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Classical Statistical Thermodynamics and Electromagnetic Zero-Point Radiation

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Classical statistical thermodynamics in the presence of electromagnetic radiation is reanalyzed, and is reformulated to give a natural classical description of the phenomena which originally led to the introduction of the idea of quanta. The traditional classical ideal gas fails to exist *in principle* for particles of finite mass which have electromagnetic interactions, and hence the classical proofs of energy equipartition are all erroneous. A consistently classical treatment of thermal radiation leads to the natural introduction of temperature-independent fluctuating radiation in the universe. The spectrum of this electromagnetic zero-point radiation may be obtained from the arguments for Wien's displacement law or from the requirement of Lorentz invariance of the radiation spectrum; this zero-point spectrum agrees with the $\frac{1}{2}\hbar\omega$ per normal mode familiar in quantum theory. The presence of temperature-independent disordered energy from zero-point radiation leads to a contribution to the entropy connected with thermodynamic probability distinct from the contribution of caloric entropy. The use of quanta in calculations of the thermodynamic probability is seen as a subterfuge to account for this mismatch between caloric entropy and probability. Several examples of statistical thermodynamics, which are generally regarded as having their explanation in terms of quanta, allow natural explanations within the context of classical theory with classical electromagnetic zero-point radiation.

I. INTRODUCTION: REANALYSIS OF CLASSICAL STATISTICAL THERMODYNAMICS

THE inability of traditional classical physics to account for the spectrum of blackbody radiation has long been regarded as a signal of the fundamental failure of classical theory. The conviction of this failure led directly to the development of quantum theory. In this paper, we wish to suggest that the failure of classical theory should properly be traced to a failure of an approximation of classical theory—to the assumption that accelerated particles of an ideal gas do not radiate. Correction of this erroneous approximation allows a reformulation of classical statistical thermodynamics which correctly predicts the blackbody radiation spectrum and gives a classical interpretation of some of the ideas of quantum statistical mechanics.

In Sec. II of this paper, we point out the specific aspects of the traditional proofs for energy equipartition which must fail if the radiation emitted by particles is included in a classical theory. Despite difficulties encountered with ideal gases for finite-mass particles, the usual ideas of an ideal gas become true in the limit of massive particles. Thus it is possible to define temperature in terms of the kinetic energies of ideal-gas particles in the limit of infinite mass, and the basic ideas of thermodynamics are not altered by our investigation. However, the limitations of the traditional analysis of classical statistical mechanics require an effort to reformulate the theory. This is begun in Sec. III. Noting that the definition of heat in terms of the first law of thermodynamics requires that the thermal energy of radiation at a temperature T should be finite, we see that the radiation from particles suggests that there must be temperature-independent classical radiation in the universe. The energy spectrum of this zero-point radiation may be deduced from Lorentz invariance or from the more traditional arguments of Wien associated with the reflection of radiation from a moving mirror. The zero-point spectrum is then used to deduce the Planck radiation spectrum for electromagnetic radiation at finite temperature. In Sec. IV of this paper, we explore the connections between entropy and thermodynamic probability. We notice that the presence of zero-point radiation leads to fluctuations and hence probability notions even at the absolute zero of temperature where changes of caloric entropy are undefined. The connections between caloric entropy and probability are separated out in the last part. The idea of quanta is found to be a trick for extracting that part of the thermodynamic probability which is indeed connected with heat energy. Some examples introduced by Einstein as proofs of the need for quanta are reinterpreted as continuous processes occurring naturally in the presence of electromagnetic zero-point radiation.

Throughout the paper, comparisons are drawn with the results of both traditional classical theory and quantum theory. Specifically, the term "classical" is intended to refer to continuous processes, and is to be contrasted with quantum behavior referring to discrete processes. By traditional classical theory, we mean essentially the ideas of physics before Planck's revolutionary work on quanta in 1900. The reformulation given here makes no use whatsoever of quantum ideas, no use of any notions of discrete or discontinuous processes. Classical theory is found to give an adequate formulation of statistical thermodynamics when classical radiation is included consistently.

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II. FAILURE OF TRADITIONAL CLASSICAL ARGUMENTS FOR ENERGY EQUIPARTITION

A. Failure in Principle of the Existence of an Ideal Gas for Particles of Finite Mass

The existence in principle of an ideal gas plays a crucial rule in the traditional theory of classical statistical mechanics. The behavior of an ideal gas forms the basis for a definition of temperature in thermodynamics, and the basis for the equipartition theorem in statistical mechanics. An ideal gas is imagined to consist of hard spheres of small volume which are contained in thermodynamic equilibrium inside a box of large volume. The walls of the box are viewed as perfectly elastic reflectors of the gas molecules, and serve no function except to confine the particles to a finite volume and hence allow a discussion of equilibrium conditions. The particles are assumed to be spread randomly in space throughout the box, and to have random directions for their velocities. The analysis[1] proceeds to consider the equilibrium between particles of different mass, and later particles with internal degrees of freedom. Based on very general assumptions, it is found that in equilibrium all particles irrespective of mass have the same average kinetic energy. This allows a definition of temperature T as

$$\langle \tfrac{1}{2}mv^2\rangle = \tfrac{3}{2}kT. \tag{1}$$

The analysis also indicates that each quadratic degree of freedom has an average energy $\frac{1}{2}kT$.

The elementary account above for an ideal gas already contains the approximation which leads to the failure of traditional classical statistical thermodynamics. If we consider the fact that an accelerated particle radiates, then an ideal gas exists in principle only for infinitely heavy particles. It is not true that in equilibrium, particles of different mass have the same kinetic energy.

Let us consider a classical particle in a box. If the particle collides with a perfectly elastic wall, then on being accelerated the particle will radiate and will lose kinetic energy. Thus on leaving the perfectly elastic wall, the particle will have a lower velocity then when it approached the wall. Thus, not all directions of velocity are equally probable, near such a wall; rather the approaching velocity is likely to be higher than the receding velocity. One may enquire about the behavior of a particle which has no interaction with the electromagnetic field and hence will not introduce problems involving radiation. In this case it must be noted that the particle passes through any wall, and hence cannot be confined to a box as a particle must be before equilibrium conditions occur. Any particle suitable for discussion in statistical mechanics must have electromagnetic interactions, and hence is subject to the radiation problem noted above.

Turning very specifically to Maxwell's argument[2] for equipartition in an ideal gas, we consider the collision of two particles, of mass and velocity m_1, $\mathbf{v}_1$, and m_2, $\mathbf{v}_2$, respectively, viewing the collision in the center of mass frame. Then it seems reasonable that all directions for the relative final velocity $\mathbf{w}$ have equal probability relative to the direction of motion of the center of mass,

$$\langle \mathbf{w}\cdot\mathbf{v}_{\text{c.m.}}\rangle = 0. \tag{2}$$

Now $\mathbf{w}\cdot\mathbf{v}_{\text{c.m.}}$ can be rewritten as

$$\mathbf{w}\cdot\mathbf{v}_{\text{c.m.}} = [(m_1\mathbf{v}_1^2 - m_2\mathbf{v}_2^2) + (m_2 - m_1)(\mathbf{v}_1\cdot\mathbf{v}_2)]/(m_1+m_2). \tag{3}$$

We see that the equipartition of energy

$$\langle \tfrac{1}{2}m_1\mathbf{v}_1^2\rangle = \langle \tfrac{1}{2}m_2\mathbf{v}_2^2\rangle \tag{4}$$

holds provided $\langle \mathbf{v}_1\cdot\mathbf{v}_2\rangle = 0$, as is immediately assumed in all elementary analyses. If, however, the particles radiate, then $\langle \mathbf{v}_1\cdot\mathbf{v}_2\rangle$ is nonvanishing just outside a perfectly elastic wall, precisely because velocities leaving the wall are lower than velocities approaching the wall, and particles of different mass radiate away different fractions of their energy. In other words, the naive argument for energy equipartition collapses.

Note added in proof. This explanation apparently is not sufficiently clear to readers who do not refer to Ref. 1. What the traditional proof seeks to prove is that at any given point in the box, all particles, independent of mass, have the same kinetic energy. The average in Eq. (3) is over all possible collision configurations at the given space point. The basic concept can be understood by considering particles moving in only one dimension and assuming that each of the particles has a unique speed, $|v_1|$ for particle 1, and $|v_2|$ for particle 2. Then there are four possible configurations for collision: (i) $v_1 = |v_1|$, $v_2 = |v_2|$; (ii) $v_1 = |v_1|$, $v_2 = -|v_2|$; (iii) $v_1 = -|v_1|$, $v_2 = |v_2|$; (iv) $v_1 = -|v_1|$, $v_2 = -|v_2|$. The relative positions of the particles is always such that a collision actually occurs. The average $\langle \mathbf{v}_1\cdot\mathbf{v}_2\rangle$ taken over all collisions is

$$\langle \mathbf{v}_1\cdot\mathbf{v}_2\rangle = |v_1||v_2| + |v_1|(-|v_2|) + (-|v_1|)|v_2| + (-|v_1|)(-|v_2|) = 0.$$

This is the result required for equipartition. However, in our argument involving radiation we wish to take a point right next to a wall, so that when the particle is moving towards the wall, its average speed is larger than when it is moving away from the wall, corresponding to the energy loss due to radiation. In this case, the four possible collision configurations become (i) $v_1 = |v_1|$, $v_2 = |v_2|$; (ii) $v_1 = |v_1|$, $v_2 = -|v_2| + \epsilon_2$; (iii) $v_1 = -|v_1| + \epsilon_1$, $v_2 = |v_2|$; (iv) $v_1 = -|v_1| + \epsilon_1$, $v_2 = -|v_2| + \epsilon_2$. Now averaging over all collisions, we have

[1] See, for example, the account by R. P. Feynman, R. B. Leighton, and M. Sands, in *Feynman Lectures on Physics* (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1963), Vol. I, Chap. 39.

[2] We are following the particularly lucid description of Ref. 1.

$\langle \mathbf{v}_1 \cdot \mathbf{v}_2 \rangle = \epsilon_1 \epsilon_2 \neq 0$. If there is radiation at the walls, then energy equipartition fails.

Perhaps the reader should be reminded that an analogous situation is imagined in familiar quantum theory. A particle in a box at temperature T in the quantum analysis has a Maxwell-Boltzmann energy distribution except for quantum correction terms involving inverse powers in the mass. It is precisely such correction terms which will appear due to radiation in classical theory.

B. Ideal-Gas Behavior in the Limit of Massive Particles: Definition of Temperature

Despite the difficulties mentioned in the previous section, all is not lost regarding the idea of an ideal gas. As the particles become more massive, the particle velocities decrease and the accelerations, due, for example, to harmonic-oscillator potential walls, will decrease, so that the effects of radiation become negligible. For this situation, we indeed expect the traditional arguments of classical statistical mechanics to be valid, and energy equipartition should indeed hold. Thus, for gas molecules at reasonable temperatures and low densities, the radiation effects become tiny, and, as experimental evidence has long indicated, many gases indeed behave approximately as ideal gases. It is amusing to keep in mind the quantum analysis for this situation. Just as we have argued that radiation effects become small for massive particles, so the quantum analysis argues that quantum effects become small for such particles, although, of course, such effects are still important in principle.

In the light of this ideal-gas behavior for massive particles, we will define the temperature of a system in thermal equilibrium as the average kinetic energy of enclosed particles with weak mutual interactions in the limit as the masses of the particles go to infinity,

$$\lim_{m\to\infty} \langle \tfrac{1}{2} m \mathbf{v}^2 \rangle \equiv \tfrac{3}{2} kT . \tag{5}$$

As far as thermodynamics is concerned, the existence of ideal-gas behavior for massive particles is all that is required for the usual development of the theory. As in the traditional theory, the first law of thermodynamics

$$dU = -dW + dQ \tag{6}$$

is used to define heat dQ, and the second law relates the efficiencies of reversible engines to temperature differences. The foundations of thermodynamics are unaltered. Rather it is statistical mechanics, and in particular the ideas of energy equipartition which have been formulated incorrectly due to neglect of electromagnetic radiation from particles.

C. Failure of Classical Analyses for Thermal Radiation

The traditional analyses for thermal radiation try to include electromagnetic interactions as an appendix of the traditional statistical mechanics which is developed under the explicit assumption that such interactions vanish. Thus it can hardly be regarded as surprising that the traditional classical analyses lead to the inadequate Rayleigh-Jeans law.

For example, many classical treatments consider a charged oscillator which interacts with both radiation and, by collision, with particles of an ideal gas. Planck[3] calculated that the average energy of an oscillator of frequency ω in interaction with radiation would have an average energy

$$E = (\pi^2 c^3/\omega^2)\rho(\omega,T) , \tag{7}$$

where $\rho(\omega,T)$ is the distribution of electromagnetic energy at frequency ω and temperature T. The collisions with gas molecules demanded in traditional theory an average energy of kT, and hence imply

$$\rho(\omega,T) = (\omega^2/\pi^2 c^3) kT , \tag{8}$$

the Rayleigh-Jeans law. The erroneous argument lies in the assignment of an average energy $\frac{1}{2}kT$ per normal mode, which we have just seen, actually holds only for very massive particles where radiation by the oscillator is small. But notice that a large particle mass m leads to a low frequency of the oscillator since $\omega = (K/m)^{1/2}$ (where K is the spring constant), and in this limit of low frequency the Rayleigh-Jeans law is indeed correct. All the analyses of thermal radiation in traditional classical theory fail for high frequencies where particle radiation is high and the usual ideal-gas analysis breaks down.

III. CLASSICAL THEORY SUGGESTS PRESENCE OF ELECTROMAGNETIC ZERO-POINT RADIATION

A. Incoming Radiation in Classical Statistical Mechanics

All formulations of classical electromagnetism allow unspecified incoming radiation in the far past. In Maxwell electrodynamics, this radiation is arbitrarily set equal to zero. Similarly, traditional classical statistical mechanics assumes that this radiation vanishes and that all radiation of interest in statistical physics is thermal in origin. We will argue that contrary to these traditional choices, classical theory requires the presence of electromagnetic radiation at the absolute zero of temperature.

B. Aside: Dynamic Equilibrium for Radiation

We will first present some heuristic ideas on this problem of incoming radiation which are unconnected

[3] M. Planck, *The Theory of Heat Radiation* (Dover Publications, Inc., New York, 1959), Part IV. A concise treatment appears in M. Abraham and R. Becker, *Theorie der Elekrizität* (B. G. Teubner Verlag, Leipzig, 1933), Vol. II, 6th ed., pp. 373–5. A treatment without explicit use of radiation damping appears in M. Born, *Atomic Physics* (Hofner Publishing Co., New York, 1966), pp. 254, 424.

with our principal line of argument. In any thermodynamic system, particle collisions will lead to sudden accelerations of particles, producing radiation into arbitrarily high ranges of frequency. The system will gradually lose energy to these high frequencies unless there is already radiation present which is reexchanged with the lower frequencies, leading to equilibrium. But the energy at these arbitrarily high frequencies will constitute an infinite amount of energy which accordingly cannot be associated with thermal energy. Thus, there must be temperature-independent radiation in the universe.

The heuristic idea of radiation equilibrium being proposed here is that of a dynamical equilibrium where the exchange of energy between frequencies is balanced due to fundamental electromagnetic processes. It seems particularly satisfying to note that exactly such a dynamical equilibrium can be found in the case of zero-point radiation. Thus on reflection from a moving mirror, as given in the arguments for Wein's displacement law, the effect of Doppler shift is such as to redistribute the energy of radiation of the zero-point spectrum in precisely such a way as to preserve the spectrum.[4] It is tempting to speculate that a purely electromagnetic analysis leading to the blackbody spectrum should be possible for thermal radiation.

C. Thermodynamics and Thermal Radiation

We have already remarked that the difficulties in traditional theory regarding radiation by particles do not influence the results of thermodynamics. Thus, appropriately enough, when we turn to the thermodynamic arguments involving thermal radiation, we find that they are indeed in accord with experiment. We will review these results here because we will find their information useful on several occassions in what follows.

In all cases, the arguments combine the specific properties of electromagnetic radiation with the ideas of thermodynamics to obtain results appropriate for electromagnetic heat radiation. A purely electromagnetic calculation gives Maxwell's radiation pressure as $\frac{1}{3}$ of the energy density of thermal radiation

$$p=\tfrac{1}{3}u. \tag{9}$$

Combining this result with thermodynamic arguments, including use of a Carnot cycle, leads to the Stefan-Boltzmann law for the energy density u of thermal radiation

$$u=\sigma T^4, \tag{10}$$

where σ is a constant and T is the absolute temperature.

An ingenious argument involving thermodynamics and reflections of electromagnetic waves from a moving mirror can be used to obtain Wein's law concerning the spectral distribution of energy. Thus on an adiabatic expansion of a cavity, the thermal energy is redistributed among the frequencies in such a way that the change in the spectral energy density $\rho(\omega,T)$ is given by

$$\delta\rho(\omega,T)=\left[\frac{\omega}{3}\frac{\partial\rho(\omega,T)}{\partial\omega}-\rho(\omega,T)\right]\frac{\delta V}{V}, \tag{11}$$

where

$$u=\int_{\omega=0}^{\infty} d\omega\, \rho(\omega,T), \tag{12}$$

and V is the volume of the cavity. This result can be combined with the assertions that for blackbody radiation u is a function of T alone, and that under an adiabatic expansion the total thermal energy in the cavity is changed by the negative of the work done, to give

$$\rho(\omega,T)=\omega^3F(\omega/T), \tag{13}$$

which is the familiar general form of Wein's displacement law.

D. Container Walls and Thermodynamic Equilibrium

The great power of statistical thermodynamics lies in the simplicity of its assumptions and the resulting generality of its conclusions. At the outset, we called into question some of the assumptions of the statistical model, in particular the description of the interaction of a particle and the enclosing walls. We will now try to reinterpret this interaction, still speaking in the most general terms possible.

In our discussion of the traditional ideal gas, we spoke of the walls in the traditional context of perfectly elastic reflecting potentials. We should note immediately that such walls are not relevant for statistical mechanics since they cannot lead to thermal equilibrium. Whereas equilibrium will require a balance of both particle energy and radiation, there is no wall which is at once a perfectly elastic potential for both particles and radiation. Thus, equilibrium requires that the walls as well as the gas particles confined consist of physical particles which can exchange both mechanical energy and electromagnetic radiation. This mechanical behavior of the wall will be seen later to be of importance in simplifying the considerations regarding thermal radiation.

E. Particle with Oscillator in a Box

The failure of the equipartition theorem for particles means that we do not at present know the average kinetic energy of a gas molecule, or of a particle on a spring, or of any of the traditional mechanical systems used in statistical mechanics. What we do know is the mean kinetic energy of a point particle in the limit of infinitely heavy mass since this is the limit in which radiation effects become negligible and, hence, traditional statistical mechanics holds; this was the limit in which temperature was defined. Therefore, we take advantage of this limiting situation.

[4] See Eq. (11) and Sec. III H.

Following a suggestion of Einstein and Hopf,[5] we insert a classical dipole oscillator inside a very massive particle. The massive nature of the particle connects its kinetic energy to the temperature, while the oscillator gives an explicit connection to the unknown radiation field. For simplicity of analysis, the oscillator is assumed to move only in the x direction while the oscillator points along the z direction. The crucial idea is that at equilibrium, the average kinetic energy due to interaction with the radiation field must be the same as $\frac{1}{2}kT$ which is known from the definition of temperature.

While the particle is in the interior of the box, the interaction of the oscillator and the thermal radiation present in the box can be separated roughly into two aspects: (i) There is a rapid fluctuating force due to the interaction of the oscillating dipole with fluctuations in radiation. This leads to a fluctuating impulse Δ during a short time interval τ. (ii) There is a general "frictional" force on the oscillator as it moves through the radiation. This force tends to slow the particle down and, in effect, is a radiation reaction force associated with the work done by thermal radiation. At low velocities appropriate for massive particles, this frictional force may be taken as linear in the velocities $F=-Pv$.

There is a further force on the particle corresponding to interactions with the walls. This must include a contribution from the radiation reaction force since the particle radiates due to the acceleration at the walls. During a short time interval τ, the walls give the particle an impulse J.

In equilibrium, the mean-square momentum of the particle is a constant. Thus,

$$\langle(mv_t)^2\rangle=\langle(mv_{t+\tau})^2\rangle=\langle(mv_t+\Delta-Pv_t\tau+J)^2\rangle. \quad (14)$$

We next expand the right-hand side. Remembering that Δ is as likely to be positive as negative during the time interval τ, we see that

$$\langle v_t\Delta\rangle=0 \quad \text{and} \quad \langle\Delta J\rangle=0. \quad (15)$$

Also recalling that the mass of the particle must be chosen large while the time interval τ will be small, we may neglect $P^2\tau^2\langle v^2\rangle$ compared to $-2mP\tau\langle v^2\rangle$, and $\langle J^2\rangle$ and $-2P\tau\langle vJ\rangle$ compared to $2m\langle vJ\rangle$. The resulting expression is thus

$$-2m\langle vJ\rangle=\langle\Delta^2\rangle=-2mP\tau\langle v^2\rangle. \quad (16)$$

Using classical electromagnetic theory, Einstein and Hopf[5] evaluated[6] $\langle\Delta^2\rangle$ and P as

$$\langle\Delta^2\rangle=(4\Gamma\pi^4c^4\tau/5\omega^2)\rho^2(\omega,T), \quad (17)$$

$$P=c\pi^2\frac{6}{5}\Gamma\left[\rho(\omega,T)-\frac{\omega}{3}\frac{\partial}{\partial\omega}\rho(\omega,T)\right], \quad (18)$$

where Γ is the damping parameter for the oscillator, corresponding in the case of a physical oscillator to

$$\Gamma=\tfrac{2}{3}(e^2/mc^3). \quad (19)$$

Substituting the results of Eqs. (17) and (18) into (16), and setting $\frac{1}{2}m\langle v^2\rangle=\frac{1}{2}kT$ for the one degree of freedom, we obtain

$$-2m\langle vJ\rangle=\frac{4\Gamma\pi^4c^4\tau}{5\omega^2}\rho^2-2kT\tau c\pi^2\frac{6}{5}\Gamma\left(\rho-\frac{\omega}{3}\frac{\partial\rho}{\partial\omega}\right). \quad (20)$$

F. Problem of Particle Interactions with Walls

In the analysis up to this point, we have left unspecified the form of the interaction between a particle and a wall appearing through the impulse J. In traditional classical theory, the interaction with the walls may be regarded as a completely elastic rebound of a particle, and hence $\langle vJ\rangle$ vanishes, corresponding to the absence of net work performed by the walls on the gas particles. Thus Eq. (20) describes an energy balance for the particle which during a time interval τ balances the energy $\langle\Delta^2\rangle/2m$, gained through random impulses, against the energy $P\tau\langle v^2\rangle$ lost to work done by the velocity-dependent radiation forces. If we make this assumption that $\langle vJ\rangle$ vanishes, then the unique solution of the differential equation for $\rho(\omega,T)$ is the Rayleigh-Jeans distribution

$$\rho(\omega,T)=(\omega^2/\pi^2c^3)kT. \quad (21)$$

This result fails to agree with experiment. It also leads to an infinite thermal-energy change connected with a finite temperature change—a result which is in disagreement with the ideas of energy conservation expressed in the first law of thermodynamics.

However, we note that classical statistical theory does not force us into this result; the term $\langle vJ\rangle$ in Eq. (20) is actually not convincingly determined, corresponding to our lack of knowledge of interactions at the walls. In particular, we note that the contribution from a perfectly elastic potential wall is negative definite, since the velocity of a particle which radiates is smaller on leaving the wall than on approaching, while the impulse J is always directed toward the interior of the box. Thus in our view, the real problem of classical theory is the determination of the unknown term $\langle vJ\rangle$.

G. Need for Temperature-Independent Radiation

Let us go back to the earlier classical results in the Stefan-Boltzmann law and in the Wein displacement law, which involve only radiation, in order to see what information can be obtained regarding the interaction with the walls. We first note that the Stefan-Boltzmann law implies that

$$\lim_{\omega\to\infty}\omega\rho(\omega,T)=0, \quad (22)$$

since otherwise the energy density of thermal radiation would be infinite. Looking at the right-hand side of

[5] A. Einstein and L. Hopf, Ann. Physik **33**, 1105 (1910).

[6] T. H. Boyer, Phys. Rev. **182**, 1374 (1969). Evaluations of $\langle\Delta^2\rangle$ and P in more recent notation appear in the appendix of this paper.

Eq. (20), we find that for large ω, $\rho^2(\omega,T)$ is negligible compared to $\rho(\omega,T)$, and $\rho(\omega,T)$ must be decreasing so that $\partial\rho(\omega,T)/\partial\omega$ is negative. But then the right-hand side is negative, corresponding to the particle *absorbing* energy at the wall. This is the antithesis of our notion of a loss of kinetic energy of the particle due to radiation.

This apparent contradiction requires that we step back and look again at our assumptions. The derivation of Eq. (20) made use of classical electrodynamics and the definition of temperature. The Stefan-Boltzmann law follows from classical thermodynamics, Maxwell's pressure of electromagnetic radiation, and the assumption that all radiation in a blackbody cavity is thermal in origin.

The one hypothesis which is not of a fundamental nature is the last-mentioned assumption that all radiation of interest is thermal in origin. Indeed we pointed out earlier that the incoming radiation in the universe in classical theory has never been fixed from any experimental data. Thus the possibility which we are compelled to consider is that there is temperature-independent radiation; i.e., in the terminology of electromagnetism, incoming radiation in the universe in the far past. (The term "incoming" apparently confused readers of the prepublication report of this paper. What is meant is that the zero-point radiation is a solution of the homogeneous Maxwell equations. The universe is regarded as initially containing both matter and radiation.)

At this point, we must be very careful with our notation. As in all our previous calculations, we will denote by $\rho(\omega,T)$ the *full* spectral density of fluctuating electromagnetic radiation. The purely electromagnetic calculations such as those for $\langle\Delta^2\rangle$ and P will be unaltered in their dependence on $\rho(\omega,T)$. At temperature $T=0$, the full electromagnetic radiation becomes the incoming zero-point radiation $\rho(\omega)=\rho(\omega, T=0)$. Thus the thermodynamic arguments apply only to the additional heat energy $\rho(\omega,T)-\rho(\omega)$. The Stefan-Boltzmann law for heat energy is not of the form given by Eqs. (10) and (12), but rather now reads

$$\int_{\omega=0}^{\infty}[\rho(\omega,T)-\rho(\omega)]d\omega=\sigma T^4. \tag{23}$$

H. Spectrum Required of Radiation at Absolute Zero of Temperature

The immediate question for our theory is whether there exists any form of zero-point radiation which will fit consistently with the traditional ideas of classical theory. We first remark that zero-point radiation throughout the universe will not alter Maxwell's radiation pressure, because it gives balancing forces on the inside and outside of any wall of a cavity. But how about Wein's arguments involving a moving mirror? Turning to Eq. (11), we find the change in energy density of radiation due to adiabatic compression. At temperature $T=0$, we expect no change of electromagnetic energy no matter what expansion of the cavity is made. Thus at $T=0$, we require from (11) that

$$0=\delta\rho(\omega)=(\tfrac{1}{3}\omega[d\rho(\omega)/d\omega]-\rho(\omega))\delta V/V \tag{24}$$

or

$$\rho(\omega)=\text{const}\times\omega^3. \tag{25}$$

Looked at from another point of view, this corresponds to setting $F(\omega/T)=\text{const}$ in the usual form (13) of Wein's law. If we go back to the original arguments leading to Wein's law, we find that if $\rho(\omega)$ has the form in (25), then no net work is performed on a moving mirror, and hence there is indeed no change in energy density of radiation. Turning to consider the velocity-damping force $F=-Pv$, where P is given in (18), we discover that $\rho(\omega)$ gives rise to no contributions to this force. In fact, it can be shown[7] that $\rho(\omega)$ of Eq. (25) corresponds to the unique Lorentz-invariant spectrum of electromagnetic radiation and hence cannot give rise to *any* velocity-damping force. It is a natural incoming radiation in a Lorentz-invariant universe.

The result obtained here for classical zero-point radiation has an immediate connection with quantum electromagnetic zero-point energy appearing in the quantization of the electromagnetic field by analogy with an assembly of simple harmonic oscillators. Indeed, if we multiply the quantum zero-point energy $\frac{1}{2}\hbar\omega$ per normal mode by the number of normal modes per unit frequency interval per unit volume, then we find

$$\rho_{\text{quantum}}(\omega)=\hbar\omega^3/2\pi^2c^3. \tag{26}$$

This is of precisely the required form (25), with the constant given in terms of Planck's constant $h=2\pi\hbar$,

$$\text{const}=\hbar/2\pi^2c^3. \tag{27}$$

In order to express experimental results in the forms now familiar from quantum expressions, we will henceforth adopt this value for the constant in (25). Thus in subsequent work, we will find that $\hbar$ enters our theory, not as any quantum of action, but solely as the constant setting the scale of the zero-point electromagnetic radiation spectrum.

I. Interaction of a Particle with a Wall at Zero Temperature

Having found the form required of acceptable classical radiation at temperature $T=0$, we now turn back to Eq. (20) so as to consider the interactions $\langle vJ\rangle$ at the wall at this temperature. Substituting $\rho(\omega)$ from Eqs. (25) and (27), we find $P=0$, corresponding to the absence of velocity-damping forces, and

$$-2m\langle vJ\rangle=\langle\Delta^2\rangle_{T=0}. \tag{28}$$

Thus at equilibrium, the particle picks up energy due to

[7] T. H. Boyer, Phys. Rev. **182**, 1374 (1969). T. W. Marshall, Proc. Camb. Phil. Soc. **61**, 537 (1965).

fluctuating radiation while in the box, and then gives up the kinetic energy on striking a wall. The energy can be given up in dipole radiation by the particle's oscillator and also through mechanical excitations of the particles forming the walls. The mechanical excitations will decay through the emission of radiation, and hence the energy absorbed by the particle in the box will eventually be returned to the radiation field completing the energy balance.

J. Derivation of the Spectrum of Blackbody Thermal Radiation

The next step in the analysis requires the explanation of the interaction of the particles and the walls at finite temperature. We have already remarked that the walls must be regarded as formed of mechanical systems which can absorb mechanical energy. For equilibrium, the walls must be at the same temperature as the enclosed particles and radiation. Hence it is natural to assume that as far as purely thermal energy is involved, there is no net transfer of energy between a gas of particles and the walls. In other words, particles in collision with the walls will emerge with a spectrum of kinetic energies the same as the entering particles—except for the crucial contribution of zero-point energy which can be removed only at the walls. Thus we are suggesting

$$-2m\langle vJ\rangle_T=-2m\langle vJ\rangle_{T=0}=\langle\Delta^2\rangle_{T=0}. \qquad (29)$$

The result can be inserted in Eq. (20) together with the value of $\langle\Delta^2\rangle_{T=0}$ obtained from $\rho(\omega)$ in (25). The resulting differential equation for $\rho(\omega,T)$ has a unique solution continuous at $T=0$, and for the choice of constant as in (27), gives

$$\rho(\omega,T)=(\omega^2/\pi^2c^3)[\hbar\omega/(e^{\hbar\omega/kT}-1)+\tfrac{1}{2}\hbar\omega], \quad (30)$$

which is the experimentally observed Planck spectrum.

A derivation of the blackbody radiation spectrum along the lines followed here has been given previously.[6] However, in the earlier analysis, electromagnetic zero-point radiation was regarded as a new postulate to be combined with traditional theory. Here we wish to emphasize that a careful analysis of classical theory naturally suggests the presence of precisely this zero-point radiation. Rather than forming a new *ad hoc* postulate on top of classical theory, temperature-independent radiation fits in neatly to fill one of the gaps in our knowledge of classical behavior.

K. Repetition of Analysis for Classical Oscillator

At the outset of this work, we pointed out that the traditional ideas of energy equipartition fail in general, holding only for massive particles where radiation is negligible. The crucial aspect of the accurate arguments involving thermal radiation is the use of a heavy particle to give the temperature in terms of the definition (5), and then a coupling to the electromagnetic field, through an entirely independent oscillator whose average energy is unknown at the start. In the analysis above, we chose a particle in a box. Here we point out that the analysis is exactly the same for a heavy particle on the end of a spring.

In equilibrium, the average energy of a spring oscillator of mass m and spring constant K is time-independent so that

$$\langle\tfrac{1}{2}Kx_t^2+\tfrac{1}{2}mv_t^2\rangle=\langle\tfrac{1}{2}Kx_{t+\tau}^2+\tfrac{1}{2}mv_{t+\tau}^2\rangle=\langle\tfrac{1}{2}K(x_t+v_t\tau)^2 +(1/2m)(mv_t+\Delta-Pv_t\tau-Kx_t\tau+\mathcal{J})^2\rangle, \quad (31)$$

where the impulse terms have been separated out roughly as follows: (i) Δ represents the fluctuating impulse in the time interval τ due to the interaction of electromagnetic radiation with the dipole oscillator inside the particle. (ii) $-Pv$ is the velocity-dependent damping force due to radiation. (iii) $-Kx$ is the force due to the spring. (iv) $\mathcal{J}$ is the impulse delivered to the particle in the time interval τ due to radiation reaction not associated with thermal radiation.

We must be somewhat careful on this separation of terms. In Eq. (14), we spoke of J as an impulse due to the walls including unspecified effects due to radiation reaction. Here the impulse from the spring is separated out; $-kx\tau+\mathcal{J}$ corresponds to what was termed J. Also, we should realize clearly that both $-Pv\tau$ and $\mathcal{J}$ are impulses associated with radiation reaction. The average velocity-dependent force $-Pv$ due to radiation does work on the particle with the energy reappearing in the thermal radiation field, and hence is a radiation reaction force. The crucial point is that the radiation reaction associated with zero-point radiation must be separated out as a distinct term not included in the velocity-dependent damping force $-Pv$.

Here again we expand the right-hand expression. Noting that $\langle v\Delta\rangle$, $\langle x\Delta\rangle$, and $\langle\Delta J\rangle$ vanish due to the fluctuating character of Δ, and dropping terms which are essentially second order in τ, we arrive at

$$0=\langle\Delta^2\rangle-2mP\tau\langle v^2\rangle+2m\langle v\mathcal{J}\rangle, \qquad (32)$$

which is exactly of the form (16). Again an analysis of the situation at temperature $T=0$ in the presence of zero-point radiation where $\langle\Delta^2\rangle_{T=0}\neq 0$ but $P_{T=0}=0$, reminds us of the need for the further radiation reaction impulse $\mathcal{J}$. Without this term, the oscillator would perform a random walk to arbitrarily high energy and equilibrium would be impossible.

L. Zero-Point Energy of Particles

The presence of zero-point electromagnetic radiation will lead to zero-point energy of all particles interacting with radiation. Since we have previously remarked that all particles satisfying the properties demanded of traditional statistical mechanics, such as being confined by a box, must interact with radiation, we conclude that all particles of interest here acquire zero-point energy. The introduction of thermal energy of particles must be in addition to the zero-point energy.

In the case of a particle at temperature $T=0$, we saw that the particle acquired energy through interaction with the internal dipole oscillator. The fluctuating impulses $\langle\Delta^2\rangle_{T=0}$ acquired in a time τ, as indicated in Eq. (17), depends only on the parameter Γ of the oscillator and not on the mass of the particle. Relative to the frame in which the particle was at rest at time t, the particle acquires a kinetic energy $\frac{1}{2}\langle\Delta^2\rangle_{T=0}/m$ in a time τ. This kinetic energy varies inversely as the mass and, hence, becomes vanishingly small for a very massive particle. The actual zero-point energy of a particle in a box will depend on the detailed shape of the box which will dictate the stationary probability distribution for the particle undergoing Brownian motion due to the fluctuations of zero-point radiation. The situation here is quite analogous to quantum mechanics where the ground-state energy eigenvalue of a particle in a box depends crucially on the shape of the box through the boundary conditions on the Schrödinger equation.

It should be apparent to the reader that there is a distinction of the most fundamental sort between thermal radiation and zero-point radiation. Thus the contribution to particle kinetic energy by heat radiation goes to a finite nonzero limit $\frac{3}{2}kT$ in Eq. (5) as the mass of the particle becomes infinite. By contrast, for a particle in a box, the contribution to particle kinetic energy from zero-point radiation vanishes as the mass of the particle becomes infinite. The fluctuating electromagnetic radiation contributes to all electromagnetic processes in the form of the energy density $\rho(\omega,T)$ without any distinction between the zero-point and heat contributions. The contrast in the behavior of the two components of the radiation can immediately be traced to the difference in spectral distributions of energy.

The heat contribution to radiation depends specifically on the reference frame of the container and gives rise to velocity-dependent forces. These velocity-dependent forces are what act to impose the tendency toward equipartition of energy. The radiation fluctuations must give rise to mean-square velocity determined by the external-temperature parameter T before the velocity-dependent forces act to rob the particle of its energy.

By contrast, the zero-point spectrum is Lorentz-invariant and cannot give rise to velocity-dependent forces. A particle in free space would perform a random walk in energy without any limitation. It is the container which enforces equilibrium on a particle at absolute zero. In total contrast to the heat contribution, the parameters determining the equilibrium energy involve the mass of the particle and the dimensions of the box. The equilibrium equation (28) does not describe the particle energy at equilibrium; it merely notes that the energy absorbed from the zero-point radiation is given up through radiation reaction at the walls of the box. Electromagnetic zero-point radiation is fundamentally connected with some of the most profound problems of physics, with the ideas of radiation reaction, of inertial frames of reference, and of particle mass.

IV. ENTROPY AND THERMODYNAMIC PROBABILITY

A. Traditional Classical Ideas of Entropy

Heat energy is associated with random motions of the elements of a system. Hence in thermodynamics, it is not enough to specify the energy of a system; we must also describe the state of disorder, the entropy.

The idea of entropy is approached from two distinct directions in statistical thermodynamics. On the one hand, thermodynamics introduces the caloric entropy

$$dS_{\text{caloric}}=dQ/T \quad \text{(reversible process)}, \qquad (33)$$

where S_{caloric} is found to be a state function for the system. On the other hand, for systems with finitely many degrees of freedom, a probabilistic entropy S_{prob} is introduced connected to the thermodynamic probability W,

$$S_{\text{prob}}=(S_{\text{prob}})_0+k\ln W, \qquad (34)$$

where W is the number of microstates of the system which give the same macrostate. It was one of the triumphs of Boltzmann's work in classical statistical physics to suggest the coincidence of these two notions of entropy. However, we propose that this idea is erroneous. In the view presented in this paper, it is precisely the failure to distinguish clearly between these two forms of entropy in the context of electromagnetism which was one of the contributing factors leading to the rejection of classical theory and to the quantum revolution of the beginning of the century.

B. Caloric Entropy

In the context of our classical reformulation, the traditional ideas of caloric entropy appearing in thermodynamics are unchanged. The definition of S_{caloric} in (33) is still acceptable. Again, we find the usual ideas of the conservation of entropy in a reversible cycle. However, we must recall that the usual calculations of entropy for an ideal gas hold only in the case of very heavy gas particles where the zero-point energy of the particles is small.

C. Reconsideration of Traditional Connection Between Caloric Entropy and Thermodynamic Probability

The limitation that the traditional entropy calculations for ideal gases hold only for massive particles is of considerable importance. It reminds us that the traditional arguments connecting caloric entropy and probability must be reexamined in the light of zero-point energy arising from electromagnetic radiation.

Let us consider two specific examples. The first involves a traditional ideal gas of particles of mass m at temperature T in an insulated volume V, where, following the traditional outlook, we ignore the thermal radiation. If a wall of the box is suddenly removed, then the gas undergoes a free expansion to a volume V'. The entropy of the gas is a state function of the gas and hence depends only on the initial and final equilibrium conditions of the gas. Replacing the irreversible expansion by a reversible expansion with the same initial and final conditions, we obtain[8] the change in entropy of the gas of N molecules:

$$\Delta S = Nk \ln(V'/V). \tag{35}$$

However, this change in entropy can be compared with the change in thermodynamic probability of the system. The traditional argument remarks that the kinetic energy of the particles must be the same in the initial and final states, and hence that the change in thermodynamic probability is related solely to the change in volume accessible to the system. A short calculation[8] provides

$$W/W' = V^N/V'^N, \tag{36}$$

and hence in this case

$$\Delta S_{\text{caloric}} = +k(\ln W' - \ln W). \tag{37}$$

There is an immediate connection between caloric entropy and thermodynamic probability.

This argument is wrong for particles of finite mass. Zero-point radition gives rise to a zero-point kinetic energy of the particles which will depend upon the configuration of the volume V and V'. Thus there is a change in kinetic energy of the particles. By the first law of thermodynamics, heat ΔQ has entered the system and the caloric entropy has changed. Also, the change in kinetic energy means a further change in the thermodynamic probability. The detailed connection between caloric entropy and thermodynamic probability is not immediately clear.

A more striking example involves a free expansion of a gas of weakly interacting particles at temperature $T=0$. In this case, the change in caloric entropy cannot be defined, but, in principle, it is possible to compute the change in thermodynamic probability of the system as a change of the zero-point energy distribution in classical phase space.

D. Classical Thermodynamic Probability of Harmonic Oscillator

Having touched upon the concept of caloric entropy, we now wish to approach the problem of thermodynamic probability. The fundamental postulate in classical thermodynamic probability, is that all states accessible to the system are equally probable. The system considered first in traditional developments is that of a particle in a box under the assumption that all positions in the box are equally probable, as are all directions of particle velocity. However, we have emphasized that any classical statistical system of finite mass must be considered in conjunction with the associated equilibrium radiation. In the case of a particle in a box, the description of a state of the system is complicated by the fact that the energy transfer between the particle and radiation parts of the system includes a transfer at the walls so that it is difficult to describe the states of the entire system in the crucially important region of low temperature.

The case of a harmonic oscillator may be somewhat easier to handle. It is frequently pointed out that radiation behaves in many respects as a collection of harmonic oscillators. The exchange of energy is merely between a physical oscillator and the radiation oscillators. Following traditional arguments, we assume that the probability distribution on the oscillators' phase space makes the probabilistic entropy or, equally well, the thermodynamic probability a maximum. In the present case, maximum entropy of the system should mean maximum entropy per oscillator. Dividing phase space for the mechanical oscillator into boxes of (temporarily) finite volume G, we require that the probabilistic entropy should be a maximum in conjunction with the restrictions that the average energy be given by E, and that the total probability on phase space be unity. Thus denoting by w_i the probability of the particle being in the ith box with energy E_i, we have

$$1 = \sum w_i, \tag{38}$$

$$E = \sum_i E_i w_i, \tag{39}$$

$$S_{\text{prob}} = (S_{\text{prob}})_0 - k \sum_i w_i \ln w_i, \tag{40}$$

$$\delta S_{\text{prob}} = 0 = -k(\ln w_i + 1)\delta w_i. \tag{41}$$

Because of the form of the constraint equations, it is clear that the probability w_i can depend only upon the energy E_i of the ith element in phase space, and the solution is the familiar canonical distribution[9]

$$w_i = e^{-E_i/E} \Big/ \sum_i e^{-E_i/E}, \tag{42}$$

or, in the limit as the volume G of the elements in phase space go to zero,

$$w(q,p) = \exp\left[-\frac{(\frac{1}{2}Kx^2 + \frac{1}{2}p^2/m)}{E}\right] \Big/ \int\int dq dp \exp\left[-\frac{(\frac{1}{2}Kx^2 + p^2/2m)}{E}\right]. \tag{43}$$

[8] R. Resnick and H. Halliday, *Physics* (John Wiley & Sons, Inc., New York, 1967), Vol. I. pp. 640, 642.

[9] See the work of Planck in Ref. 3, Part III. Also, F. W. Sears, *Thermodynamics, the Kinetic Theory of Gases, and Statistical Mechanics* (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1953), Chap. 14.

The total probabilistic entropy for the system is given by

$$S_{\text{prob}} = (S_{\text{prob}})_0 + k \ln E, \tag{44}$$

with the additive constant $(S_{\text{prob}})_0$ unknown as in all of classical theory.

E. Phase-Space Distribution of Classical Oscillator in Fluctuating Classical Electromagnetic Field

It is of some interest to note that we can check the purely statistical argument given above against a calculation for the phase-space distribution of the oscillator based purely on classical electromagnetic theory. It was noted long ago that a classical oscillator of frequency ω in equilibrium with random electromagnetic radiation will have an average energy related to the spectrum of electromagnetic energy at frequency ω as in Eq. (7). However, more recently, Marshall[10] has extended the calculation to obtain the phase-space density of such a classical oscillator, and, in a velocity-damping approximation, he obtains precisely the Gaussian shape in Eq. (43). Thus, the classical arguments from thermodynamic probability are in complete agreement with purely electromagnetic calculations.

F. Discrepancy between Caloric and Probabilistic Entropy

The reader may be somewhat surprised by the situation at which we have arrived. On the one hand, we have applied classical statistical thermodynamics to electromagnetic radiation, have seen the need for zero-point radiation, and have then derived the Planck distribution law. Thus taking $\rho(\omega,T)$ according to Eq. (30), we have the average of an oscillator in the electromagnetic field given by (7) as

$$E = \hbar\omega/(e^{\hbar\omega/kT}-1) + \tfrac{1}{2}\hbar\omega, \tag{45}$$

which is precisely the result of quantum theory. On the other hand, we have applied classical statistical thermodynamics to an oscillator in the electromagnetic field and have found the classical entropy (44). If we merely differentiate this result, we find

$$\frac{\partial S_{\text{prob}}}{\partial E} = \frac{k}{E}. \tag{46}$$

Now from the first law of thermodynamics and the definition (33), it follows that

$$\frac{\partial S_{\text{caloric}}}{\partial E} = \frac{1}{T}. \tag{47}$$

[10] T. W. Marshall, Proc. Roy. Soc. (London) **A276**, 475 (1963). The aspect of Marshall's work which interest us here is the calculation in Secs. 3 and 4.

If we identify S_{prob} with S_{caloric}, we find the result of traditional energy equipartition

$$E = kT. \tag{48}$$

How are we to reconcile the contradictory results of Eqs. (45) and (48)? The answer to this old dilemma has been hinted at throughout this section; it is the fact that S_{prob} and S_{caloric} *cannot* be identified. Entropy is associated with disordered energy. However, we have found that the universe contains two sources of disordered energy. One source is the radiation associated with finite temperature; the other is associated with the random fluctuations of the incoming electromagnetic zero-point radiation. Both sources of random energy contribute to the entropy S_{prob} associated with thermodynamic probability.

V. HISTORICAL EXAMPLES OF THE MISMATCH BETWEEN S_{caloric} AND S_{prob}: ALTERNATIVE INTERPRETATIONS THROUGH QUANTA OR ZERO-POINT RADIATION

A. Historical Comment

Planck discovered[11] his famous radiation law without the idea of quanta. Rather, the formula arose as a natural modification of the connection between the entropy and energy of thermal radiation, which formed an interpolation between the contradictory entropy-energy formulas of Wein and of Rayleigh and Jeans. Only after noting the success of this modification did Planck return to try to find a theoretical justification for the modification. He found an explanation in the notion of discrete units of energy, the notion of quanta. The experimentally verified radiation formula was obtained provided he modified the classical calculation of thermodynamic probability by the use of quanta. The essence of quantum statistical mechanics lies right here; it is a modification of the traditional classical connection between entropy and thermodynamic probability.

The point of view advanced in the present paper is that the introduction of the notion of quanta is quite unnecessary. Just as the blackbody radiation spectrum can be explained by the presence of classical electromagnetic zero-point radiation, so also the discrepancy between entropy and thermodynamic probability follows naturally from this same zero-point radiation. The idea of quanta forms a subterfuge for what is a natural part of a theory of classical statistical thermodynamics including electromagnetism.

In this last part of our analysis, we turn to consider a number of examples of historical importance in the development of quantum theory. Early in this century,

[11] The historical arguments regarding thermal radiation are traced in E. T. Whittaker, *A History of the Theories of Aether and Electricity, Modern Theories 1900–1926* (Philosophical Library, New York, 1954), Chap. III. See also M. J. Klein, in *The Natural Philosopher* (Blaisdell Publishing Co., New York, 1964), **Vol. 3**, p. 1.

Einstein proposed a number of situations which he concluded had their most natural interpretations in terms of quanta. Inasmuch as Einstein became skeptical of later formulations of quantum theory, it is ironic that it is precisely his incisive and fundamental examples which we will review and then reinterpret without quanta in terms of classical electromagnetic zero-point radiation.

B. Fluctuations in Thermal Radiation

It was in 1909 that Einstein[12] proposed the following problem involving fluctuations of electromagnetic radiation which proved a quandary for traditional classical theory. Although Einstein found the resolution of the problem in terms of quanta, our point of view suggests that the difficulty should be properly traced to the mismatch between caloric entropy and thermodynamic probability. We will first summarize the original arguments and our reinterpretation, and then will consider each aspect in more detail. Einstein considered the fluctuation of radiation in an enclosure and, using the traditional connection of entropy and probability, derived a relation between entropy and mean-square energy fluctuations of the radiation,

$$\frac{\partial^2 S_{\text{prob}}}{\partial E^2}=\frac{-k}{\langle e^2\rangle} \tag{49}$$

per normal mode. Now in the wave theory of electromagnetism $\langle e^2\rangle=E^2$. Einstein then assumed $S_{\text{prob}}=S_{\text{caloric}}$, and saw that his derivation had led to the erroneous Rayleigh-Jeans law of radiation. Einstein concluded that the error lay in the assumed wave nature of light. Accordingly, he recomputed $\langle e^2\rangle$ for the case of photons, finding a new value such that the assumption $S_{\text{prob}}=S_{\text{caloric}}$ led satisfactorily so the Planck radiation law without zero-point radiation. The point of view advanced in the present analysis is that the modification required is the recognition that $S_{\text{prob}}\neq S_{\text{caloric}}$. The connection of S_{caloric} must be with the purely thermal fluctuations. Regarding the thermal and zero-point fluctuations as independent so that the mean squares of the fluctuations add, we have a relation for S_{caloric} which leads to the Planck law with zero-point energy. We do *not* modify the wave nature of electromagnetism but rather note the mismatch between S_{prob} and S_{caloric} which arises from the presence of fluctuating radiation at the zero of temperature.

C. Relation between Entropy and Fluctuations in Energy

Einstein proposed to consider a cavity containing thermal radiation, which could be regarded as separated into a large volume V and a small volume $\mathcal{V}$. The energy $\mathfrak{U}$ of electromagnetic radiation in $\mathcal{V}$ between frequencies ω and $\omega+d\omega$ undergoes spontaneous fluctuations. Along with this fluctuation in the energy is a change in the associated entropy. Denoting by Σ the entropy contribution between ω and $\omega+d\omega$ for the larger volume and by $\mathcal{S}$ that of the smaller volume, we have

$$\mathfrak{S}(\epsilon)=\Sigma+\mathcal{S}=\Sigma_0+\mathcal{S}_0+\left(\frac{\partial\Sigma}{\partial\epsilon}+\frac{\partial\mathcal{S}}{\partial\epsilon}\right)\epsilon+\frac{1}{2}\left(\frac{\partial^2\Sigma}{\partial\epsilon^2}+\frac{\partial^2\mathcal{S}}{\partial\epsilon^2}\right)\epsilon^2+\cdots, \tag{50}$$

where Σ_0 and $\mathcal{S}_0$ are the equilibrium entropies, ϵ is the fluctuation in energy of $\mathcal{V}$, and all derivatives are evaluated when the fluctuation is zero. The first derivatives vanish because the entropy is a maximum for the energy distribution at equilibrium. If the volume V is much larger than $\mathcal{V}$, the second derivative of Σ may be neglected compared to that of $\mathcal{S}$. Noting, further, that the derivative with respect to ϵ may be changed to one with respect to $\mathfrak{U}$, the expression becomes

$$\mathfrak{S}(\epsilon)\cong\Sigma_0+\mathcal{S}_0+\frac{1}{2}\left(\frac{\partial^2\mathcal{S}}{\partial\mathfrak{U}^2}\right)\epsilon^2. \tag{51}$$

Now we connect entropy with probability. From (34), we have the probability for various fluctuations ϵ in energy given by

$$dW=\text{const}\times\exp[\mathfrak{S}_{\text{prob}}(\epsilon)/k]d\epsilon=\text{const}'\times\exp\left[\frac{1}{2k}\frac{\partial^2\mathcal{S}_{\text{prob}}}{\partial\mathfrak{U}^2}\epsilon^2\right]d\epsilon. \tag{52}$$

The value of const′ may be found by taking the integral over all energy fluctuations as giving unit total probability. Since the entropy will be sharply peaked at the equilibrium energy, we may here take the limits of integration for the energy fluctuation from $-\infty$ to ∞. Finally, from the definition of mean-square fluctuation as

$$\langle\epsilon^2\rangle=\int\epsilon^2\,dW, \tag{53}$$

it follows that

$$\frac{\partial^2\mathcal{S}_{\text{prob}}}{\partial\mathfrak{U}^2}=\frac{-k}{\langle\epsilon^2\rangle}. \tag{54}$$

The magnitude of the fluctuations can be obtained[13] from purely electromagnetic theory of waves giving

$$\langle\epsilon^2\rangle=(\pi^2c^3/\omega^2)\rho^2d\omega, \tag{55}$$

where

$$\mathfrak{U}=\rho d\omega. \tag{56}$$

For convenience of analysis, we now use Eq. (7) to

[12] A. Einstein, Phys. Zeits. **10**, 185 (1909).

[13] H. A. Lorentz, *Les Théories Statistiques en Thermodynamique* (B. G. Teubner Verlag, Leipzig, 1916), p. 114. See also, S. Tomonaga, *Quantum Mechanics* (North-Holland Publishing Co., Amsterdam, 1962), Vol. I, p. 298.

remove the factors corresponding to the number of normal modes per unit frequency interval, and go over to the fluctuations per normal mode, or, equally well, to the corresponding fluctuations of an oscillator in the radiation field. We then find

$$\frac{\partial^2 S_{\text{prob}}}{\partial E^2}=\frac{-k}{E^2} \tag{57}$$

for the relationship between S_{prob} and the average oscillator energy E. We should note that if we turn back to our calculations for the probabilistic entropy of a harmonic oscillator, then we find that this fluctuation equation (57) is identical with the derivative of Eq. (46) with respect to E.

The crucial point in Einstein's analysis is the next step, the assumption that $S_{\text{prob}}=S_{\text{caloric}}$ in Eq. (57). Thus with this assumption, Einstein concludes that $E=kT$ for an oscillator and that the erroneous Rayleigh-Jeans law holds for the spectral energy distribution $\rho(\omega,T)$. In searching for an explanation of the failure of the analysis, Einstein directed his attention to the right-hand side of Eq. (54), to the calculation of $\langle\epsilon^2\rangle$. In particular, Einstein showed that if we reject the wave nature of light and regard it rather as composed of discrete quanta of energy $\hbar\omega$, then $\langle\epsilon^2\rangle$ is not the value given in (55), but rather is

$$\langle\epsilon^2\rangle_{\text{light quanta}}=[\hbar\omega\rho+(\pi^2c^3/\omega^2)\rho^2]d\omega. \tag{58}$$

Again removing the factor of the number of normal modes per unit frequency interval, this corresponds for an oscillator to

$$\partial^2 S_{\text{prob}}/\partial E^2=-k/(E^2+\hbar\omega E). \tag{59}$$

Now making the identification $S_{\text{prob}}=S_{\text{caloric}}$, Einstein found

$$1/T=\partial S_{\text{caloric}}/\partial E=\partial S_{\text{prob}}/\partial E=(k/\hbar\omega)\ \ln[(E+\hbar\omega)/E] \tag{60}$$

and

$$E=\hbar\omega/(e^{\hbar\omega/kT}-1), \tag{61}$$

in agreement with Planck's formula. Here was another piece of circumstantial evidence condemning classical theory and favoring a quantum analysis.

The reformulation of classical theory proposed here offers an alternative explanation of the situation. In this view, Eq. (57) is entirely correct. It follows from fundamental classical ideas involving thermodynamic probability and also the wave nature of light. In order to use this equation to derive the relationship of energy and temperature, we must first separate out the contributions to the entropy from thermal fluctuations and zero-point fluctuations. Since the fluctuations are from independent sources, they add as mean squares

$$\langle\epsilon^2\rangle_{\text{total}}=\langle\epsilon^2\rangle_{\text{zero-point}}+\langle\epsilon^2\rangle_{\text{caloric}}. \tag{62}$$

Thus the caloric entropy is associated with the mean-square fluctuations associated with temperature as

$$\frac{\partial^2 S_{\text{caloric}}}{\partial \mathfrak{U}^2}=\frac{-k}{\langle\epsilon^2\rangle_{\text{caloric}}}=\frac{-k}{\langle\epsilon^2\rangle_{\text{total}}-\langle\epsilon^2\rangle_{\text{zero-point}}}. \tag{63}$$

Using the result in (55) based on the wave nature of light, we have

$$\langle\epsilon^2\rangle_{\text{total}}=(\pi^2c^3/\omega^2)\rho^2(\omega,T)d\omega, \tag{64}$$

$$\langle\epsilon^2\rangle_{\text{zero-point}}=(\pi^2c^3/\omega^2)\rho^2(\omega)d\omega. \tag{65}$$

This gives the relation per normal mode

$$\frac{\partial^2 S_{\text{caloric}}}{\partial E^2}=\frac{-k}{E^2-E_0{}^2}=\frac{-k}{E^2-(\frac{1}{2}\hbar\omega)^2}, \tag{66}$$

where in the last expression we have substituted the specific result of (25) and (27) for the zero-point spectrum $\rho(\omega)$. Thus integrating once, we have

$$\frac{1}{T}=\frac{\partial S_{\text{caloric}}}{\partial E}=\frac{+k}{2E_0}\ln\left(\frac{E+E_0}{E-E_0}\right)=\frac{k}{\hbar\omega}\ln\left(\frac{E+\frac{1}{2}\hbar\omega}{E-\frac{1}{2}\hbar\omega}\right) \tag{67}$$

and

$$E=\hbar\omega/(e^{\hbar\omega/kT}-1)+\tfrac{1}{2}\hbar\omega, \tag{68}$$

which is the same result as Einstein's argument except for the zero-point energy $\frac{1}{2}\hbar\omega$. We conclude that an acceptable alternative interpretation is possible within classical theory.

D. Classical Explanation for Planck's Quanta

The example above provides a detailed description of how the idea of quanta forms a subterfuge for handling a situation in which two independent energies contribute to entropy but only one energy is recognized in the theory. Thus we have noted that the fluctuations associated with the caloric part of the entropy are given per oscillator by

$$\langle e^2\rangle_{\text{caloric}}=E^2-E_0{}^2, \tag{69}$$

where E^2 gives the total mean-square fluctuation including that due to zero-point energy. If, however, we are aware only of the caloric energy, then we would try to write this fluctuation entirely in terms of E_{caloric}, where

$$E=E_{\text{caloric}}+E_0, \tag{70}$$

and would find the association

$$\langle e^2\rangle_{\text{caloric}}=E^2-E_0{}^2=E_{\text{caloric}}{}^2+2E_0E_{\text{caloric}}. \tag{71}$$

But this latter expression has an immediate interpretation in terms of fluctuations of the caloric energy arising from quanta of magnitude $2E_0$. Thus the idea of quanta of magnitude $\hbar\omega$ follows actually from the presence of classical zero-point energy $E_0=\frac{1}{2}\hbar\omega$. The success of Planck's idea of quanta can thus be accounted for on the classical level precisely in terms of the failure to recognize the contribution of fluctuating zero-point radiation.

It is clear that a classical explanation for photon statistics can be given along these same lines. We must first recognize that counting according to indistinguishibility of particles is *not* the essence of Bose-Einstein statistics. In order to avoid the Gibbs paradox in classical theory,[14] the identical-particle counting usually associated with Bose-Einstein statistics must be used in classical statistical mechanics if the particles are classically identical. The distinguishibility associated with nonmechanical properties, such as the color of the paint on the molecule, are associated with noncaloric aspects of entropy. Properly, identical-particle counting is an entirely classical idea; for example, with this counting the traditional Maxwell-Boltzmann distribution of particle velocities is obtained in the classical limit of a continuous distribution on phase space. The essence of Bose-Einstein statistics is not the identical-particle counting; it is the use of phase-space cells of finite size in calculating the partition function. This leads to the dispersion formula for photons in Bose-Einstein statistics[15]

$$\langle(\Delta n_r)^2\rangle=\langle n_r\rangle^2+\langle n_r\rangle\,, \tag{72}$$

where here n_r is the number of photons of frequency ω_r. However, we can see this quantum statistical result follows directly from the classical interpretations given above in (71). Quantum statistical mechanics regards E_{caloric} as arising from photons of energy $2E_0$. Thus

$$E_{\text{caloric}}=\langle n_r\rangle 2E_0. \tag{73}$$

Then the fluctuation in caloric energy $\langle e^2\rangle_{\text{caloric}}$ will be associated with fluctuations in the number of photons

$$\langle e^2\rangle_{\text{caloric}}=\langle(\Delta n_r 2E_0)^2\rangle.$$

Writing the classical fluctuation equation (71) in this photon notation and dividing out the common factor of $(2E_0)^2$, we have exactly Eq. (72), usually derived from quantum statistics. Details of the analysis for both photon and general Bose-Einstein statistics will be presented in another publication.

E. Equilibrium of Particle in a Box: Einstein's *A*- and *B*-Coefficient Analysis

In the early part of this paper, we used as a fundamental starting point the ideas of Einstein for a heavy particle with a dipole oscillator confined to a box. We have shown that a purely classical analysis of this problem leads naturally to the idea of electromagnetic zero-point radiation and hence to an explanation of equilibrium which is in agreement with experiment. The distribution of thermal radiation found is the Planck law with zero-point radiation.

We should point out that this physical system also has a description in terms of quanta. In the original 1910 work,[5] Einstein and Hopf followed traditional classical theory in neglecting interactions of the particles with the walls of the box, and derived the equilibrium equation

$$\langle\Delta^2\rangle=2mP\tau\langle v^2\rangle=2PkT\tau\,, \tag{74}$$

corresponding to our Eq. (16) without the term $-2m\langle vJ\rangle$. Substituting in the classical electromagnetic values for $\langle\Delta^2\rangle$ and P given here in (17) and (18), they arrived at the Rayleigh-Jeans law. They regarded the analysis as an unimpeachable argument for the failure of classical theory.

Since the physics leading to (74) seemed fundamental, Einstein[16] returned to the problem from a quantum perspective in 1917. Accepting the equilibrium balance (74), he computed new values for $\langle\Delta^2\rangle$ and P based upon the interaction of the particle (now containing a discrete two-level system), with photons carrying energy $\hbar\omega$ and linear momentum $\hbar\omega/c$. This is part of the famous A and B coefficient derivation of Planck's law. Now (74) leads to the Planck distribution without zero-point energy.

Thus we note that this physical situation also allows analysis alternatively in terms of light quanta or of electromagnetic zero-point radiation. It is, however, interesting to note the difference in the analyses in the limit of small temperature $T\rightarrow 0$. In this limit, Einstein's quantum values for $\langle\Delta^2\rangle$ and for P both vanish so that the particle is subject to no fluctuations. The inherent quantum motion at $T=0$ must in this argument be added as a further *ad hoc* postulate. However, in the argument through classical radiation, the zero-point motion is consistently accounted for since the particle is still subject to the fluctuations $\langle\Delta^2\rangle_{T=0}$ due to random radiation at the absolute zero of temperature.

VI. SUMMARY OF CLASSICAL AND QUANTUM INTERPRETATIONS OF THERMAL RADIATION

The following equations outline the different interpretations of thermal radiation, including the traditional classical theory leading to the Rayleigh-Jeans law, the Planck derivation involving finite cell size on phase space, and the classical interpretation in the presence of electromagnetic zero-point radiation. The equations common to the three points of view are centered on the page, while the divergent interpretations are given in three parallel columns. The thermodynamic and statistical mechanical aspects are treated separately.

[14] See, for example, F. Reif, *Fundamentals of Statistical and Thermal Physics* (McGraw-Hill Book Co., Inc., New York, 1965), pp. 243–246.

[15] See Ref. 14, p. 346.

[16] A. Einstein, Phys. Zeits. **18**, 121 (1917). See also Abraham and Becker, *Theorie der Elekrizität* (B. G. Teubner Verlag, Leipzig, 1933), Vol. II, 6th ed., pp. 388–395.

Blackbody Radiation

Traditional classical	Quantum	Zero-point classical

1. Thermodynamics

Energy density

$$u_c=\int_{\omega=0}^{\infty}\frac{\omega^2}{\pi^2c^3}E_c(\omega,T)\,d\omega=\sigma T^4. \tag{75}$$

Entropy density

$$s_c=\int_{\omega}^{\infty}\frac{\omega^2}{\pi^2c^3}S_c(\omega,T)\,d\omega=\tfrac{4}{3}\sigma T^3. \tag{76}$$

Connection with temperature

$$\left(\frac{\partial S_c(\omega,T)}{\partial E_c(\omega,T)}\right)_\omega=\frac{1}{T}. \tag{77}$$

All energy in statistical mechanics is heat (caloric) energy. $E_c(\omega,T)$ is the only fluctuating energy and $S_c(\omega,T)$ is the associated entropy per normal mode.

There is fluctuating electromagnetic radiation $E(\omega)$ at zero temperature. The caloric energy is defined as $E_c(\omega,T)\equiv E(\omega,T)-E(\omega)$, where $E(\omega,T)$ is the total fluctuating radiation at temperature T, and $E(\omega)\equiv E(\omega, T=0)$. Caloric entropy is associated with caloric energy only.

2. Statistical Mechanics

Each normal mode is considered separately, corresponding to a single oscillator of the same frequency. The explicit labels ω and T are dropped so that $E\equiv E(\omega,T)$, $E_0\equiv E(\omega)$, $S_{c,p}\equiv S_{c,p}(\omega,T)$. Derivatives are written as partial derivatives with ω held fixed. *All* sources of disordered energy are involved in the phase-space distributions. Bose-Einstein counting is used in all cases. Initially the phase-space cell size is finite with value G, and ellipses following lines of constant energy are used to subdivide phase space.

The entropy associated with thermodynamic probability is

$$S_p=-k\sum_i w_i\ln w_i, \tag{78}$$

where w_i is the probability of the particle being in the ith cell with energy E_i.

$$\delta S_p=0,\quad E=\sum_i E_iw_i,\quad 1=\sum_i w_i. \tag{79}$$

Then

$$w_i=\alpha e^{-\beta E_i}, \tag{80}$$

$$S_p=k\ln\sum_i e^{-\beta E_i}-k\beta E, \tag{81}$$

$$(\partial S_p/\partial E)=k\beta, \tag{82}$$

$$E=G/(e^{\beta G}-1). \tag{83}$$

Cell size $G\to 0$. Eq. (83) becomes

$$E=\beta^{-1}. \tag{84a}$$

Cell size $G\to \hbar\omega$. Eq. (83) becomes

$$E=\hbar\omega/(e^{\hbar\omega\beta}-1). \tag{84b}$$

Cell size $G\to 0$. Eq. (83) becomes

$$E=\beta^{-1}. \tag{84c}$$

3. Connection between Thermodynamics and Statistical Mechanics

The thermodynamic equation (77) becomes in the oscillator notation,

$$(\partial S_c/\partial E_c)=1/T. \tag{85}$$

$$E=E_c, \tag{86a}$$

$$S_p=S_c, \tag{87a}$$

Eqs. (82), (85), (86a), and (87a) imply

$$\beta=(kT)^{-1}. \tag{88a}$$

Eqs. (84a) and (88a) imply

$$E=kT. \tag{89a}$$

Energy equipartition.

$$E=E_c, \tag{86b}$$

$$S_p=S_c, \tag{87b}$$

Eqs. (82), (85), (86b), and (87b) imply

$$\beta=(kT)^{-1}. \tag{88b}$$

Eqs. (84b) and (88b) imply

$$E=\hbar\omega/(e^{\hbar\omega/kT}-1) \tag{89b}$$

Planck oscillator formula.

$$E=E_c+E_0, \tag{86c}$$

where $E_0=\frac{1}{2}\hbar\omega$ from arguments for Wein's law.

$$S_p\neq S_c, \tag{87c}$$

$$\beta\neq(kT)^{-1}. \tag{88c}$$

There are now two sources of disordered energy. Mean-square energy fluctuations $\langle e^2\rangle$ add, giving

$$\langle e^2\rangle=\langle e_c^2\rangle+\langle e_0^2\rangle. \tag{90}$$

From (78) and (79),

$$-(1/k)(\partial^2 S_p/\partial E^2)=\langle e^2\rangle. \tag{91}$$

For the thermal fluctuations alone,

$$-(1/k)(\partial^2 S_c/\partial E_c^2)=\langle e_c^2\rangle. \tag{92}$$

Now for harmonic vibrations,

$$\begin{aligned}\langle e^2\rangle&=E^2,\\ \langle e_0^2\rangle&=E_0^2.\end{aligned} \tag{93}$$

Integrating (92), using (90), (93), comparing with (85), and requiring (89a) in the $T\rightarrow\infty$ limit, gives

$$E_c=\hbar\omega/(e^{\hbar\omega/kT}-1),$$
$$E=\hbar\omega/(e^{\hbar\omega/kT}-1)+\tfrac{1}{2}\hbar\omega.$$

Zero-point classical formula.

VII. CLOSING SUMMARY

Although the physicists at the beginning of the present century regarded the difficulties in explaining the blackbody-radiation pattern as symptomatic of a fundamental failure of classical theory, actually classical theory is well able to account for thermal radiation. The radiation by particles on striking walls forces us to regard the interaction with the walls as not completely known. In particular, the traditional ideas of ideal gases and of energy equipartition must be reexamined. Operating within this framework, we are led naturally to the introduction of temperature-independent radiation corresponding to incoming radiation in the universe in the far past. The presence of such radiation means that the disordered energy of the universe is not purely thermal and hence that caloric entropy and thermodynamic probability are not connected by the logarithmic relation of Boltzmann. In particular, quantum theory is seen as a subterfuge so as to maintain the Boltzmann relation, but, in effect, to separate out only the caloric contributions to thermodynamic probability. Some examples of Einstein which were used as arguments for light quanta can be seen to be easily interpreted in terms of classical electromagnetic zero-point radiation.

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