\pm$ the momenta conjugate to field amplitudes $\beta_\pm$. Thus Hamilton's equations read $d\beta_\pm/d\Omega=\partial H/\partial p_\pm=p_\pm/H$, $dp_\pm/d\Omega=-\partial H/\partial\beta_\pm=0$, and $dH/d\Omega=\partial H/\partial\Omega=0$. This is the familiar problem of a free particle, and the classical solutions can be written down immediately, but in quantum theory it has the drawback that H has a continuous rather than a discrete spectrum. To avoid this technical difficulty and make it possible to speak of discrete eigenfunctions and of quantum numbers, we go to a closed (type IX) cosmological model for which we will derive the Hamiltonian of Eq. (1.1). Another reason for preferring the closed-universe model is that this model offers some prospects for understanding the homogeneity of the Universe,[5,6] so the closed universe models may turn out to be physically more significant.

II. CANONICAL FORMALISM

The approach to quantum theory of gravity with which I am most familiar is the ADM canonical method.[1,15] This approach in many ways closely parallels the earlier Hamiltonian methods of Dirac.[11,16] The correspondence has been described in greatest detail by Kimura[17] and Anderson.[18] The principal differences are that ADM imagine that the constraints have been solved, eliminating the extraneous degrees of freedom, and show how the remaining independent degrees of freedom can be put in canonical form, with a Hamiltonian giving rise to equations of motion in the conventional way. Dirac assumes that the constraints remain a part of the statement of the theory, the amplitudes and their conjugate momenta are thus not independent, and the way in which the Hamiltonian is used to construct the equations of motion is modified. The ADM approach anticipates quantization of the Schwinger type[19] without subsidiary conditions; the Dirac approach anticipates a quantization with subsidiary conditions[20] defining the physical subspace of state vectors, and uses an indefinite metric in Hilbert space. Both methods have been used successfully and are equivalent for electrodynamics. The Dirac approach was used by DeWitt[13] in the only previous study of quantum cosmology that I am aware of. Other closely related methods are Schwinger's work[21] along ADM lines, and the Feynman sum-over-histories approach to quantizing gravity[22] which Wheeler[12] uses and whose relationship to Hamiltonian methods is discussed by Leutwyler.[23] Manifestly covariant quantization methods have been developed by Feynman,[24] DeWitt,[25] and Mandelstam.[26] For a discussion of the applicability of these different approaches, and for references to the pioneering work of Rosenfeld and of Bergmann, consult DeWitt's paper.[13]

In the ADM method which I adopt, the variational principle for Einstein's equations, $\delta I=0$ with

$$I=(16\pi)^{-1}\int R(-g)^{1/2}d^4x,$$

is written in the form $I=\int \mathfrak{L}d^4x$, with[27]

$$(16\pi)\mathfrak{L}=\pi^{ij}(\partial g_{ij}/\partial t)-NC^0-N_iC^i. \tag{2.1}$$

In this variation principle one varies g_{ij}, π^{ij}, N, and N_i independently, and the C^μ are defined by

$$C^0\equiv-(g)^{1/2}\{{}^3R+g^{-1}[\tfrac{1}{2}(\pi^k{}_k)^2-\pi^{ij}\pi_{ij}]\},$$
$$C^i\equiv-2\pi^{ij}{}_{|j}. \tag{2.2}$$

All notations in Eq. (2.1) and subsequent equations are three dimensional with g_{ij} the spatial components (i, j, k, etc.$=1, 2, 3$) of the space-time metric, so g^{ij} is the matrix reciprocal to g_{ij} and ${}^3R$ is the scalar curvature computed from g_{ij}. Indices are raised and lowered with g_{ij}, and the vertical bar in $\pi^{ij}{}_{|j}$ indicates a covariant

[13] B. S. DeWitt, Phys. Rev. **160**, 1113 (1967).
[14] E. Kasner, Am. J. Math. **43**, 217 (1921).
[15] R. Arnowitt, S. Deser, and C. W. Misner, Nuovo Cimento **15**, 487 (1960).
[16] P. A. M. Dirac, Proc. Roy. Soc. (London) **A246**, 333 (1958).
[17] T. Kimura, Progr. Theoret. Phys. (Kyoto) **27**, 747 (1962).
[18] J. L. Anderson, Rev. Mod. Phys. **36**, 929 (1964).
[19] J. Schwinger, Phys. Rev. **82**, 914 (1951); **91**, 713 (1953).
[20] See, for example, S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Row, Peterson and Co., Evanston, Ill., 1961), Chap. 9.
[21] J. Schwinger, Phys. Rev. **130**, 1253 (1963); **132**, 1317 (1963).
[22] C. W. Misner, Rev. Mod. Phys. **29**, 497 (1957).
[23] H. Leutwyler, Phys. Rev. **134**, B1155 (1964).
[24] R. P. Feynman, Acta Phys. Polon. **24**, 697 (1963).
[25] B. S. DeWitt, Phys. Rev. **162**, 1195 (1967); **162**, 1239 (1967).
[26] S. Mandelstam, Ann. Phys. (N. Y.) **19**, 25 (1962).
[27] Units are chosen so that $G=\hbar=c=1$.

derivative formed using g_{ij}. Also, g is the determinant of g_{ij}, and satisfies ${}^4g\equiv\det g_{\mu\nu}=-N^2g$. As described in Ref. (1), varying the ten quantities g_{ij}, N, and N_i in this Lagrangian gives the ten Einstein equations, while varying the six π^{ij} gives equations relating the π^{ij} and the $\partial g_{ij}/\partial t$.

The four equations $C^\mu=0$ obtained by varying N and N_i in Eq. (2.1) are the constraints or initial-value equations. They show that the g_{ij} and the π^{ij} cannot be specified arbitrarily, even as initial conditions. Let us assume then that four quantities formed from the g_{ij} and π^{ij} are expressed in terms of eight others by solving the constraint equations $C^\mu=0$ given by Eq. (2.2). The variational principle then has a very simple appearance, for from Eq. (2.1) we now find

$$(16\pi)\mathfrak{L}=\pi^{ij}(\partial g_{ij}/\partial t)\,, \tag{2.3}$$

but the π^{ij} and g_{ij} can no longer be varied independently. The Lagrange multipliers N and N_i which one uses to go from Eq. (2.3) to Eq. (2.1) also play another role in the theory; they are used to construct the full metric ${}^4g_{\mu\nu}$ from a knowledge of g_{ij} and π^{ij}. The relationship is

$${}^4g_{0i}=N_i\,,\quad {}^4g_{00}=-(N^2-N_iN^i)\,. \tag{2.4}$$

The reduced Lagrangian (2.3) does not give any information about N and N_i, but actually the Lagrangian (2.1) also allows arbitrary values for N and N_i in the solutions. The values of N and N_i are determined only by coordinate conditions. In the Lagrangian (2.1) one might choose $N_i=0$, $N=1$ as coordinate conditions. In the Lagrangian (2.3) the coordinate conditions will be expressed in terms of the g_{ij} and π^{ij} and chosen in such a way as to assist in reducing the variational principle to canonical form. In this case the combinations of g_{ij} and π^{ij} restricted by the coordinate conditions can no longer be varied, and the equations for their time derivatives must be stated separately from the variational principle and serve to determine the N and N_i. In our applications these will be combinations of the equations

$$\partial g_{ij}/\partial t=2Ng^{-1/2}(\pi_{ij}-\tfrac{1}{2}g_{ij}\pi^k{}_k)+N_{i|j}+N_{j|i}\,, \tag{2.5}$$

which follow from Eq. (2.1) by varying π^{ij}.

A canonical form is a restatement of the variational principle $\delta I=0$ in a way that puts the action integral in the form

$$I=\int[\pi^A(\partial\phi_A/\partial t)-\mathcal{H}(\pi^B,\phi_B)]d^4x\,, \tag{2.6}$$

where $\mathcal{H}$ is then the Hamiltonian density. For our applications to homogeneous cosmologies, the field amplitudes and momenta are defined as functions of time only, so that the integration over space coordinates is carried out and the canonical form is that familiar from particle mechanics,

$$I=\int[p^Adq_A-H(p,q)\,dt]\,, \tag{2.7}$$

rather than Eq. (2.6) for field theory. The steps by which Eq. (2.3) is reduced to the form (2.7) are given in Sec. III for two particular types of cosmological metrics.

III. COSMOLOGICAL MODELS

We will consider two classes of homogeneous cosmological models, one for which the computations are extremely simple, another for which the results are more interesting. The metric for the simpler model is[28]

$$ds^2=-dt^2+R^2(t)(e^{2\beta})_{ij}dx^idx^j\,. \tag{3.1}$$

Here $\beta_{ij}(t)$ is a diagonal traceless matrix, so $e^{2\beta}=\mathrm{diag}\,(e^{2\beta_{11}},e^{2\beta_{22}},e^{2\beta_{33}})$ and $\det(e^{2\beta})=e^{2\mathrm{tr}\beta}=1$. Thus $R^3(t)$ is proportional to the volume of the universe at time t. The homogeneity of this space is expressed by the invariance of the metric (3.1) under the Abelian group of simple space translations $(t,x^i)\rightarrow(t,\,x^i+a^i)$. This group is type I in Bianchi's classification[29] of the three-dimensional local Lie groups, so that Eq. (3.1) is said to define a type-I homogeneous cosmology.[30]

In the action integral

$$I=(16\pi)^{-1}\int\pi^{ij}(\partial g_{ij}/\partial t)dtd^3x\,, \tag{3.2}$$

the integration over space coordinates can be done, since π^{ij} as well as g_{ij} is taken to be a function of t only. To obtain the same numerical factors here as in the closed universe we will treat later, let the standard coordinate volume in this space be chosen[31] so that $\int d^3x=(4\pi)^2$, rather than, for instance, using a unit volume. Then Eq. (3.2) becomes

$$I=\pi\int\pi^{ij}dg_{ij}\,, \tag{3.3}$$

and in this we can insert the form (3.1) chosen for the metric, namely, $g_{ij}=R^2(e^{2\beta})_{ij}$, to find

$$dg_{ij}=2g_{ij}d\ln R+2g_{ik}d\beta_{kj} \tag{3.4}$$

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[28] C. W. Misner, Astrophys. J. **151**, 431 (1968).

[29] L. Bianchi, Mem. Soc. It. della Sc. (dei XL), (3) **11**, 267–352 (1897); reprinted in L. Bianchi, *Opere* (Edizioni Cremonese, 1952–1958), Vol. 9, pp. 16–109.

[30] O. Heckmann and E. Schucking, in *Gravitation: An Introduction to Current Research*, edited by L. Witten (Wiley-Interscience, Inc., New York, 1962), Chap. 11.

[31] This choice affects the commutation relation and would have to be reconsidered before quantitative results could be deduced for this type-I universe. In an open universe where there is no discrete lowest-mode graviton, one should consider all sufficiently weak infrared gravitons together. Alternately, one could suppose x^i are coordinates on a three-torus (periodic boundary conditions) to make this universe closed and the mode with g_{ij} independent of x^i becomes a discrete mode.

and

$$I=2\pi\int \pi^i{}_k d\beta_{ki}+\pi^k{}_k d\ln R. \quad (3.5)$$

In this action integral not all the variables are independent, and variations must respect the constraints

$$^3Rg+\tfrac{1}{2}(\pi^k{}_k)^2-\pi^i{}_k\pi^k{}_i=0 \quad (3.6)$$

and

$$-2\pi^{ik}{}_{|k}=0. \quad (3.7)$$

The momentum constraints (3.7) are satisfied identically in the present case since the π^{ik} are functions of t only, and the three-space covariant derivative reduces to an ordinary derivative (g_{ij} is independent of the x^i). The Hamiltonian constraint (3.6) also simplifies because the three-space metric from Eq. (3.1) is flat, giving $^3R=0$. We can therefore solve Eq. (3.6) algebraically to give $\pi^k{}_k$ in terms of the other traceless components of $\pi^i{}_k$, and thus reduce Eq. (3.5) to a form without constraints. To do this we set

$$\pi^i{}_k=(2\pi)^{-1}p^i{}_k+\tfrac{1}{3}\delta^i{}_k\pi^l{}_l, \quad (3.8)$$

where $p^k{}_k=0$, and then write[32]

$$\begin{aligned}\beta_{ik}&=\mathrm{diag}(\beta_++\beta_-\sqrt{3},\,\beta_+-\beta_-\sqrt{3},\,-2\beta_+),\\ 6p^i{}_k&=\mathrm{diag}(p_++p_-\sqrt{3},\,p_+-p_-\sqrt{3},\,-2p_+),\end{aligned} \quad (3.9)$$

as well as

$$R\propto e^{-\Omega} \quad (3.10)$$

and

$$H=(2\pi)\pi^k{}_k. \quad (3.11)$$

The result is

$$I=\int p_+d\beta_++p_-d\beta_--Hd\Omega. \quad (3.12)$$

This variational principle is now in the canonical form of Eq. (2.7) and β_+, β_- are the two field amplitudes (generalized coordinates), p_+ and p_- are their conjugate momenta, H is the Hamiltonian, and Ω is the independent (coordinate time) variable. The solution of Eq. (3.6) with $^3R=0$ for $H=(2\pi)\pi^k{}_k$ in terms of $p_\pm$ and $\beta_\pm$ is

$$H=(p_+^2+p_-^2)^{1/2}. \quad (3.13)$$

Hamilton's equations, $dp_\pm/d\Omega=-\partial H/\partial\beta_\pm$ and $dH/d\Omega=\partial H/\partial\Omega$, show that p_+, p_-, and H are all constants of motion. The velocity equations, $d\beta_\pm/d\Omega=\partial H/\partial p_\pm$ give

$$\beta_\pm'\equiv(d\beta_\pm/d\Omega)=p_\pm/H, \quad (3.14)$$

which can be used in Eq. (3.13) to obtain the condition

$$|\beta'|\equiv(\beta_+'^2+\beta_-'^2)^{1/2}=1. \quad (3.15)$$

The general solution of the variational principle $\delta I=0$ from Eq. (3.12) is therefore a motion of the point $\beta\equiv(\beta_+,\beta_-)$ with unit Ω velocity along any line in the β plane. Since β can be set to zero at any one time by rescaling the coordinates $x^i\to(e^{-\beta_0})_{ij}x^j$ in Eq. (3.1), it is sufficient to consider only lines through the origin $\beta=0$. In one standard parameterization, then, the solutions are

$$\begin{aligned}\beta_+&=\Omega(u^2+u-\tfrac{1}{2})/(u^2+u+1),\\ \beta_-&=\Omega\sqrt{3}(u+\tfrac{1}{2})/(u^2+u+1),\end{aligned} \quad (3.16)$$

where u is a constant, $-\infty<u\leq+\infty$.

Although the above discussion gives the complete solution of the variation problem (3.3) subject to the constraints (3.6) and (3.7) in this case, it does not determine the metric (3.1) completely, as the original time coordinate t dropped out of the discussion when we wrote $(\partial g_{ij}/\partial t)dt=dg_{ij}$ in Eq. (3.3), and a different choice of Ω for the time coordinate was suggested by our determination to write Eq. (3.5) in the canonical form (3.12). There are two equivalent ways to proceed. We can try to find Ω as a function of t, or we can accept Ω as the time coordinate, write the metric in the form

$$ds^2=-N^2d\Omega^2+R_0^2e^{-2\Omega}(e^{2\beta})_{ij}dx^idx^j, \quad (3.17)$$

and try to find N as a function of Ω (R_0 is a constant, $R=R_0e^{-\Omega}$). We choose the latter course as described in connection with Eqs. (2.5). The metric (3.17) has been chosen so that $\sqrt{g}=R_0^3e^{-3\Omega}$ is a specified function of the time coordinate Ω. Thus we need from Eqs. (2.5) an equation for $\partial(\sqrt{g})/\partial\Omega$. For any time coordinate $\bar{t}$, one has

$$\frac{\partial\sqrt{g}}{\partial\bar{t}}=\tfrac{1}{2}(\sqrt{g})g^{ij}\frac{\partial g_{ij}}{\partial\bar{t}}=-\tfrac{1}{2}N\pi^k{}_k+(\sqrt{g})N^k{}_{;k}. \quad (3.18)$$

In our case with $\bar{t}=\Omega$, $\sqrt{g}=R_0^3e^{-3\Omega}$, and $N^i=0$, this gives

$$N=H^{-1}e^{-3\Omega}(12\pi R_0^3). \quad (3.19)$$

This equation, together with Eqs. (3.16), then specifies the metric (3.17) completely. To recover the usual forms, we solve $dt=-Nd\Omega$ to find

$$t=(4\pi R_0^3/H)e^{-3\Omega}. \quad (3.20)$$

Let us choose R_0 so that this reads $t=e^{-3\Omega}$; then the metric (3.17) using the solutions (3.16) and the definitions (3.9) becomes[14]

$$ds^2=-dt^2+R_0^2(t^{2p_1}dx^2+t^{2p_2}dy^2+t^{2p_3}dz^2), \quad (3.21)$$

with[33]

$$\begin{aligned}p_1&=-u/(u^2+u+1),\\ p_2&=(u+1)/(u^2+u+1),\\ p_3&=u(u+1)/(u^2+u+1).\end{aligned} \quad (3.22)$$

We now wish to repeat much of the preceding analysis for a metric of Bianchi type-IX corresponding to an

[32] Note that these choices differ by a factor of 2 from Refs. 5 and 28, but agree with Ref. 6.

[33] E. M. Lifshitz and I. M. Khalatnikov, Advan. Phys. **12**, 185 (1963).

anisotropic closed universe. The metric in this case is

$$ds^2 = -N^2 d\Omega^2 + \tfrac{1}{4}R^2(e^{2\beta})_{ij}\sigma_i\sigma_j\,, \tag{3.23}$$

where again $R \propto e^{-\Omega}$, and

$$\begin{aligned}\sigma_1 &= \sin\psi d\theta - \cos\psi \sin\theta d\phi\,,\\ \sigma_2 &= \cos\psi d\theta + \sin\psi \sin\theta d\phi\,,\\ \sigma_3 &= -(d\psi + \cos\theta d\phi)\,.\end{aligned} \tag{3.24}$$

The space dependence of the metric here is the only difference from the previous case, and reflects a more complicated homogeneity group, which is now left translations in the group of unit quaternions,[34] the covering group for the familiar rotation group $SO(3)$, whose structure constants appear in the relations (involving the exterior derivative d and the Grassman wedge or exterior product $\wedge$)

$$d\sigma_i = \tfrac{1}{2}\epsilon_{ijk}\sigma_j \wedge \sigma_k\,, \tag{3.25}$$

which the differential forms σ_i satisfy. The coordinates $\psi\theta\phi$ are Euler angle coordinates on $SO(3)$, taken over to the covering group by letting ψ have the range $0 \le \phi < 4\pi$, while $0 \le \theta \le \pi$ and $0 \le \phi < 2\pi$ as usual. The numeric $\frac{1}{4}$ in Eq. (3.23) is chosen so that when $\beta_{ij}=0$, the space part of the metric is just the standard metric for a three-sphere of radius R and circumference $2\pi R$. Thus for $\beta=0$ this metric is the Robertson-Walker positive curvature metric.

The action integral (3.2) in this case reads

$$I = (16\pi)^{-1}\int \pi^{ij} dg_{ij} \wedge \sigma_1 \wedge \sigma_2 \wedge \sigma_3\,,$$

with the metric components $g_{ij} = \frac{1}{4}R^2(e^{2\beta})_{ij}$ referred to the σ_i basis vectors. Since

$$\int \sigma_1 \wedge \sigma_2 \wedge \sigma_3 = \int \sin\theta d\phi d\theta d\psi = (4\pi)^2\,,$$

and since π^{ij} and g_{ij} in this basis are functions of t or Ω only, the space integration is carried out with Eq. (3.3) and subsequent steps leading to Eq. (3.12) proceeding as in the previous case. The constraint equations (3.6) and (3.7) must be restudied, however. The forms of Eqs. (3.6) and (3.7), which are the $G_0{}^\mu = 0$ Einstein equations, are known from previous studies of this metric,[35] or they can be computed directly. One finds that $\pi^{ij}{}_{|j}$ vanishes identically when both g_{ij} and π^{ij} are diagonal in the σ_i frame and independent of the space coordinates. Thus Eq. (3.7) is satisfied automatically as in type I, although this would no longer be true if we were to consider a nondiagonal β_{ij} matrix. In Eq. (3.6) we set $\sqrt{g} = (\frac{1}{8})R^3$ since $\det e^{2\beta} = 1$ as a result of $\beta_{kk} = 0$ as before, but we now have

$${}^3R = (6/R^2)(1-V)\,, \tag{3.26a}$$

where

$$\begin{aligned}V(\beta) &= \tfrac{1}{3}\,\mathrm{tr}(e^{4\beta} - 2e^{-2\beta} + 1)\\ &= \tfrac{2}{3}e^{4\beta_+}(\cosh 4\sqrt{3}\beta_- - 1) + 1\\ &\quad - \tfrac{4}{3}e^{-2\beta_+}\cosh 2\sqrt{3}\beta_- + \tfrac{1}{3}e^{-8\beta_+}.\end{aligned} \tag{3.26b}$$

Thus when we solve Eq. (3.6) for $H = (2\pi)\pi^k{}_k$, additional terms arising from ${}^3R$ arise, and we now find

$$H = [p_+{}^2 + p_-{}^2 + e^{-4\Omega}(V-1)]^{1/2}. \tag{3.27}$$

To avoid numerical factors in this Hamiltonian, we have set

$$R = (2/3\pi)^{1/2}e^{-\Omega} \equiv (2G\hbar/3\pi c^3)^{1/2}e^{-\Omega}. \tag{3.28}$$

The determination of N proceeds as before from Eq. (3.18), again with $N^k = 0$, and gives

$$N = H^{-1}e^{-3\Omega}(2/3\pi)^{1/2}. \tag{3.29}$$

The metric is, therefore,

$$ds^2 = -(2/3\pi)H^{-2}e^{-6\Omega}d\Omega^2 + (6\pi)^{-1}e^{-2\Omega}(e^{2\beta})_{ij}\sigma_i\sigma_j. \tag{3.30}$$

The equation $dt = -Nd\Omega$ cannot be integrated explicitly in this case to give the cosmic time t, since H is no longer a constant.

The classical solutions of the dynamical problem posed by the Hamiltonian of Eq. (3.27) are discussed by these Hamiltonian methods in Ref. 6 and by other methods in Ref. 5. We point out only a few of the most essential features here. The velocity equations $\beta_\pm' = \partial H/\partial p_\pm$ give as before

$$d\beta_\pm/d\Omega = p_\pm/H. \tag{3.31}$$

By using these equations to eliminate p_+ and p_- from Eq. (3.27), one finds the relationship

$$1 = \beta_+'^2 + \beta_-'^2 + H^{-2}e^{-4\Omega}(V-1) \tag{3.32}$$

in place of Eq. (3.15). Also, the equation $dH/d\Omega = \partial H/\partial\Omega$ now gives

$$d\ln H^2/d\Omega = -4(1-\beta'^2). \tag{3.33}$$

Near the singularity, i.e., for $\Omega \to \infty$, Eq. (3.32) reduces in first approximation to $\beta'^2 \simeq 1$, and so Eq. (3.33) gives $H = \mathrm{const}$ as for the type-I metric. A closer inspection will show that these approximations are valid only for finite Ω intervals, roughly while $|\beta| < \frac{1}{2}\Omega$, interrupted by epochs where the potential V does play a role. The anisotropy potential $V(\beta_+,\beta_-)$ is positive definite with $V \approx 8(\beta_+{}^2 + \beta_-{}^2)$ near $\beta=0$. The potential walls rise steeply away from $\beta=0$, with the equipotentials forming equilateral triangles in the $\beta_+\beta_-$ plane, as shown in Fig. 1. One of the three equivalent sides of the triangle is

[34] See, for example, C. W. Misner and A. H. Taub, Zh. Eksperim. i Teor. Fiz. **55**, 233 (1968) [English transl.: Soviet Phys.—JETP **28**, 122 (1969)].

[35] C. W. Misner, Gravity Award Essay, Gravity Research Foundation, New Boston, N. H., 1967 (unpublished); see Refs. 5 and 6; M. Ryan, J. Math. Phys. **10**, 1724 (1969); C. G. Behr, Astron. Abh. der Hamburger Sternwarte **7**, No. 5 (1965); Heckmann and Schucking, Ref. 30; A. Taub, Ann. Math. **53**, 472 (1951).

described by the asymptotic form

$$V\sim\tfrac{1}{3}e^{-8\beta_+},\quad \beta_+\to-\infty\,, \tag{3.34}$$

which is valid in the sector $|\beta_-|<-\sqrt{3}\beta_+$. As $\Omega\to\infty$, the space curvature terms $e^{-4\Omega}(V-1)$ in H can only play a role if $V\gg1$, so we will use the asymptotic form (3.34). The condition that V be important is easily seen from Eq. (3.32) to be $H^{-2}e^{-4\Omega}V\approx1$ or $e^{-4(\Omega+2\beta_+)}\approx3H^2$ or

$$\beta_+\approx\beta_{\text{wall}}=-\tfrac{1}{2}\Omega-\tfrac{1}{8}\ln(3H^2)\,. \tag{3.35}$$

Thus β_{wall} defines an equipotential in the β plane bounding the region in which the potential (space curvature) terms are significant. When β is well inside this equipotential, one has $|\beta'|=1$ and $H=\text{const}$, consequently, from Eq. (3.35), $|\beta_{\text{wall}}'|=\frac{1}{2}$. Thus the β point moves twice as fast as the receding potential wall, and at finite intervals as $\Omega\to\infty$, the $\beta(\Omega)$ trajectory will collide with the potential wall and be deflected from one straight-line (type I) motion to another.

We will need a few details of this "bounce" later, so let us derive them from the Hamiltonian using the asymptotic form (3.34). Then

$$H=[p_+^2+p_-^2+\tfrac{1}{3}\exp(-8\beta_+-4\Omega)]^{1/2} \tag{3.36}$$

shows that H is independent of β_- in this approximation (i.e., the equipotential is asymptotically a straight line) so p_- will be constant during the bounce. Another constant can be found by comparing the equations

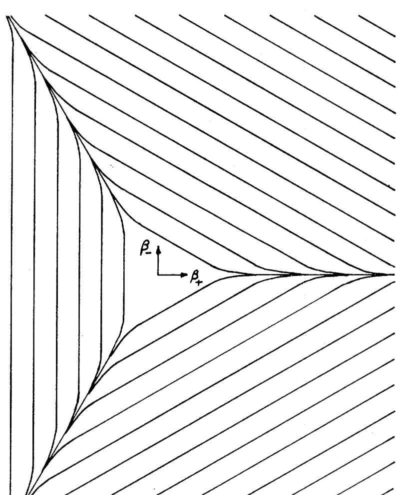

FIG. 1. Equipotentials of the function $V(\beta)$ are sketched here in the β plane from the asymptotic form of Eq. (3.34). (Equipotentials near the origin, not shown, are closed curves for $V<1$.) Between successive equipotentials on this diagram, which have separations $\Delta\beta=1$, V increases by a *factor* of $e^8\approx3\times10^3$. From Eq. (3.32), the system point $\beta(\Omega)$ moves with velocity $|d\beta/d\Omega|=1$ except when it approaches a limiting equipotential $V=H^2e^{4\Omega}$. This limiting equipotential moves outward with velocity $|d\beta_{\text{wall}}/d\Omega|=\frac{1}{2}$ except during the brief period when the system point β bounces against it. The velocity $d\beta/d\Omega$ changes its direction in an ergodic way as a result of these bounces.

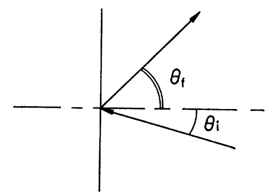

FIG. 2. Angles θ_i and θ_f are the angles of incidence and of reflection for a bounce from one of the three equivalent walls of the triangular potential $V(\beta)$. Because this wall moves to the left with speed $|\beta_{\text{wall}}'|=\frac{1}{2}$ while the system point $\beta(\Omega)$ moves with speed $|\beta'|=1$, one has a limit $|\theta_i|<60°$ in order for a bounce against this wall to occur. In other cases, the bounce will occur on a different wall (not shown). Note also that $\theta_f>90°$ is possible because of the motion of the wall.

$$p_+'=-\partial H/\partial\beta_+=+4(3H)^{-1}e^{-8\beta_+-4\Omega}$$

and

$$H'=\partial H/\partial\Omega=-2(3H)^{-1}e^{-8\beta_+-4\Omega}$$

with the result $K=\text{const}$, where

$$K=\tfrac{1}{2}p_++H=\tfrac{1}{2}p_++(p_+^2+p_-^2+\tfrac{1}{3}e^{-8\beta_+-4\Omega})^{1/2}. \tag{3.37}$$

These two constants of motion allow us to find β_+' and β_-' after the bounce in terms of their values before. Since $|\beta'|=1$ well before and well after the bounce, we can parametrize β' as follows (cf. Fig. 2): initially $(\beta_-')_i=\sin\theta_i$, $(\beta_+')_i=-\cos\theta_i$, and in the final state $(\beta_-')_f=\sin\theta_f$, $(\beta_+')_f=+\cos\theta_f$. Then the constancy of p_- and K gives, respectively, since $\beta_\pm'=p_\pm/H$,

$$H_i\sin\theta_i=H_f\sin\theta_f \tag{3.38}$$

and

$$H_i(-\tfrac{1}{2}\cos\theta_i+1)=H_f(\tfrac{1}{2}\cos\theta_f+1)\,. \tag{3.39}$$

These can be combined to give an equation for θ_f in terms of θ_i,

$$\sin\theta_f-\sin\theta_i=\tfrac{1}{2}\sin(\theta_i+\theta_f)\,, \tag{3.40}$$

which is sufficient for the application in this paper. [In terms of the Lifshitz-Khalatnikov parameter u of Eqs. (3.16) and (3.22), the relationship (3.40) is just[5,6] $u_f=u_i-1$ to within some permutations of the axes, and[6] $H_f/H_i=(u_f^2+u_f+1)/(u_i^2+u_i+1)$. The first computation of a formula for H_f/H_i was given by Ryan.[36]]

IV. QUANTIZATION

The canonical form (3.12) for the classical equations leads us to choose as basic commutation relations

$$[\beta_a,p_b]=i\delta_{ab}\,, \tag{4.1}$$

which can be satisfied by choosing

$$p_+=\frac{1}{i}\frac{\partial}{\partial\beta_+}\,,\quad p_-=\frac{1}{i}\frac{\partial}{\partial\beta_-}\,. \tag{4.2}$$

[36] M. Ryan (private communication).

The eigenfunctions ϕ_n of the Hamiltonian (3.27) are then just the eigenfunctions of

$$H^2 = -\frac{\partial^2}{\partial\beta_+^2} - \frac{\partial^2}{\partial\beta_-^2} + e^{-4\Omega}[V(\beta)-1], \qquad (4.3)$$

and the eigenvalues E_n of H are obtained from those of H^2,

$$H^2\phi_n = E_n^2\phi_n. \qquad (4.4)$$

Since H is explicitly a function of Ω, the eigenvalues will also depend on Ω, $E_n = E_n(\Omega)$. The wave function will be a linear combination of eigenfunctions

$$\psi(\beta,\Omega) = \sum_n a_n(\Omega)\phi_n(\beta,\Omega) \qquad (4.5)$$

with time-dependent amplitudes. We will not attempt here to obtain a wave function ψ with any precision. We are merely interested in a first glance approximation to indicate the conditions under which the classical limit (high quantum numbers $n \gg 1$) is inadequate.

For large Ω, near the singularity at $\Omega = \infty$, only the asymptotic form of the potential $V(\beta)$ is relevant, and we see from Fig. 1 and Eq. (3.34) that $V(\beta)$ is a potential well with a triangular base and very steep (exponential) walls. It will be a good approximation to treat the potential walls as infinitely steep, and then the eigenvalue problem (4.4) is just that which the Schrödinger equation gives for a nonrelativistic single particle confined in a triangular box in two dimensions. For a square box the eigenvalues and eigenfunctions are elementary, and one has $H^2\phi = (\pi^2/L^2)(m^2+n^2)\phi$, where L^2 is the area of the square, and the quantum numbers m and n are positive integers. We estimate that the eigenvalues for a triangle will be essentially the same as those for a square, and thus we take $E_n = |n|/L$, where $|n|$ corresponds to $(m^2+n^2)^{1/2}$. The area of the triangle we compute from Eq. (3.35) in the simplified form $\beta_{\text{wall}} \approx -\frac{1}{2}\Omega$ which holds for large Ω since $\ln H$ is nearly constant. The area of the triangle is therefore $L^2 = 3\sqrt{3}\beta_{\text{wall}}^2 = \frac{3}{4}\sqrt{3}\Omega^2$, and our estimate of the eigenvalues becomes

$$E_n \sim (\tfrac{2}{3}\pi)^{3/4}|n|\Omega^{-1}. \qquad (4.6)$$

We will use this formula to deduce how the quantum number n changes as a consequence of the changes in H or E_n, which can be computed from the classical theory when n is large.

Let us return, therefore, to the classical solutions for the Hamiltonian H of Eq. (3.27). We have seen that as $\Omega \to \infty$, $\beta(\Omega)$ changes in a series of constant-velocity "runs" from the potential wall on one side of the triangle to that on another, and that these runs are each terminated by a bounce against the potential wall. The potential wall is constantly moving outward as described by Eq. (3.35). During the runs H is constant, $|\beta'|=1$, and the direction of the velocity β' can be specified by a parameter u as in Eqs. (3.16). At each bounce, H decreases as described by Eq. (3.37) and the direction of β' changes as described by Eq. (3.40). Our problem is to estimate the long-term mean rate of decrease of H, averaged over a large number of runs and bounces as $\Omega \to \infty$. For almost all initial conditions, the direction of β' changes ergodically as a consequence of this bouncing around. Thus the long-term average rate of decrease of H will be the same for all these ergodic motions. We make the assumption that the long-term average rate of decrease of H is in fact the same for *all* initial conditions [excepting only $(\beta_-)_0 = 0 = (\beta_-')_0$ and two permutation equivalent sets of initial conditions for which β is always confined to a single axis of the triangle giving a one-dimensional motion]. By this assumption we are able to compute the average behavior of H by following it for any single *conveniently* chosen set of initial conditions.

Consider therefore an orbit $\beta(\Omega)$ for which, at the first bounce, $\theta_i + \theta_f = 60°$. Then, by the geometry shown in Fig. 3, θ_i for the second bounce will be exactly the same as it was for the first bounce, and similarly every subsequent bounce will have, conveniently, the same θ_i. From Eq. (3.40) we find that $\theta_i+\theta_f = 60°$ implies $\theta_i = 15.5°$. (The corresponding u parameter is[5,6] $u = \frac{1}{2}+\frac{1}{2}\sqrt{5}$.) A sketch of this quasiperiodic orbit is given in Fig. 4. As a further simplification we let the bouncee take place at the midpoints of the midpoints of ths sides of the triangular equipotentials. At each bounce H decreases according to Eq. (3.38) by a factor $(H_f/H_i) = (\sin\theta_i/\sin\theta_f) = 0.382$. For an accurate way to see the average behavior of H on this orbit after many bounces, I am indebted to a suggestion from Dr. Jacobs[37] who pointed out on the basis of Fig. 5 that the times Ω_i and Ω_f which measure the duration of the "runs" before and after a bounce are in the same ratio $\Omega_i/\Omega_f = \sin\theta_i/\sin\theta_f$ as H_f/H_i. Thus

$$H_f\Omega_f = H_i\Omega_i. \qquad (4.7)$$

But Fig. 5 also shows that $\Omega_i = \Omega\sqrt{3}/4\,\sin\theta_f = 0.62\,\Omega$, where Ω gives the time at which the bounce occurs.

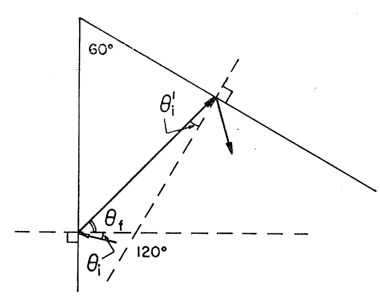

FIG. 3. The normals to two sides of an equilateral triangle meet at 120°, so the angle of reflection θ_f after one bounce, and the angle of incidence θ'_i for the following bounce are two angles in a triangle whose third angle is 120°. Consequently, $\theta_f+\theta_i' = 60°$. The condition for the simplest quasiperiodic orbit, namely, $\theta_i' = \theta_i$ where all bounces have the same incident angle, thus requires $\theta_f+\theta_i = 60°$, and leads from Eq. (3.39) to $\theta_i = 15.5°$.

[37] K. Jacobs (private communication).

Suppose this bounce was the nth in the sequence, with $\Omega=\Omega_n$ the time at which the bounce occurred, and $H_n=H_i$ the value of H just before this nth bounce. Then we have $\Omega_i=0.62\Omega_n$. But Ω_f is the duration of the run preceding the $(n+1)$th bounce, so $\Omega_f=0.62\,\Omega_{n+1}$; and $H_f=H_{n+1}$ is the value of H just before the $(n+1)$th

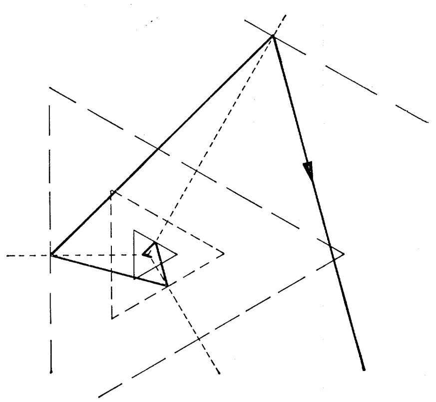

FIG. 4. The simplest quasiperiodic orbit is sketched in the β plane. The system point $\beta(\Omega)$ approaches the midpoint of the potential wall with an incident angle of $\theta_i=15.5°$, and is reflected at $\theta_f=44.5°$. Then it proceeds toward the next side of the triangular potential. Because the strength of the potential is decreasing (equivalently, the potential walls are moving outward), the next bounce, although the angles are as before, occurs at an increased β distance from the origin.

bounce. In these terms, Eq. (4.7) reads

$$H_{n+1}\Omega_{n+1}=H_n\Omega_n\,, \qquad (4.8)$$

and shows that $H\Omega$ returns to a fixed, constant value just before each bounce. Thus, although $H\Omega$ is not a constant, it is an adiabatic invariant whose value does not drift in a secular way as $\Omega\to\infty$. By taking an average over many runs and bounces, we may conclude that

$$\langle H\Omega\rangle=\text{const.} \qquad (4.9)$$

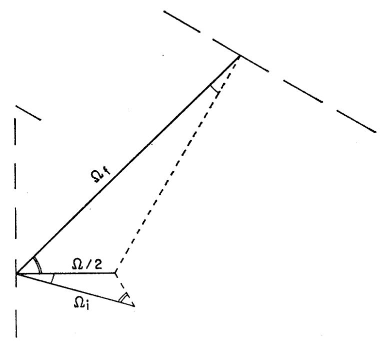

FIG. 5. Geometrical relations between two successive steps in the quasiperiodic orbit of Fig. 4. Because the potential walls move out with velocity $|d\beta_{\text{wall}}/d\Omega|=\frac{1}{2}$, we can take the position of the left hand wall to be $|\beta_{\text{wall}}|=\frac{1}{2}\Omega$ at the time Ω at which the bounce occurs. This distance $\frac{1}{2}\Omega$ in the β plane here is seen to be a common side for the two similar triangles shown here, whose large angles are 120°, and whose other angles are $\theta_i=15.5°$ and $\theta_f=44.5°$. The lengths of the sides opposite the 120° angles are computed from the velocity $|d\beta/d\Omega|\equiv(\beta_+'^2+\beta_-'^2)^{1/2}=1$ to be just the elapsed Ω time between successive bounces, i.e., Ω_i prior to the bounce shown (at time Ω) and Ω_f after. The law of sines then gives $(\sin 120°)/\Omega_i=(\sin\theta_f)/(\frac{1}{2}\Omega)$ for the smaller triangle and $(\sin 120°)/\Omega_f=(\sin\theta_i)/(\frac{1}{2}\Omega)$ for the larger triangle so that $\Omega_i\sin\theta_f=\Omega_f\sin\theta_i$ holds, leading to the adiabatic condition of Eq. (4.7).

We will now use the above results to estimate how quantum numbers change as $\Omega\to\infty$ at the singularity. From Eq. (4.6) we see that $E_n\Omega=(2\pi/3^{3/4})|n|$, so that when n is large, the classical calculations with $H\simeq E_n$ apply. These calculations tell us that n is an adiabatic invariant. Thus the time dependence of the Hamiltonian will cause transitions from one (instantaneous) eigenstate of H to others, and n will vary with time, but not in such a way as to give long term secular changes of n. On the average, n remains constant as $\Omega\to\infty$:

$$\langle n\rangle=\text{const.} \qquad (4.10)$$

In particular, if it is assumed that the present state of anisotropy in the expansion of the Universe is classical, so $|n|\gg 1$ now, then it follows that as we extrapolate back toward the initial singularity $\Omega\to\infty$, $|n|$ remains roughly constant and the quantum state of the Universe remains classical ($|n|\gg 1$) all the way back to the earliest times when the radius of the Universe was very much smaller than 10^{-33} cm.