Quantum Theory of Superfluid Vortices. III. Inelastic Phonon Scattering*

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The quantum theory of vortex waves is used to study phonon scattering by a vortex in He II. If the scattering amplitude is expanded in the number of quanta of vortex waves emitted or absorbed, the resulting inelastic amplitude diverges at long wavelengths. Inclusion of inelastic effects to all orders avoids this unphysical behavior and yields a differential cross section essentially equal to that for elastic phonon scattering.

I. INTRODUCTION

Below 1°K, phonons and rotons in He II form a dilute quasiparticle gas with long mean free paths and lifetimes.[1] In this limit, the interaction between quasiparticles becomes negligible, but scattering can still occur because of spatial inhomogeneities. An example of great interest is the scattering by quantized vortices,[2,3] which has been observed experimentally both through the energy loss of large vortex rings[4] and through the attenuation of second sound in rotating He II.[5] Previous calculations[6–10] have treated the vortex as rigid and thus consider only elastic scattering of quasiparticles. Such a description cannot be wholly accurate, however, because a vortex has internal degrees of freedom corresponding to waves propagating along its axis.[11,12] Consequently, inelastic events that alter the internal state of the vortex must be included along with the usual elastic ones. Indeed, inelastic roton scattering has previously been suggested[8,9] to account for the discrepancy between theoretical and experimental values of mutual friction in rotating He II, but no detailed calculation has yet been carried out.

Inelastic scattering is most simply studied by quantizing the internal states of the scatterer. The present problem, therefore, requires a quantum theory of superfluid vortices, which has recently been proposed and applied both to He II[13] and to type-II superconductors.[14] This formalism will now be used to study the inelastic scattering of quasiparticles by a vortex in He II. For definiteness, the present paper is restricted to phonon scattering, where the interaction is sufficiently weak to permit a first-order calculation in Born approximation. The same basic formalism also applies to rotons or He^3 impurities, but the detailed calculation of the scattering cross section would be considerably more complicated.

When the vortex is excited to one of its internal oscillation modes, the altered velocity field modifies the interaction with the incident quasiparticles. As shown in Sec. II, it is possible to incorporate both elastic and inelastic scattering in a single interaction Hamiltonian. The transition amplitude for phonon scattering is evaluated in Sec. III and then used to determine the differential cross section averaged over the thermal distribution function of the internal vortex modes (Sec. IV).

II. INTERACTION HAMILTONIAN

Consider an unbounded fluid of density ρ containing a single vortex line with circulation κ. For numerical purposes, we shall take $\kappa = h/m \approx 10^{-3}\ \text{cm}^2\,\text{sec}^{-1}$, which is appropriate for He II, but the theory is more generally applicable. If the fluid is at rest at infinity, then the velocity field arises solely from the presence of the vortex and is uniquely determined by its instantaneous configuration. The position of the vortex will be specified by a three-dimensional coordinate vector $\vec{R}'$, which is a function of the arc length along the vortex. It will be convenient to resolve all vectors in cylindrical polar coordinates $\vec{R}' = (\vec{r}', z')$, where $\vec{r}'$ is a two-dimensional vector in the xy plane, perpendicular to the undeformed vortex axis. The fluid velocity at point $\vec{R}$ is then given by[15]

$$\vec{v}(\vec{R}) = (\kappa/4\pi)\int d\vec{s}\,'\times(\vec{R}-\vec{R}')|\vec{R}-\vec{R}'|^{-3}, \tag{1}$$

where the line integral is along the axis of vortex. Equation (1) is a direct analog of the Biot-Savart law.

We assume that the vortex initially lies along the polar axis with undeformed position $(\vec{0}, z')$. If the axis is now slightly deformed to the position $[\vec{u}(z'), z']$, Eq. (1) may then be written

$$\vec{v}(\vec{R}) = \frac{\kappa}{4\pi}\int_{-\infty}^{\infty} dz' \frac{(\hat{z}+d\vec{u}/dz')\times[\vec{r}-\vec{u}(z')+\hat{z}(z-z')]}{|\vec{r}-\vec{u}(z')+\hat{z}(z-z')|^3}\,. \tag{2}$$

Although this expression is rather complicated, it becomes simple in two limiting cases: If $\vec{u}=0$, the velocity reduces to that of a rectilinear vortex at the origin

$$\vec{v}_0(\vec{r}) = (\kappa/4\pi)\int dz'\hat{z}\times\vec{r}[r^2+(z-z')^2]^{-3/2} = (\kappa/2\pi r^2)(\hat{z}\times\vec{r})\,. \tag{3}$$

If $\vec{u}$ is finite but constant, the total velocity is given by

$$\vec{v}(\vec{R})=\vec{v}_0(\vec{r}-\vec{u})=(\kappa/2\pi|\vec{r}-\vec{u}|^2)[\hat{z}\times(\vec{r}-\vec{u})]\,, \tag{4}$$

because the vortex is displaced with no bending. This result also follows directly from Eq. (2) with $d\vec{u}/dz'=0$. For $|\vec{r}|\gg|\vec{u}|$, Eq. (4) may be expanded in powers of $\vec{u}$, and the linear term provides a good description of the first far-field correction. Such an expansion clearly fails for $|\vec{r}|<|\vec{u}|$, however, because each term becomes singular at $\vec{r}=\vec{0}$, rather than at $\vec{r}=\vec{u}$.

When a quasiparticle is added to the system, it interacts with the vortex line both through the circulating velocity field and through the altered density near the vortex core. Unfortunately, any study of the effect of density variations requires a detailed theory of the vortex core. For this reason, we here consider only the scattering by the fluid velocity and assume that the interaction energy is given by[5]

$$H_{\text{int}} = \tfrac{1}{2}[\vec{P}\cdot\vec{v}(\vec{R})+\vec{v}(\vec{R})\cdot\vec{P}]\,, \tag{5}$$

where $\vec{v}(\vec{R})$ is the total velocity field of the vortex and $\vec{R}$ and $\vec{P}$ are the position and momentum of the quasiparticle. If the quasiparticle is scattered from an initial momentum $\hbar\vec{K}_i$ to a final momentum $\hbar\vec{K}_f$, then the corresponding matrix element of Eq. (5) becomes

$$H_{fi} = (2\Omega)^{-1}\hbar(\vec{K}_f+\vec{K}_i)\cdot\int d^3\vec{R}\,e^{-i(\vec{K}_f-\vec{K}_i)\cdot\vec{R}}\,\vec{v}(\vec{R})\,, \tag{6}$$

where the plane-wave states are normalized in a volume Ω. A combination of Eqs. (2) and (6) yields

$$H_{fi} = \frac{\hbar\kappa}{8\pi\Omega}(\vec{K}_f+\vec{K}_i)\cdot\int d^3R\int dz'e^{-i(\vec{K}_f-\vec{K}_i)\cdot\vec{R}}\left(\hat{z}+\frac{d\vec{u}(z')}{dz'}\right)\times\left(\frac{\vec{r}-\vec{u}(z')+\hat{z}(z-z')}{|\vec{r}-\vec{u}(z')+\hat{z}(z-z')|^3}\right), \tag{7}$$

which may be rewritten with a simple change of variables

$$H_{fi} = (\hbar\kappa/8\pi\Omega)(\vec{K}_f+\vec{K}_i)\cdot\int d^3R\int dz'[\hat{z}+d\vec{u}(z')/dz']\times\vec{R}R^{-3}\exp\{-i(\vec{K}_f-\vec{K}_i)\cdot[\vec{R}+\vec{u}(z')+z'\hat{z}]\}. \tag{8}$$

It is now possible to carry out the integral over $\vec{R}$ explicitly

$$\int d^3R\,e^{-i\vec{K}\cdot\vec{R}}\vec{R}R^{-3} = i\int d^3R\,R^{-3}\vec{\nabla}_{\vec{K}}e^{-i\vec{K}\cdot\vec{R}} = 2\pi i\int_0^\infty R^{-1}dR\,\vec{\nabla}_{\vec{K}}\int_{-1}^{1}d(\cos\Theta)\,e^{-iKR\cos\Theta}$$

$$= 2\pi i\int_0^\infty R^{-1}dR\,\vec{\nabla}_{\vec{K}}[2(KR)^{-1}\sin KR] = 4\pi i\hat{K}\int_0^\infty\frac{dR}{R}\frac{\partial}{\partial K}\left(\frac{\sin KR}{KR}\right) = -4\pi i\vec{K}K^{-2}\,. \tag{9}$$

Substitution of Eq. (9) into Eq. (8) gives

$$H_{fi} = \frac{\hbar\kappa i}{2\Omega}\,\frac{(\vec{K}_f+\vec{K}_i)\times(\vec{K}_f-\vec{K}_i)}{|\vec{K}_f-\vec{K}_i|^2}\cdot\int dz\left[\hat{z}+\frac{d\vec{u}(z)}{dz}\right]\exp[-i(k_f-k_i)z-i(\vec{l}_f-\vec{l}_i)\cdot\vec{u}(z)]\,, \tag{10}$$

where we have resolved the three-dimensional wave vector in cylindrical polar coordinates $\vec{K}=(\vec{l},k)$ and rearranged the scalar triple product. Only a single integration remains, and the superfluous prime has now been omitted.

To this point, $\vec{u}(z)$ has been treated as a given classical function with H_{fi} as the matrix element for a transition induced by the deformed vortex. We now quantize the theory by interpreting $\vec{u}(z)$ as an operator that acts on the internal states of the vortex; in this way, H_{fi} becomes an operator that causes transitions between the various excited internal states. The details of the quantization procedure derived in Ref. 13, and only the relevant results will be given here. If the system is assumed to obey periodic boundary conditions over a length L along the z axis, the oscillation modes of the vortex may be labeled by a one-dimensional set of quantum numbers $k=2\pi s/L$, where s is a positive or negative integer. These internal states are equivalent to a set of independent harmonic oscillators with the second-quantized Hamiltonian

$$H_v = \tfrac{1}{2}\sum_k \hbar\omega_k (a_k^\dagger a_k + a_k a_k^\dagger). \tag{11}$$

Here the operators $a_k^\dagger$ and a_k create and destroy one quantum in the kth mode, and they satisfy the usual boson commutation relations

$$[a_k, a_{k'}^\dagger] = \delta_{kk'}. \tag{12}$$

The frequency of the kth mode is given by

$$\omega_k = (\kappa k^2/4\pi)\ln(1/|k|a), \tag{13}$$

where a is a length characterizing the radius of the vortex core. Although the precise value of a depends on the model used for the core, vortices in He II are well described by the value $a \approx 1$ Å. The operators a and $a^\dagger$ enable us to express the displacement operator $\vec{u}(z)$ as an expansion in normal modes

$$\vec{u}(z) = (\hbar/2\rho\kappa L)^{1/2}\sum_k[(\hat{x}+i\hat{y})a_k^\dagger e^{-ikz} + (\hat{x}-i\hat{y})a_k e^{ikz}]. \tag{14}$$

It is interesting to note that this expression differs considerably from that arising in the theory of elastic waves[16]; its form reflects the peculiar Hamiltonian in the vortex system, where u_x and u_y themselves constitute the conjugate variables.

As an example of the utility of the second-quantized formalism, we compute the following correlation function of the displacements:

$$\langle|\vec{u}(z)-\vec{u}(z')|^2\rangle = \langle[\vec{u}(z)-\vec{u}(z')]\cdot[\vec{u}(z)-\vec{u}(z')]\rangle = \langle|\vec{u}(z)|^2 + |\vec{u}(z')|^2 - [\vec{u}(z)\cdot\vec{u}(z')+\vec{u}(z')\cdot\vec{u}(z)]\rangle. \tag{15}$$

Here the angular brackets denote an ensemble average at temperature $T=(k_B\beta)^{-1}$ over the states of the vortex

$$\langle\cdots\rangle = \mathrm{Tr}[\exp(-\beta H_v)\cdots]/\mathrm{Tr}[\exp(-\beta H_v)] = \mathrm{Tr}[\hat{\rho}_v\cdots], \tag{16}$$

where $\hat{\rho}_v = \exp(-\beta H_v)/\mathrm{Tr}[\exp(-\beta H_v)]$. (17)

The evaluation of Eq. (15) is straightforward and gives

$$\begin{aligned}\langle|\vec{u}(z)-\vec{u}(z')|^2\rangle &= (\hbar/\rho\kappa L)\sum_k(e^{ikz}-e^{ikz'})(e^{-ikz}-e^{-ikz'})\langle a_k^\dagger a_k + a_k a_k^\dagger\rangle \\ &= (2\hbar/\rho\kappa L)\sum_k \coth(\tfrac{1}{2}\beta\hbar\omega_k)[1-\cos k(z-z')].\end{aligned} \tag{18}$$

With parameters appropriate for He II ($\kappa \approx h/m \approx 10^{-3}$ cm^2 sec^{-1}, $a \approx 1$ Å, $T\approx 1$°K), it is easily seen that the classical limit is correct for all but the shortest wavelengths ($\hbar\omega_k \ll k_BT$ if $k \lesssim 10^7$ cm^{-1}). Equation (18) may then be evaluated approximately

$$\begin{aligned}\langle|\vec{u}(z)-\vec{u}(z')|^2\rangle &\approx (4/\beta\rho\kappa L)\sum_k\omega_k^{-1}[1-\cos k(z-z')] \approx \frac{16\pi}{\beta\rho\kappa^2 L}\sum_k \frac{1-\cos k(z-z')}{k^2\ln(1/|k|a)} \\ &\approx \frac{32\pi}{\beta\rho\kappa^2 L\ln(L/a)}\,\frac{L}{\pi}\int_0^\infty dk\,\frac{\sin^2[\frac{1}{2}k(z-z')]}{k^2} = 8\pi|z-z'|/\rho\kappa^2\beta\ln(L/a),\end{aligned} \tag{19}$$

where the weak logarithmic dependence on k has been neglected in the third line. The above calculation assumes that $|z-z'| \ll L$, because the integral approximation destroys the periodicity of Eq. (18) in $|z-z'|$; a more exact expression is obtained by evaluating the sum directly, but Eq. (19) suffices for our purposes.

The displacement operator [Eq. (14)] may now be substituted into Eq. (10) to give the transition operator H_{fi}, which we separate into two terms

$$H_{fi} = H_{fi}^{(1)} + H_{fi}^{(2)}, \tag{20}$$

where
$$H_{fi}^{(1)} = \frac{\hbar\kappa i}{2\Omega} \frac{(\vec{K}_f+\vec{K}_i)\times(\vec{K}_f-\vec{K}_i)\cdot\hat{z}}{|\vec{K}_f-\vec{K}_i|^2} \int dz\, e^{-i(k_f-k_i)z} e^{-i(\vec{1}_f-\vec{1}_i)\cdot\vec{u}(z)}, \tag{21}$$

$$H_{fi}^{(2)} = \frac{\hbar\kappa i}{2\Omega} \frac{(\vec{K}_f+\vec{K}_i)\times(\vec{K}_f-\vec{K}_i)}{|\vec{K}_f-\vec{K}_i|^2} \cdot \int dz\, e^{-i(k_f-k_i)z} \frac{1}{2}\left[e^{-i(\vec{1}_f-\vec{1}_i)\cdot\vec{u}(z)} \frac{d\vec{u}(z)}{dz} + \frac{d\vec{u}(z)}{dz} e^{-i(\vec{1}_f-\vec{1}_i)\cdot\vec{u}(z)}\right], \tag{22}$$

and Eq. (22) has been symmetrized in the noncommuting operators $\vec{u}$ and $d\vec{u}/dz$. These two terms are quite different, as can be seen from the following argument. In the limit $\vec{u}\to 0$, the term $H_{fi}^{(1)}$ contains the scattering by the undeformed vortex [Eq. (3)] while $H_{fi}^{(2)}$ vanishes. Furthermore, $H_{fi}^{(2)}$ involves the operator $d\vec{u}/dz$ and is thus insensitive to the long-wavelength modes; in particular, $H_{fi}^{(2)}$ vanishes if $\vec{u}$ reduces to a constant displacement, when $H_{fi}^{(1)}$ again contains the total scattering [Eq. (4)]. We noted previously that Eq. (4) cannot be expanded in powers of $\vec{u}$; for the same reason, Eq. (21) must be treated as given, *retaining terms of all orders in* $\vec{u}$. In contrast, Eq. (22) may be expanded in the displacements, neglecting all but the leading contribution. With this approximation, we obtain the transition operators

$$H_{fi}^{(1)} = V_1 \int dz\, e^{-i(k_f-k_i)z} e^{-i(\vec{1}_f-\vec{1}_i)\cdot\vec{u}(z)}, \tag{23}$$

$$H_{fi}^{(2)} = \sum_k \int dz\, e^{-i(k_f-k_i)z} (\hbar/2\rho\kappa L)^{1/2} [kV_2 e^{ikz} a_k - kV_2^* e^{-ikz} a_k^\dagger], \tag{24}$$

where
$$V_1 \equiv V_1(\vec{K}_f, \vec{K}_i) = (\hbar\kappa i/2\Omega)[(\vec{K}_f+\vec{K}_i)\times(\vec{K}_f-\vec{K}_i)\cdot\hat{z}/|\vec{K}_f-\vec{K}_i|^2], \tag{25}$$

$$V_2 \equiv V_2(\vec{K}_f, \vec{K}_i) = -(\hbar\kappa/2\Omega)[(\vec{K}_f+\vec{K}_i)\times(\vec{K}_f-\vec{K}_i)\cdot(\hat{x}-i\hat{y})/|\vec{K}_f-\vec{K}_i|^2]. \tag{26}$$

The physical difference between $H_{fi}^{(1)}$ and $H_{fi}^{(2)}$ can also be understood by noting that $\vec{u}(z)$ is linear in the creation and destruction operators. Consequently, an expansion in powers of $\vec{u}$ is permissible only if dominant scattering process corresponds to small changes in the occupation numbers of the various normal modes. In the long-wavelength limit, however, the energy $\hbar\omega_k$ per quantum [Eq. (13)] becomes very small, and the transition involves many "soft" quanta. Thus, any expansion based on small quantum numbers is bound to fail whenever long-wavelength modes are important, as in Eq. (21). The present situation is similar to the "infrared catastrophe" in electrodynamics,[17] where an expansion in the number of photons leads to divergences at long wavelengths.

III. TRANSITION AMPLITUDE

In Born approximation, the transition amplitude is proportional to the matrix element of the interaction Hamiltonian. Equations (23) and (24) already incorporate the initial and final states of the phonon, and we must now consider the internal states of the vortex. Since H_v represents an assembly of independent harmonic oscillators, the corresponding complete set of states may be expressed as a direct product over all normal modes. It is convenient to work in the occupation-number representation, but other choices are also possible.[18] If $\{n\}$ and $\{n'\}$ denote the set of all initial and final occupation numbers, then the particular transition $\{n\}\to\{n'\}$ is specified by the matrix element $\langle\{n'\}|H_{fi}|\{n\}\rangle$. Of the two terms in Eq. (20), the operator $H_{fi}^{(2)}$ can only alter a single occupation number, and we readily find

$$\langle\{n'\}|H_{fi}^{(2)}|\{n\}\rangle = \int dz\, e^{-i(k_f-k_i)z}(\hbar/2\rho\kappa L)^{1/2} \times \sum_k \{[\prod_{p\neq k} \delta_{n_p', n_p}][V_2 k e^{ikz} n_k^{1/2} \delta_{n_k', n_k-1} - V_2^* k e^{-ikz}(n_k+1)^{1/2}\delta_{n_k', n_k+1}]\}. \tag{27}$$

In contrast, $H_{fi}^{(1)}$ contains all powers of $\vec{u}$ and requires a more difficult calculation.

Equation (23) involves the operator $(\vec{1}_f - \vec{1}_i)\cdot\vec{u}(z)$, which may be rewritten

$$(\vec{1}_f-\vec{1}_i)\cdot\vec{u}(z) = (\hbar/2\rho\kappa L)^{1/2}\sum_k[(\vec{1}_f-\vec{1}_i)\cdot(\hat{x}-i\hat{y})e^{ikz}a_k + (\vec{1}_f-\vec{1}_i)\cdot(\hat{x}+i\hat{y})e^{-ikz}a_k^\dagger]$$

$$= \sum_k (\lambda e^{ikz} a_k + \lambda^* e^{-ikz} a_k^\dagger], \tag{28}$$

where $\lambda = (\hbar/2\rho\kappa L)^{1/2}(\vec{1}_f - \vec{1}_i)\cdot(\hat{x} - i\hat{y})$. (29)

Operators referring to different normal modes commute, and the exponential in Eq. (23), therefore, factors

$$\exp[-i(\vec{1}_f - \vec{1}_i)\cdot \vec{u}(z)] = \prod_k \exp[-i(\lambda e^{ikz} a_k + \lambda^* e^{-ikz} a_k^\dagger)]. \tag{30}$$

Furthermore, the eigenstates form a direct product, so that the transition amplitude becomes

$$\langle\{n'\}|H_{fi}^{(1)}|\{n\}\rangle = V_1 \int dz\, e^{-i(k_f - k_i)z} \prod_k \langle n_k'|\exp[-i(\lambda e^{ikz} a_k + \lambda^* e^{-ikz} a_k^\dagger)]|n_k\rangle \tag{31}$$

involving matrix elements of the form

$$M_{n'n}(\alpha, \beta) = \langle n'|e^{-i(\alpha a + \beta a^\dagger)}|n\rangle. \tag{32}$$

This quantity is studied in Appendix A, where it is shown that $M_{n'n}$ is proportional to a Laguerre polynomial.[19] In particular, Eq. (31) may be written

$$\langle\{n'\}|H_{fi}^{(1)}|\{n\}\rangle = V_1 \int dz\, e^{-i(k_f - k_i)z} \prod_k M_{n_k'n_k}(\lambda e^{ikz}, \lambda^* e^{-ikz}) \tag{33}$$

containing a product of Laguerre polynomials of argument

$$|\lambda e^{ikz}|^2 = |\lambda|^2 = (\hbar/2\rho\kappa L)|\vec{1}_f - \vec{1}_i|^2. \tag{34}$$

For an order of magnitude estimate, we may take $|\vec{1}_f - \vec{1}_i|$ as the thermal wave number of the phonon $\approx k_B T/\hbar c$; with typical numerical values for He II ($c \approx 238$ m sec^{-1}, $\rho \approx 0.145$ g cm^{-3}, $L \approx 10^{-4}$ cm), Eq. (34) becomes $|\lambda|^2 \approx \frac{1}{4}\times 10^{-6}$ at $T \approx \frac{1}{2}$ °K. Since $|\lambda|^2$ is very small, it is tempting to expand Eq. (31)

$$\begin{aligned}
\langle\{n'\}|H_{fi}^{(1)}|\{n\}\rangle &= V_1 \int dz\, e^{-i(k_f - k_i)z} \langle\{n'\}|1 - i\sum_k(\lambda e^{ikz} a_k + \lambda^* e^{-ikz} a_k^\dagger) + \dots |\{n\}\rangle = V_1 \int dz \\
&\times e^{-i(k_f - k_i)z}\{\prod_p \delta_{n_p'n_p} - i\sum_k[\prod_{p\neq k} \delta_{n_p'n_p}][\lambda e^{ikz} n_k^{1/2}\delta_{n_k', n_k - 1} + \lambda^* e^{-ikz}(n_k + 1)^{1/2}\delta_{n_k', n_k + 1}] + \dots\} \\
&= V_1 \prod_p \delta_{n_p'n_p} L\delta_{k_f k_i} - iV_1 \sum_k [\prod_{p\neq k} \delta_{n_p'n_p}][\lambda n_k^{1/2}\delta_{n_k', n_k - 1} L\delta_{k_f - k, k_i} \\
&+ \lambda^*(n_k + 1)^{1/2}\delta_{n_k', n_k + 1} L\delta_{k_f + k, k_i}] + \dots .
\end{aligned} \tag{35}$$

Here, the first term corresponds to elastic phonon scattering that leaves the vortex in its initial state, while the inelastic corrections are formally of order $|\lambda|$. We shall see below, however, that such an expansion leads to a long-wavelength divergence, thereby showing that the inelastic processes play an essential role.

In the present approximation, the probability per unit time for a particular scattering event $\vec{K}_i\{n\} \to \vec{K}_f\{n'\}$ is given by

$$2\pi\hbar^{-1}\delta(E_f - E_i)|\langle\{n'\}|H_{fi}|\{n\}\rangle|^2 = 2\pi\hbar^{-1}\delta(E_f - E_i)|\langle\{n'\}|H_{fi}^{(1)} + H_{fi}^{(2)}|\{n\}\rangle|^2, \tag{36}$$

where E_f and E_i are the final and initial total energies. An incident phonon at $\frac{1}{2}$ °K has a wave number $\approx k_B T/\hbar c \approx \frac{1}{3}\times 10^7$ cm^{-1}, which characterizes the maximum wave vector $|\vec{K}_f - \vec{K}_i|$ that can be transferred to the vortex. For elastic scattering, the energy of the phonon is conserved, so that $K_f = K_i$. For inelastic scattering, however, the change in the phonon energy must equal the net energy of the vortex wave quanta emitted or absorbed. It is convenient to consider separately the single-quantum processes and multiquantum processes. (i) If only a single quantum is involved, then the energy of the quantum must be

$$(\hbar\kappa/4\pi)(\vec{K}_f - \vec{K}_i)^2 \ln(1/|\vec{K}_f - \vec{K}_i|a) \lesssim (\hbar\kappa/4\pi)(k_B T/\hbar c)^2 \ln(\hbar c/k_B T a) \approx 3\times 10^{-18} \text{ erg},$$

which is much less then the thermal energy of the incident phonon ($\approx 5\times 10^{-17}$ erg). Since $H_{fi}{}^{(2)}$ is linear in the creation and destruction operators, it follows that any terms of Eq. (36) involving $H_{fi}{}^{(2)}$ may be simplified with the approximation $\delta(E_f - E_i) \to \delta(\hbar c K_f - \hbar c K_i)$. (ii) Multiquantum processes are considerably more complicated, because the exact conservation of momentum and energy are expressed by the equations

$$k_f - k_i = \sum_k (n_k - n_k')k,$$

$$\hbar c K_f - \hbar c K_i = \sum_k \hbar\omega_k (n_k - n_k') = \sum_k (n_k - n_k')(\hbar\kappa k^2/4\pi)\ln(1/|k|a). \tag{37}$$

Since k can be either positive or negative, there are always processes for which the change in the phonon's energy $\hbar c(K_f - K_i)$ becomes comparable with the initial energy $\hbar c K_i$. As a result, the energy of the vortex waves must be included explicitly in the term of Eq. (36) that involves $|\langle\{n'\}|H_{fi}{}^{(1)}|\{n\}\rangle|^2$. This calculation is carried out in Appendix B; the probability of a particular energy transfer turns out to be distributed about $K_f = K_i$ with a width $\hbar\omega_q$, where q is a characteristic wave number $q \approx 2\pi|\vec{1}_f - \vec{1}_i|^2/\rho\kappa^2\beta\ln \times (L/a) \lesssim 10^4$ cm^{-1}. For all cases of interest, $\hbar\omega_q \lesssim 10^{-22}$ erg is much less than $k_B T$, and the physically important inelastic processes transfer a negligible amount of energy. The important conclusion is that the right-hand side of Eq. (36) may be approximated as

$$2\pi\hbar^{-2}c^{-1}\delta(K_f - K_i)|\langle\{n'\}|H_{fi}|\{n\}\rangle|^2, \tag{38}$$

even when all inelastic contributions are included. This result is understandable because the dominant multiquantum processes involve "soft" quanta with small wave numbers. The precise form of the associated energy $[\propto k^2\ln(1/|k|a)]$ then guarantees that the total energy is also small.

In any practical experiment, the final state of the vortex is not observed directly, and the physically interesting quantity is obtained by summing Eq. (38) over all quantum numbers $\{n'\}$

$$2\pi\hbar^{-2}c^{-1}\delta(K_f - K_i)\sum_{\{n'\}}|\langle\{n'\}|\,|H_{fi}|\{n\}\rangle|^2 = 2\pi\hbar^{-2}c^{-1}\delta(K_f - K_i)\langle\{n\}|H_{fi}{}^\dagger H_{fi}|\{n\}\rangle, \tag{39}$$

where the second line follows from the completeness of the oscillator states. Although the matrix element in Eq. (39) can be evaluated in its present form, it is simpler to realize that the measured cross section represents an ensemble average over the initial states of the vortex. We, therefore, compute

$$\begin{aligned}&2\pi\hbar^{-2}c^{-1}\delta(K_f - K_i)\sum_{\{n\}}\langle\{n\}|\hat{\rho}_v|\{n\}\rangle\langle\{n\}|H_{fi}{}^\dagger H_{fi}|\{n\}\rangle\\ &\quad= 2\pi\hbar^{-2}c^{-1}\delta(K_f - K_i)\mathrm{Tr}[\hat{\rho}_v H_{fi}{}^\dagger H_{fi}] = 2\pi\hbar^{-2}c^{-1}\delta(K_f - K_i)\langle H_{fi}{}^\dagger H_{fi}\rangle,\end{aligned} \tag{40}$$

where the statistical operator $\hat{\rho}_v$ is given in Eq. (17).

The operator H_{fi} is a sum of two terms, and Eq. (40) thus contains three different contributions

$$\langle|H_{fi}{}^{(1)}|^2\rangle + \langle|H_{fi}{}^{(2)}|^2\rangle + \langle H_{fi}{}^{(1)\dagger}H_{fi}{}^{(2)} + H_{fi}{}^{(2)\dagger}H_{fi}(1)\rangle.$$

The first term is the most difficult and will now be evaluated in detail

$$\begin{aligned}\langle|H_{fi}{}^{(1)}|^2\rangle &= |V_1|^2\int dz\,dz'\,e^{-i(k_f - k_i)(z-z')}\\ &\quad\times\prod_k\langle\exp[i(\lambda e^{ikz'}a_k + \lambda^* e^{-ikz'}a_k{}^\dagger]\exp[-i(\lambda e^{ikz}a_k + \lambda^* e^{-ikz}a_k{}^\dagger)]\rangle.\end{aligned} \tag{41}$$

This expression can be simplified with the identity[20]

$$e^A e^B = e^{A+B}\exp(\tfrac{1}{2}[A,B]), \tag{42}$$

which is valid whenever $[A, B]$ commutes with A and B. A simple calculation gives

$$\begin{aligned}\langle|H_{fi}{}^{(1)}|^2\rangle &= |V_1|^2\int dz\,dz'e^{-i(k_f-k_i)(z-z')}\prod_k\{\exp[-i|\lambda|^2\sin k(z-z')]\,\langle\exp[-i\lambda(e^{ikz} - e^{ikz'})a_k - i\lambda^*(e^{-ikz}\\ &\quad - e^{-ikz'})a_k{}^\dagger]\rangle\} = |V_1|^2\int dz\,dz'\,e^{-i(k_f-k_i)(z-z')}\prod_k\langle\exp[-i\lambda(e^{ikz} - e^{ikz'})a_k - i\lambda^*(e^{-ikz} - e^{-ikz'})a_k{}^\dagger]\rangle,\end{aligned} \tag{43}$$

where the phase factor cancels identically in the product over positive and negative values of k. The final ensemble average may be evaluated with a theorem of Bloch,[21] or with Eqs. (A16) and (A17)

$$\langle|H_{fi}^{(1)}|^2\rangle = |V_1|^2\int dz\,dz'e^{-i(k_f-k_i)(z-z')}\Pi_k \exp\{-|\lambda|^2[1-\cos k(z-z')]\coth(\tfrac{1}{2}\beta\hbar\omega_k)\}$$

$$= |V_1|^2\int dz\,dz'e^{-i(k_f-k_i)(z-z')}\exp\{-|\lambda|^2\textstyle\sum_k \coth(\tfrac{1}{2}\beta\hbar\omega_k)[1-\cos k(z-z')]\}$$

$$= |V_1|^2\int dz\,dz'e^{-i(k_f-k_i)(z-z')}\exp\{-\tfrac{1}{4}|\vec{1}_f-\vec{1}_i|^2\langle|\vec{u}(z)-\vec{u}(z')|^2\rangle\}. \tag{44}$$

Here, the last line has been rewritten with Eqs. (18) and (34). The summation over k has already been computed in Eq. (19), and we finally obtain

$$\langle|H_{fi}^{(1)}|^2\rangle = |V_1|^2\int dz\,dz'e^{-i(k_f-k_i)(z-z')}e^{-q|z-z'|} = 2L|V_1|^2q[q^2+(k_f-k_i)^2]^{-1}, \tag{45}$$

where $$q = 2\pi|\vec{1}_f-\vec{1}_i|^2/\rho\kappa^2\beta\ln(L/a) \tag{46}$$

is the characteristic wave number for axial momentum transfer. In the limit $|\vec{1}_f-\vec{1}_i|\to 0$, the right-hand side of Eq. (45) is sharply peaked about $k_f\approx k_i$, which reflects the dominant role of the elastic processes; nevertheless, it is clearly wrong to expand Eq. (45) in powers of $|\vec{1}_f-\vec{1}_i|^2$ because the denominator would then vanish for $k_f=k_i$. This same long-wavelength divergence would occur if $H_{fi}^{(1)}$ were expanded as in Eq. (35). It is evident from Eq. (44) that inelastic effects reduce the matrix element $\langle|H_{fi}^{(1)}|^2\rangle$ from its value $2\pi L|V_1|^2\delta(k_f-k_i)$ for purely elastic scattering.

The remaining contributions to $\langle|H_{fi}|^2\rangle$ are easily computed. Consider first $\langle|H_{fi}^{(2)}|^2\rangle$ which follows from Eq. (24):

$$\langle H_{fi}^{(2)\dagger}H_{fi}^{(2)}\rangle = (\hbar/2\rho\kappa L)\int dz\,dz'e^{-i(k_f-k_i)(z-z')}$$

$$\times\textstyle\sum_{pp'}pp'\langle[V_2^*e^{-ip'z'}a_{p'}^\dagger - V_2e^{ip'z'}a_{p'}][V_2e^{ipz}a_p - V_2^*e^{-ipz}a_p^\dagger]\rangle$$

$$= (\hbar/2\rho\kappa L)\int dz\,dz'e^{-i(k_f-k_i)(z-z')}\textstyle\sum_p p^2|V_2|^2[\langle a_p^\dagger a_p\rangle e^{ip(z-z')}+\langle a_pa_p^\dagger\rangle e^{-ip(z-z')}]$$

$$= (L\hbar/2\rho\kappa)|V_2|^2(k_f-k_i)^2\coth(\tfrac{1}{2}\beta\hbar\omega_{fi}). \tag{47}$$

Here ω_{fi} denotes the frequency of the vortex wave with wave number $|k_f-k_i|$. In a similar way, the cross terms between $H_{fi}^{(1)}$ and $H_{fi}^{(2)}$ become

$$\langle H_{fi}^{(1)\dagger}H_{fi}^{(2)}+H_{fi}^{(2)\dagger}H_{fi}^{(1)}\rangle = 2\,\mathrm{Re}\langle H_{fi}^{(2)\dagger}H_{fi}^{(1)}\rangle$$

$$= 2\mathrm{Re}\{V_1\int dz\,dz'e^{-i(k_f-k_i)(z-z')}(\hbar/2\rho\kappa L)^{1/2}\textstyle\sum_p p[V_2^*e^{-ipz'}\langle a_p^\dagger e^{-i(\vec{1}_f-\vec{1}_i)\cdot\vec{u}(z)}\rangle$$

$$- V_2e^{ipz'}\langle a_pe^{-i(\vec{1}_f-\vec{1}_i)\cdot\vec{u}(z)}\rangle]\}. \tag{48}$$

The ensemble averages may be computed with Eqs. (A16) and (A17)

$$\langle a_p^\dagger e^{-i(\vec{1}_f-\vec{1}_i)\cdot\vec{u}(z)}\rangle = \langle a_p^\dagger\Pi_k\exp[-i(\lambda e^{ikz}a_k+\lambda^*e^{-ikz}a_k^\dagger)]\rangle$$

$$= \langle a_p^\dagger\exp[-i(\lambda e^{ipz}a_p+\lambda^*e^{-ipz}a_p^\dagger)]\rangle\prod_{k\neq p}\langle\exp[-i(\lambda e^{ikz}a_k+\lambda^*e^{-ikz}a_k^\dagger)]\rangle$$

$$= [(-i\lambda e^{ipz})/(e^{\beta\hbar\omega_p}-1)]\Pi_k\exp[-\tfrac{1}{2}|\lambda|^2\coth(\tfrac{1}{2}\beta\hbar\omega_k)], \tag{49a}$$

$$\langle a_pe^{-i(\vec{1}_f-\vec{1}_i)\cdot\vec{u}(z|}\rangle = [(-i\lambda^*e^{-ipz})/(1-e^{-\beta\hbar\omega_p})]\Pi_k\exp[-\tfrac{1}{2}|\lambda|^2\coth(\tfrac{1}{2}\beta\hbar\omega_k)]. \tag{49b}$$

As shown below, the final cross section converges even if Eq. (49) is expanded to first order in λ, and we find

$$\langle H_{fi}^{(1)\dagger} H_{fi}^{(2)} + H_{fi}^{(2)\dagger} H_{fi}^{(1)} \rangle \approx 2\,\mathrm{Re}\{V_1 \int dz\, dz'\, e^{-i(k_f-k_i)(z-z')} (\hbar/2\rho\kappa L)^{1/2}$$

$$\times \sum_p [-ip(V_2^* \lambda\, e^{ip(z-z')}/(e^{\beta\hbar\omega_p}-1) - V_2\lambda^* e^{-ip(z-z')}/(1-e^{-\beta\hbar\omega_p}))]\}$$

$$= 2\,\mathrm{Re}[-iV_1(\hbar/2\rho\kappa L)^{1/2} L^2 (k_f-k_i)(V_2^*\lambda/(e^{\beta\hbar\omega_{fi}}-1) + V_2\lambda^*/(1-e^{-\beta\hbar\omega_{fi}}))]$$

$$= -2(L^2/\beta\hbar\omega_{fi})(\hbar/2\rho\kappa L)^{1/2}(k_f-k_i)\,\mathrm{Re}[iV_1(V_2^*\lambda + V_2\lambda^*)], \tag{50}$$

where the condition $\hbar\omega_{fi} \ll k_B T$ has been used in the last line. Equation (25) shows that iV_1 is real, while the real part of $V_2^*\lambda$ is given by

$$\mathrm{Re}V_2^*\lambda = -\frac{\hbar\kappa}{2\Omega}\frac{(\hbar/2\rho\kappa L)^{1/2}}{|\vec{K}_f-\vec{K}_i|^2}\mathrm{Re}\{(\vec{l}_f-\vec{l}_i)\cdot(\hat{x}-i\hat{y})\,[(k_f+k_i)\hat{z}\times(\vec{l}_f-\vec{l}_i)\cdot(\hat{x}+i\hat{y}) + (k_f-k_i)(\vec{l}_f+\vec{l}_i)$$

$$\times \hat{z}\cdot(\hat{x}+i\hat{y})]\} = -\frac{\hbar\kappa}{2\Omega}\frac{(\hbar/2\rho\kappa L)^{1/2}}{|\vec{K}_f-\vec{K}_i|^2}\mathrm{Re}\{(\vec{l}_f-\vec{l}_i)\cdot(\hat{x}-i\hat{y})$$

$$\times i[(k_f+k_i)(\vec{l}_f-\vec{l}_i)\cdot(\hat{x}+i\hat{y}) + (k_f-k_i)(\vec{l}_f+\vec{l}_i)\cdot(\hat{x}+i\hat{y})]\}$$

$$= -\frac{\hbar\kappa}{2\Omega}\frac{(\hbar/2\rho\kappa L)^{1/2}}{|\vec{K}_f-\vec{K}_i|^2}[(\vec{l}_f+\vec{l}_i)\times(\vec{l}_f-\vec{l}_i)\cdot\hat{z}]\,(k_f-k_i) = i(\hbar/2\rho\kappa L)^{1/2}(k_f-k_i)V_1\,. \tag{51}$$

In this way, we find

$$\langle H_{fi}^{(2)\dagger} H_{fi}^{(1)} + H_{fi}^{(1)\dagger} H_{fi}^{(2)} \rangle = [2L(k_f-k_i)^2/\beta\rho\kappa^2\omega_{fi}]\,|V_1|^2 \approx 8\pi L|V_1|^2/\rho\kappa^2\beta\ln(L/a), \tag{52}$$

where the logarithmic factor has again been approximated as $\ln(L/a)$. The sum of Eqs. (45), (47), and (52) finally gives

$$\langle H_{fi}^{\dagger} H_{fi}\rangle \simeq 2L|V_1|^2 q[q^2+(k_f-k_i)^2]^{-1} + [4\pi L/\rho\kappa^2\beta\ln(L/a)](2|V_1|^2+|V_2|^2), \tag{53}$$

which includes both elastic and inelastic processes. The last two terms remain finite as $k_f \to k_i$, thereby justifying our approximate treatment of $H_{fi}^{(2)}$ in Eq. (24). It is possible to repeat the above calculation with the more exact expression Eq. (22), but the additional corrections are negligible.

IV. PHONON DRAG FORCE ON A VORTEX

In Sec. III, the transition probability was summed over the final states of the vortex and averaged over the corresponding initial states. It is also necessary to sum over the final phonon states, which are usually not detected; hence the total transition rate becomes

$$2\pi\hbar^{-2}c^{-1}\sum_{\vec{l}_f}\sum_{k_f}\delta(K_f-K_i)\,\langle|H_{fi}|^2\rangle = \Omega(2\pi\hbar)^{-2}c^{-1}\int l_f\,dl_f\,d\chi_f\,dk_f\,\delta(K_f-K_i)\langle|H_{fi}|^2\rangle$$

$$= \Omega K_i(2\pi\hbar)^{-2}c^{-1}\int dk_f\langle|H_{fi}|^2\rangle\Big|_{K_f=K_i} d\chi_f \equiv \langle w\rangle d\chi_f L\,, \tag{54}$$

where $\langle w\rangle$ is the rate of transitions per unit length of vortex line into an angular interval between χ_f and $\chi_f+d\chi_f$. The incident flux of phonons on the vortex is given by $F = cl_i/\Omega K_i$, and the differential cross section for scattering of a phonon with incident wave vector K_i reduces to

$$\left\langle\frac{d\sigma}{d\chi_f}\right\rangle = \frac{\langle w\rangle}{F} = \left(\frac{\Omega K_i}{2\pi\hbar c}\right)^2 \frac{1}{Ll_i}\int_{-K_i}^{K_i} dk_f\langle|H_{fi}|^2\rangle\Big|_{K_f=K_i}, \tag{55}$$

where $l_f^2 = K_i^2 - k_f^2$ is determined by energy conservation and the angular brackets again denote an ensemble average over the initial states of the vortex.

We now examine the various terms of Eq. (53). It can be seen from Eqs. (25) and (26) that $|V_1|^2$ and $|V_2|^2$ are comparable in magnitude. Furthermore, they both have the same denominator $|\vec{K}_f - \vec{K}_i|^4 = [(k_f - k_i)^2 + (\vec{l}_f - \vec{l}_i)^2]^2$ and are peaked functions of $|k_f - k_i|$ with a natural width $|\vec{l}_f - \vec{l}_i|$. The additional factor in the first term of Eq. (53) arises from the inclusion of inelastic effects; it is also peaked as a function of $|k_f - k_i|$, but its width is given by $q = 2\pi|\vec{l}_f - \vec{l}_i|^2/\rho\kappa^2\beta\ln(L/a)$. The ratio of these quantities is

$$q/|\vec{l}_f - \vec{l}_i| = 2\pi|\vec{l}_f - \vec{l}_i|/\rho\kappa^2\beta\ln(L/a) \lesssim 2\pi(k_BT)^2/\hbar c\rho\kappa^2\ln(L/a) \approx 10^{-3}, \tag{56}$$

because $|\vec{l}_f - \vec{l}_i| \lesssim K_i \approx k_BT/\hbar c$. We see that q is much smaller than $|\vec{l}_f - \vec{l}_i|$; hence the slowly varying factor $|V_1|^2$ in the first term of Eq. (53) may be evaluated at $k_f = k_i$, while the second factor is approximately a Dirac δ function $\pi\delta(k_f - k_i)$.

The differential cross section may now be found explicitly from Eqs. (53), (55), and (56)

$$\left\langle \frac{d\sigma}{d\chi_f} \right\rangle = \left(\frac{\Omega K_i}{2\pi\hbar c}\right)^2 \frac{1}{l_i} \int_{-K_i}^{K_i} dk_f \{2|V_1|^2[\pi\delta(k_f - k_i) + 4\pi/\rho\kappa^2\beta\ln(L/a)] + 4\pi|V_2|^2/\rho\kappa^2\beta\ln(L/a)\}\Big|_{K_f=K_i}$$

$$= \left(\frac{\Omega K_i}{2\pi\hbar c}\right)^2 \frac{1}{l_i}\left[2\pi|V_1|^2\Big|_{k_f=k_i} + \frac{4\pi}{\rho\kappa^2\beta\ln(L/a)}\int_{-K_i}^{K_i} dk_f(2|V_1|^2 + |V_2|^2)\right]_{K_f=K_i}. \tag{57}$$

Here it is essential to remember that l_f^2 is equal to $K_i^2 - k_f^2$, so that the integrand in the last term of Eq. (57) is a very complicated function of k_f. Fortunately, we need not evaluate the integral explicitly, because it is of order

$$[8\pi K_i/\rho\kappa^2\beta\ln(L/a)](2|V_1|^2 + |V_2|^2) \approx 4\times 10^{-3}(2|V_1|^2 + |V_2|^2), \tag{58}$$

and hence negligible relative to the first term of Eq. (57). Consequently, the final differential cross section is given by

$$\langle d\sigma/d\chi_f\rangle = (2\pi/l_i)(\Omega K_i/2\pi\hbar c)^2(|V_1|^2)_{K_f=K_i,\,k_f=k_i}[1 + O[k_BTK_i/\rho\kappa^2\ln(L/a)]]$$

$$\approx \frac{\pi}{2}\left(\frac{\kappa}{2\pi c}\right)^2 \frac{K_i^2}{l_i} \frac{|(\vec{l}_f + \vec{l}_i)\times(\vec{l}_f - \vec{l}_i)\cdot\hat{z}|^2}{|\vec{l}_f - \vec{l}_i|^4}\Bigg|_{l_f=l_i} = \frac{\pi}{2}\left(\frac{\kappa}{2\pi c}\right)^2 \frac{K_i^2}{l_i}\cot^2\tfrac{1}{2}(\chi_f - \chi_i), \tag{59}$$

which reduces to that obtained previously for elastic scattering of a phonon in the xy plane.[6–8] We see that the inclusion of inelastic processes to all orders alters the differential cross section only by a small correction of order $\approx(k_BT)^2/\hbar c\rho\kappa^2\ln(L/a) \ll 1$.

When a vortex moves through the fluid, it experiences a retarding force owing to collisions with the thermally excited quasiparticles. For low velocities, this frictional force $\mathfrak{F}'$ per unit length is proportional to the translation velocity v and is given by[4]

$$\mathfrak{F}' = (2\pi\hbar)^{-3}v\int d^3P_i(\partial f_0/\partial E_i)cP_i^2\sin^2\Theta_i\cos\chi_i\int_{-\pi}^{\pi} d\chi_f\langle d\sigma/d\chi_f\rangle(\cos\chi_f - \cos\chi_i), \tag{60}$$

where $\vec{P}_i = (P_i, \Theta_i, \chi_i)$ is the incident momentum of the phonon in spherical polar coordinates and f_0 is the stationary equilibrium distribution function $f_0(E_i) = [\exp(\beta E_i) - 1]^{-1}$. The integration over χ_f is expressible in terms of the "transport" cross section

$$\langle\sigma^*\rangle = \int_{-\pi}^{\pi} d\chi_f[1 - \cos(\chi_f - \chi_i)]\langle d\sigma/d\chi_f\rangle, \tag{61}$$

and a combination with Eq. (60) yields

$$\mathfrak{F}' = cv(2\pi\hbar)^{-3}\pi\int_0^\infty dP_i(-\partial f_0/\partial E_i)P_i^4\int_0^\pi d\Theta_i\sin^4\Theta_i\langle\sigma^*\rangle. \tag{62}$$

The cross section contains elastic and inelastic scattering, both of which contribute to the frictional force. Since the inelastic corrections are negligible, however, the elastic scattering is dominant, and an easy integration with Eq. (59) leads to

$$\langle\sigma^*\rangle = (\kappa/2c)^2 P_i(\hbar \sin\Theta_i)^{-1}. \tag{63}$$

In this way, the phonon drag force per unit length becomes

$$\mathfrak{F}'_p = (\kappa^2 v/32\pi^2\hbar^4c^2)\int_0^\infty dP\,P^5(-\partial f_0/\partial P)\int_0^\pi d\Theta \sin^3\Theta = 5(\kappa^2 v/24\pi^2\hbar^4 c^2)$$
$$\times \int_0^\infty dP\,P^4[\exp(\beta c P)-1]^{-1} = (\kappa^2 v/24\pi^2\hbar^4c^7)(k_B T)^5 5!\,\zeta(5), \tag{64}$$

where $\zeta(5)=1.037\ldots$ is the Riemann ζ function. The experimentally useful quantity is the energy lost by a vortex ring in traveling 1 cm,[4] denoted by

$$\alpha_p = \tfrac{1}{2}\kappa \mathfrak{F}'_p v^{-1} = \kappa^3 \hbar c^{-2} 5\zeta(5)(2\pi^2)^{-1}(k_B T/\hbar c)^5 \approx 1.6T^5 \text{ eV/cm}, \tag{65}$$

where T is in °K. This expression is slightly larger than that given in Ref. 4 because the differential cross section [Eq. (59)] depends on Θ_i, but the numerical difference is negligible.

The above calculation demonstrates how to include inelastic scattering effects associated with internal vibration modes of the vortex. In the particular case of phonon scattering, such inelasticity is negligible because the mean phonon momentum is determined by the relation $k_B T \approx \hbar c K$. As a result, the maximum wave number $|\vec{1}_f - \vec{1}_i|$ transferred to the vortex is much smaller than the quantity $(\rho\kappa^2/k_B T)\ln(L/a)$ in all situations of experimental interest. A more favorable case occurs for rotons, where the typical wave number is 2×10^8 cm^{-1}, independent of temperature. Thus, the momentum transferred by a roton can be considerably larger than for a phonon, and, correspondingly, inelastic effects should be more important.[8,9] Unfortunately, a theoretical study of roton scattering is also more difficult than that discussed here, because the roton-vortex interaction is too strong to use the Born approximation. Nevertheless, the present quantum-mechanical analysis of vortex waves should provide a suitable basis for such a calculation, which would be of great experimental interest.

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APPENDIX A

We here prove several identities that were used in Sec. II. Let $|n\rangle$ be a normalized eigenstate of the number operator $a^\dagger a$; it then follows directly from the commutation relation

$$[a, a^\dagger] = 1 \tag{A1}$$

that $a|n\rangle = n^{1/2}|n-1\rangle$,

$$a^\dagger|n\rangle = (n+1)^{1/2}|n+1\rangle. \tag{A2}$$

Consider the quantity

$$M_{nn'}(\alpha,\beta) = \langle n|e^{-i(\alpha a+\beta a^\dagger)}|n'\rangle, \tag{A3}$$

where α and β are arbitrary complex numbers. Direct calculation shows that M has the following generalized Hermitian property

$$[M_{nn'}(\alpha,\beta)]^* = M_{n'n}(-\beta^*, -\alpha^*). \tag{A4}$$

The evaluation of Eq. (A3) can be simplified with the identity[20]

$$e^{A+B} = e^A e^B \exp(-\tfrac{1}{2}[A,B]) \tag{A5}$$

valid whenever $[A,B]$ commutes with both A and B. With the identification $A=-i\beta a^\dagger$, $B=-i\alpha a$, we find

$$M_{nn'}(\alpha,\beta) = \exp(-\tfrac{1}{2}\alpha\beta)\langle n|e^{-i\beta a^\dagger} e^{-i\alpha a}|n'\rangle. \tag{A6}$$

Expand the exponentials in power series

$$M_{nn'}(\alpha,\beta) = \exp(-\tfrac{1}{2}\alpha\beta)\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{(-i\beta)^s(-i\alpha)^t}{s!t!}\langle n|(a^\dagger)^s(a)^t|n'\rangle$$

$$= \exp(-\tfrac{1}{2}\alpha\beta)\sum_{s=0}^{n}\sum_{t=0}^{n'}\frac{(-i\beta)^s(-i\alpha)^t(n!n'!)^{1/2}}{s!t![(n-s)!(n'-t)!]^{1/2}}\langle n-s|n'-t\rangle$$

$$= \exp(-\tfrac{1}{2}\alpha\beta)(n!n'!)^{1/2}\sum_{s=0}^{n}\sum_{t=0}^{n'}\frac{(-i\beta)^s(-i\alpha)^t}{s!t!(n-s)!}\delta_{s-n,\,t-n'}, \tag{A7}$$

where the second line has been obtained with Eq. (A2). The remaining analysis depends on the sign of $n-n'$; for definiteness, we take $n'\geq n$. The Kronecker δ then reduces Eq. (A7) to

$$M_{nn'}(\alpha,\beta) = \exp(-\tfrac{1}{2}\alpha\beta)(n!n'!)^{1/2}(-i\alpha)^{n'-n}\sum_{s=0}^{n}\frac{(-\alpha\beta)^s}{s!(n-s)!(s+n'-n)!}, \tag{A8}$$

which is expressible as a Laguerre polynomial. With the definition[22]

$$L_p^{k}(z) = \sum_{s=0}^{p}\frac{(-z)^s[(p+k)!]^2}{s!(p-s)!(s+k)!}, \tag{A9}$$

Eq. (A8) becomes

$$M_{nn'}(\alpha,\beta) = \exp(-\tfrac{1}{2}\alpha\beta)(n'!)^{-3/2}(n!)^{1/2}(-i\alpha)^{n'-n}L_n^{n'-n}(\alpha\beta), \quad n'\geq n . \tag{A10}$$

A similar calculation for $n\geq n'$ yields

$$M_{nn'}(\alpha,\beta) = \exp(-\tfrac{1}{2}\alpha\beta)(n!)^{-3/2}(n'!)^{1/2}(-i\beta)^{n-n'}L_{n'}^{n-n'}(\alpha\beta), \quad n\geq n' . \tag{A11}$$

The quantities M obey a simple combination law that is readily derived as follows:

$$\sum_{p=0}^{\infty} M_{mp}(\alpha,\beta)M_{pn}(\gamma,\delta) = \sum_{p=0}^{\infty}\langle m|e^{-i(\alpha a+\beta a^\dagger)}|p\rangle\langle p|e^{-i(\gamma a+\delta a^\dagger)}|n\rangle$$

$$= \langle m|e^{-i(\alpha a+\beta a^\dagger)}e^{-i(\gamma a+\delta a^\dagger)}|n\rangle, \tag{A12}$$

where the last line follows from the completeness of the states $|p\rangle$. The identity (A5) again allows us to simplify the analysis, and a straightforward calculation yields

$$\sum_{p=0}^{\infty} M_{mp}(\alpha,\beta)M_{pn}(\gamma,\delta) = \exp[-\tfrac{1}{2}(\alpha\delta-\beta\gamma)]M_{mn}(\alpha+\gamma,\beta+\delta) \tag{A13}$$

valid for arbitrary complex values of α, β, γ, or δ. This result can be extended by induction to include three or more factors on the left-hand side; when combined with Eqs. (A10) and (A11), it provides an addition formula for Laguerre polynomials involving both subscripts and superscripts.

The Laguerre polynomials may be derived from a generating function[22]

$$\sum_{p=0}^{\infty}\frac{t^p}{(p+k)!}L_p^{k}(z) = \frac{\exp[-zt(1-t)^{-1}]}{(1-t)^{1+k}} \tag{A14}$$

valid for $|t|<1$. This expression allows us to derive a generalization of Bloch's theorem on the probability distribution for harmonic oscillators.[21] Consider the quantity

$$\sum_{p=0}^{\infty} t^p\langle p|a^n e^{-i(\lambda a+\mu a^\dagger)}|p\rangle = \sum_{p=0}^{\infty} t^p\left[\frac{(p+n)!}{p!}\right]^{1/2}\langle p+n|e^{-i(\lambda a+\mu a^\dagger)}|p\rangle$$

$$\sum_{p=0}^{\infty} t^p\left[\frac{(p+n)!}{p!}\right]^{1/2} M_{p+n,p}(\lambda,\mu) = \exp(-\tfrac{1}{2}\lambda\mu)(-i\mu)^n \sum_{p=0}^{\infty} t^p L_p{}^n(\lambda\mu)[(p+n)!]^{-1}$$

$$= (-i\mu)^n \exp[-\tfrac{1}{2}\lambda\mu(1+t)(1-t)^{-1}]/(1-t)^{n+1}. \qquad \text{(A15)}$$

If t is taken as $\exp(-\beta\hbar\omega)$, then a simple calculation with Eqs. (17) and (A15) gives

$$\mathrm{Tr}\{\hat{\rho}_v a^n e^{-i(\lambda a+\mu a^\dagger)}\} \equiv \langle a^n e^{-i(\lambda a+\mu a^\dagger)}\rangle = (-i\mu)^n \exp[-\tfrac{1}{2}\lambda\mu\coth(\tfrac{1}{2}\beta\hbar\omega)]/(1-e^{-\beta\hbar\omega})^n. \qquad \text{(A16)}$$

In a similar way, we find

$$\langle (a^\dagger)^n e^{-i(\lambda a+\mu a^\dagger)}\rangle = (-i\lambda)^n \exp[-\tfrac{1}{2}\lambda\mu\coth(\tfrac{1}{2}\beta\hbar\omega)]/(e^{\beta\hbar\omega}-1)^n \qquad \text{(A17)}$$

both of which were used in evaluating the ensemble averages in Sec. II.

APPENDIX B

In Sec. II, the average transition probability was evaluated by assuming that the phonon transfers negligible energy to the vortex [Eq. (40)]. Although this approximation is clearly valid for inelastic processes involving a single quantum of the internal oscillation modes, it requires a detailed justification for the multiquantum processes contained in $H_{fi}{}^{(1)}$. In particular, we consider the probability $P_{fi}{}^{(1)}$ per unit time for a scattering event $\vec{K}_i \to \vec{K}_f$, summed over all final internal states $\{n'\}$ and averaged over all initial internal states $\{n\}$

$$P_{fi}{}^{(1)} = 2\pi\hbar^{-1} \sum_{\{n\}} \sum_{\{n'\}} \delta(E_f - E_i)\langle\{n\}|\hat{\rho}_v|\{n\}\rangle\,|\langle\{n'\}|H_{fi}{}^{(1)}|\{n\}\rangle|^2, \qquad \text{(B1)}$$

where the energy $E_f - E_i$ is given by Eq. (37)

$$E_f - E_i = \hbar c(K_f - K_i) + \sum_k \hbar\omega_k(n_k' - n_k). \qquad \text{(B2)}$$

The energy-conserving δ function may be rewritten

$$2\pi\delta(E_f - E_i) = \hbar^{-1}\int_{-\infty}^{\infty} dt\, \exp[i(E_f - E_i)t/\hbar], \qquad \text{(B3)}$$

and Eq. (B1) becomes

$$P_{fi}{}^{(1)} = \hbar^{-2}\int_{-\infty}^{\infty} dt\, e^{ict(K_f-K_i)} \sum_{\{n\}} \sum_{\{n'\}} \exp[i\textstyle\sum_k \omega_k t(n_k'-n_k)]$$

$$\times \langle\{n\}|\hat{\rho}_v|\{n\}\rangle\langle\{n\}|H_{fi}{}^{(1)\dagger}|\{n'\}\rangle\langle\{n'\}|H_{fi}{}^{(1)}|\{n\}\rangle = \hbar^{-2}\int_{-\infty}^{\infty} dt\, e^{ict(K_f-K_i)} \sum_{\{n\}} \langle\{n\}|\hat{\rho}_v|\{n\}\rangle$$

$$\times \sum_{\{n'\}}\langle\{n\}|H_{fi}{}^{(1)\dagger}|\{n'\}\rangle\langle\{n'\}|\exp(iH_v t/\hbar)H_{fi}{}^{(1)}\exp(-iH_v t/\hbar)|\{n\}\rangle$$

$$= \hbar^{-2}\int_{-\infty}^{\infty} dt\, e^{ict(K_f-K_i)}\mathrm{Tr}\{\hat{\rho}_v H_{fi}{}^{(1)\dagger} H_{fi}{}^{(1)}(t)\} = \hbar^{-2}\int_{-\infty}^{\infty} dt\, e^{ict(K_f-K_i)}\langle H_{fi}{}^{(1)\dagger} H_{fi}{}^{(1)}(t)\rangle. \qquad \text{(B4)}$$

Here the second line is obtained by noting that the states $|\{n\}\rangle$ and $|\{n'\}\rangle$ are eigenstates of the unperturbed vortex Hamiltonian H_v, and $H_{fi}{}^{(1)}(t)$ is a time-dependent operator

$$H_{fi}{}^{(1)}(t) = \exp(iH_v t/\hbar)H_{fi}{}^{(1)}\exp(-iH_v t/\hbar) \qquad \text{(B5a)}$$

obtained with the substitution

$$a_k \to a_k e^{-i\omega_k t}, \quad a_k{}^\dagger \to a_k{}^\dagger e^{i\omega_k t}. \qquad \text{(B5b)}$$

The ensemble average in Eq. (B4) may be evaluated with the techniques used in Eqs. (42)–(45)

$$\langle H_{fi}^{(1)\dagger} H_{fi}^{(1)}(t)\rangle = |V_1|^2 \int dz\, dz'\, e^{-i(k_f-k_i)(z-z')} \Pi_k \langle \exp[i(\lambda e^{ikz'} a_k + \lambda^* e^{-ikz'} a_k^\dagger)]$$
$$\times \exp[-i(\lambda e^{ikz} e^{-i\omega_k t} a_k + \lambda^* e^{-ikz} e^{i\omega_k t} a_k^\dagger)]\rangle = |V_1|^2 \int dz\, dz'\, e^{-i(k_f-k_i)(z-z')}$$
$$\times \Pi_k \{\exp[-i|\lambda|^2 \sin(kz - kz' - \omega_k t)]\langle \exp[-i\lambda(e^{ikz} e^{-i\omega_k t} - e^{ikz'})a_k - i\lambda^*(e^{-ikz} e^{i\omega_k t} - e^{-ikz'})a_k^\dagger]\rangle\}$$
$$= |V_1|^2 L \int d\zeta\, e^{-i(k_f-k_i)\zeta} \exp[-|\lambda|^2 F(\zeta, t)]. \tag{B6}$$

Here $F(\zeta, t)$ is defined

$$F(\zeta, t) = \sum_k \coth(\tfrac{1}{2}\beta\hbar\omega_k)[1 - \cos(k\zeta - \omega_k t)] + i\sum_k \sin(k\zeta - \omega_k t), \tag{B7}$$

and a simple calculation shows that $|\lambda|^2 F$ may be written in terms of the time-dependent operator $\vec{u}(\zeta, t)$ [compare Eq. (B5)]

$$|\lambda|^2 F(\zeta, t) = \tfrac{1}{4}|\vec{1}_f - \vec{1}_i|^2 \langle |\vec{u}(\zeta, t) - \vec{u}(0)|^2\rangle + \tfrac{1}{4}|\vec{1}_f - \vec{1}_i|^2 \langle \vec{u}(\zeta, t)\cdot\vec{u}(0) - \vec{u}(0)\cdot\vec{u}(\zeta, t)\rangle. \tag{B8}$$

Equation (B6) is thus seen as a natural generalization of Eq. (44); the conservation of energy merely introduces an additional Fourier transform, which gives

$$P_{fi}^{(1)} = L|V_1|^2 \hbar^{-2} \int dt\, d\zeta\, e^{-i(k_f-k_i)\zeta} e^{ict(K_f-K_i)} \exp[-|\lambda|^2 F(\zeta, t)]. \tag{B9}$$

The logarithmic factor in Eq. (13) prevents an exact evaluation of Eq. (B7), and it is necessary to introduce the same approximations used in Eq. (19)

$$\omega_k \approx (\kappa k^2/4\pi)\ln(L/a) = (\hbar k^2/2m)\ln(L/a) = \hbar k^2/2m^*, \tag{B10}$$

where $\quad m^* = m[\ln(L/a)]^{-1}$ (B11)

is an effective mass. In addition, the statistical factor $\coth(\frac{1}{2}\beta\hbar\omega_k)$ may be replaced by its classical limit $2/\beta\hbar\omega_k$. We, therefore, obtain

$$F(\zeta, t) = F_1(\zeta, t) + iF_2(\zeta, t), \tag{B12}$$

where $\quad F_1(\zeta, t) = (2/\beta\hbar)\sum_k \omega_k^{-1}[1 - \cos(k\zeta - \omega_k t)]$ (B13a)

$$F_2(\zeta, t) = \sum_k \sin(k\zeta - \omega_k t) \tag{B13b}$$

are the real and imaginary parts of F. They have simple symmetry properties

$$F_1(\zeta, t) = F_1(|\zeta|, |t|), \tag{B14a}$$
$$F_2(\zeta, t) = \mathrm{sgn}\, t F_2(|\zeta|, |t|), \tag{B14b}$$

which permits us temporarily to take both ζ and t as positive. The evaluation of F_2 is straightforward

$$F_2 = (L/2\pi)\,\mathrm{Im}\int_{-\infty}^{\infty} dk \exp[-i(\hbar k^2 t/2m^* - k\zeta)] = (L/2\pi)\,\mathrm{Im}\exp(i\zeta^2 m^*/2\hbar t)$$
$$\times \int_{-\infty}^{\infty} dk \exp\{-i[k(\hbar t/2m^*)^{1/2} - \zeta(m^*/2\hbar t)^{1/2}]^2\} = (L/2\pi)(2m^*/\hbar t)^{1/2}\,\mathrm{Im}\, e^{i\sigma^2}\int_{-\infty}^{\infty} dx\, e^{-ix^2}, \tag{B15}$$

where $\quad \sigma = |\zeta|(m^*/2\hbar|t|)^{1/2}$. (B16)

The remaining definite integral may be evaluated by rotating the contour through $-\frac{1}{4}\pi$ in the complex plane, and we find for all ζ and t

$$F_2(\xi, t) = (L/2\pi)\,\mathrm{sgn}\,t(2m^*/\hbar\,|t|)^{1/2} \sin(\sigma^2 - \tfrac{1}{4}\pi). \tag{B17}$$

The real part F_1 is more difficult. It is first convenient to integrate by parts

$$F_1 = \frac{L}{2\pi}\,\frac{4m^*}{\beta\hbar^2}\int_{-\infty}^{\infty} dk\,k^{-2}\left[1 - \cos\left(k\xi - \frac{\hbar t k^2}{2m^*}\right)\right] = \frac{L}{2\pi}\,\frac{4m^*}{\beta\hbar^2}\int_{-\infty}^{\infty} dk\,k^{-1}\left(\xi - \frac{\hbar k t}{m^*}\right)\sin\left(k\xi - \frac{\hbar t k^2}{2m^*}\right)$$

$$= -\left(\frac{4t}{\beta\hbar}\right)F_2 + \frac{L}{2\pi}\,\frac{4m^*\xi}{\beta\hbar^2}\int_{-\infty}^{\infty} dk\,k^{-1}\sin\left(k\xi - \frac{\hbar t k^2}{2m^*}\right), \tag{B18}$$

where ξ and t are again assumed positive. The definite integral in the second term can be rewritten

$$\int_{-\infty}^{\infty} dk\,k^{-1}\sin\left(k\xi - \frac{\hbar t k^2}{2m^*}\right) = \mathrm{Im}\,P\int_{-\infty}^{\infty} dk\,k^{-1}\exp\left[-i\left(\frac{\hbar t k^2}{2m^*} - k\xi\right)\right]$$

$$= \mathrm{Im}\,e^{i\sigma^2} P\int_{-\infty}^{\infty} dx\,x^{-1}\exp[-i(x-\sigma)^2] = \mathrm{Im}\,e^{i\sigma^2} P\int_{-\infty}^{\infty} dx\,(x+\sigma)^{-1} e^{-ix^2}$$

$$= 2\,\mathrm{Im}\,e^{i\sigma^2} P\int_0^{\infty} dx\,\sigma(\sigma^2 - x^2)^{-1} e^{-ix^2}, \tag{B19}$$

where P denotes the Cauchy principal value at $x = \sigma$. We evaluate the integral by considering a contour integral taken along the positive real axis, indented below the point $z = \sigma$

$$\int_C dz\,\sigma(\sigma^2 - z^2)^{-1} e^{-iz^2} = P\int_0^{\infty} dx\,\sigma(\sigma^2 - x^2)^{-1} e^{-ix^2} - \tfrac{1}{2} i\pi\, e^{-i\sigma^2}. \tag{B20}$$

This contour can be rotated by $-\tfrac{1}{4}\pi$, since the arc at infinity makes no contribution

$$\int_C dz\,\sigma(\sigma^2 - z^2)^{-1} e^{-iz^2} = \exp(-i\tfrac{1}{4}\pi)\int_0^{\infty} du\,\sigma(\sigma^2 + u^2)^{-1} e^{-u^2}$$

$$= \exp(-i\tfrac{1}{4}\pi)\int_0^{\infty} dy\,(y^2+1)^{-1}\exp(-\sigma^2 y^2) = \tfrac{1}{2}\pi\exp(\sigma^2 - i\tfrac{1}{4}\pi)\,\mathrm{erfc}\,\sigma. \tag{B21}$$

Here $\quad \mathrm{erfc}\,\sigma = 2\pi^{-1/2}\int_\sigma^{\infty} dt\,e^{-t^2}$ (B22)

is the complementary error function and the integral in Eq. (B21) has been evaluated by differentiating the next to last line with respect to σ^2. A combination of Eqs. (B18)–(B21) yields

$$F_2(\xi, t) = 2Lm^*|\xi|/\beta\hbar^2 - (4L/\beta\hbar)(m^*|t|/2\pi\hbar)^{1/2}\sin(\sigma^2 - \tfrac{1}{4}\pi)[1 - \pi^{1/2}\sigma e^{\sigma^2}\,\mathrm{erfc}\,\sigma], \tag{B23}$$

which is correct for all ξ and t.

The final calculation of $P_{fi}^{(1)}$ can be simplified by introducing two characteristic wave numbers [compare Eqs. (34) and (46)]

$$q = 2L|\lambda|^2 m^*/\beta\hbar^2 = 2\pi\,|\vec{1}_f - \vec{1}_i|^2/\rho\kappa^2\beta\ln(L/a) \tag{B24}$$

$$Q = (L|\lambda|^2/\beta\hbar)^2(4m^*/\pi\hbar c) = \hbar q^2/\pi m^* c = 2\omega_q/\pi c. \tag{B25}$$

The substitutions $x = q\xi$, $y = cQt$ then gives

$$P_{fi}^{(1)} = L|V_1|^2(\hbar^2 cqQ)^{-1}\iint_{-\infty}^{\infty} dx\,dy\,\exp[-i(k_f - k_i)x/q + i(K_f - K_i)y/Q]$$
$$\times\exp\{-|x| + (2|y|)^{1/2}\sin(\sigma^2 - \tfrac{1}{4}\pi)[1 - \pi^{1/2}\sigma e^{\sigma^2}\,\mathrm{erfc}\,\sigma] - i\beta\hbar cQ\tfrac{1}{2}\,\mathrm{sgn}\,y(2|y|)^{-1/2}\sin(\sigma^2 - \tfrac{1}{4}\pi)\}, \tag{B26}$$

where $\sigma^2 = x^2/2\pi|y|$ is independent of q and Q. Since $|\vec{1}_f - \vec{1}_i| \lesssim k_B T/\hbar c \approx \tfrac{1}{3}\times 10^7\ \mathrm{cm}^{-1}$, Eqs. (B24) and (B25) show that $q \lesssim 10^4\ \mathrm{cm}^{-1}$ and $Q \lesssim 10\ \mathrm{cm}^{-1}$; consequently, the dimensionless parameter $\beta\hbar cQ$ is very small ($\beta\hbar cQ \lesssim 10^{-6}$), and we may approximate Eq. (B26)

$$P_{fi}^{(1)} \approx 4L|V_1|^2(\hbar^2 cqQ)^{-1}\iint_0^\infty dx\,dy\,\cos[(k_f-k_i)x/q]\cos[(K_f-K_i)y/Q]$$
$$\times\exp\{-|x|+(2|y|)^{1/2}\sin(\sigma^2-\tfrac{1}{4}\pi)[1-\pi^{1/2}\sigma e^{\sigma^2}\operatorname{erfc}\sigma]\}. \tag{B27}$$

This integral cannot be evaluated exactly, but it is sufficient to obtain a qualitative description of the dependence on the parameters $|k_f-k_i|/q$ and $|K_f-K_i|/Q$. The function $1-\pi^{1/2}\sigma e^{\sigma^2}\operatorname{erfc}\sigma$ decreases monotonically from 1 to 0 as σ increases from 0 to ∞, while $\sin(\sigma^2-\frac{1}{4}\pi)$ oscillates rapidly as soon as σ exceeds $(2\pi)^{1/2}$. Hence, we shall take

$$\sin(\sigma^2-\tfrac{1}{4}\pi)[1-\pi^{1/2}\sigma e^{\sigma^2}\operatorname{erfc}\sigma]\approx\sin(-\tfrac{1}{4}\pi)\theta(2\pi-\sigma^2)=-(1/\sqrt{2})\theta(2\pi-\sigma^2), \tag{B28}$$

where $\theta(x)$ is the usual step function. With this approximation, the x integration can be performed explicitly

$$P_{fi}^{(1)} = 4L|V_1|^2(\hbar^2 cQ)^{-1}q[q^2+(k_f-k_i)^2]^{-1}\int_0^\infty dy\,\cos[(K_f-K_i)y/Q]$$
$$\times\{\exp(-y^{1/2})-\exp(-2\pi y^{1/2})[\exp(-y^{1/2})-1][u\sin(2\pi u y^{1/2})-\cos(2\pi u y^{1/2})]\}, \tag{B29}$$

where $u\equiv|k_f-k_i|/q$. The remaining integral is a peaked function of $|K_f-K_i|$ that falls off for $|K_f-K_i|\gtrsim Q$; this can be shown by making the crude approximations $\exp(-y^{1/2})\approx\theta(1-y)$ and $\exp(-2\pi y^{1/2})\approx\theta(1-4\pi^2 y)$, and a simple calculation then gives

$$P_{fi}^{(1)} \approx (4L|V_1|^2/\hbar^2 c)\{q/[q^2+(k_f-k_i)^2]\}\sin(|K_f-K_i|/Q)/|K_f-K_i|\,. \tag{B30}$$

As discussed below Eq. (37), we see that the energy transfer is restricted to values $\approx\hbar cQ\approx\hbar\omega_q\lesssim 10^{-22}$ erg, which is much smaller than the energy of a thermal phonon ($\approx 10^{-16}$ erg). Since $|V_1|^2$ is a slowly varying function of $|K_f-K_i|$ and $|k_f-k_i|$, the last two factors in Eq. (B30) may be approximated by Dirac δ functions [compare the discussion below Eq. (56)]

$$P_{fi}^{(1)} \approx (2\pi/\hbar)^2 Lc^{-1}|V_1|^2\delta(k_f-k_i)\delta(K_f-K_i), \tag{B31}$$

thereby justifying the approximation introduced in Eq. (38).

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Field and Plasma in the Lunar Wake

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A theory is presented to explain the observed variations of the magnetic field and plasma in the vicinity of the moon. Under the guiding-center approximation, solutions for the plasma flow near the moon are obtained from the kinetic equation. The creation of a plasma cavity in the core region of the lunar shadow disturbs the interplanetary magnetic field. Maxwell's equations are used to study perturbations of the magnetic field in the lunar wake. The acceleration drift current, which was omitted from the earlier work, is included in the present theory in the calculation of the total electric current in the lunar wake. Numerical solutions of Maxwell's equations are obtained. When the interplanetary magnetic field lines penetrate into the lunar body, the sudden change in magnetic permeability disturbs the magnetic field at the lunar limbs. Propagations of this disturbance with magnetoacoustic speed form a Mach cone downstream, which is sometimes observed as the exterior increase of field magnitude in the lunar penumbra. Perturbations of the magnetic field are restricted to the region inside the Mach cone; the region outside remains undisturbed. The numerical results agree extremely well with experimental data from the Explorer-35 spacecraft.

I. INTRODUCTION

Measurements[1-6] of the interplanetary magnetic field and plasma in the vicinity of the moon have been made from lunar orbit on the Explorer-35 spacecraft. The purpose of this paper is to present a theory which can explain the observed variations of the field and plasma in the lunar wake.

When the solar wind interacts with the moon, no shocks are observed in the vicinity of the moon. Figure 1 shows a simultaneous measurement of the interplanetary field and plasma on Explorer 35 when the moon is outside the earth's bow shock. The major effect of the moon on the solar-wind plasma is the creation of a plasma cavity in the umbral region of the lunar shadow. In this cavity the magnitude of the magnetic field increases, i.e., the field is observed to be stronger than the undisturbed interplanetary field. On either side of the umbral increase, the field decreases, i.e., the field becomes weaker than the undisturbed condition. These penumbral decreases occur at the location where the plasma density is about half of the undisturbed plasma density, and they are often bounded on the exterior by additional small increases in the field magnitude. A positive

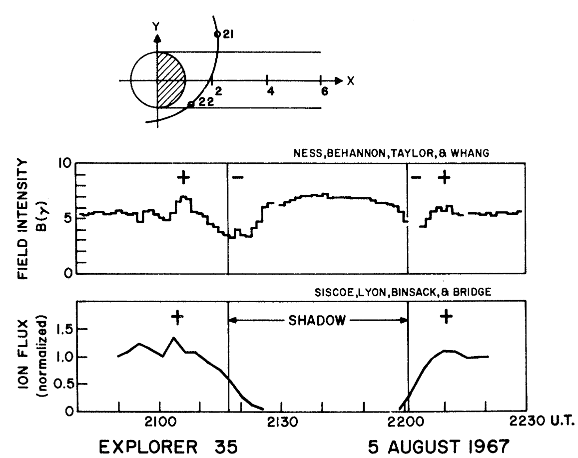

FIG. 1. Simultaneous measurements of field and plasma obtained on August 5, 1967, from lunar orbit on the Explorer-35 spacecraft. The trajectory of the spacecraft is shown projected on the ecliptic plane and positionally correlated with the data through UT annotation. The x axis is parallel to the sun-moon line.