Separable t Matrices and the Three-Body Problem

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A new, separable expansion of the two-body t matrix is presented. Such an expansion reduces the Faddeev equations to coupled equations in one continuous variable. The leading term of the expansion is the separable approximation suggested by Kowalski and Noyes. Any separable approximation to the t matrix obtained by truncating the expansion is exact half-off the energy shell, exactly satisfies the off-shell unitarity relation, and duplicates the exact t matrix in the neighborhood of two-body bound-state and resonance energies. The rate of convergence of the expansion is tested by means of examples. Two terms give a good approximation to the S-wave part of the t matrix arising from a square-well potential, which fits the low-energy two-nucleon scattering data in an average way. It is also shown that the first term of the expansion gives a very good approximation to the t matrix arising from a pure hard-core potential. Results are given for the binding energy of a system of three identical spinless particles interacting via square-well potentials with and without hard cores. The two potentials have the same scattering length and effective range. The potential without a core produces a three-body binding energy of 10.1 MeV; the potential with a core produces a three-body binding energy of 8.00 MeV.

I. INTRODUCTION

In recent years, considerable attention has been focused on the three-body problem. This has been stimulated mainly by the discovery of the Faddeev[1] equations and Mitra's[2] demonstration that separable two-body potentials[3] reduce the three-body problem to a finite number of coupled one-dimensional integral equations. The separable-potential approach in nuclear physics has been partially justified by Lovelace,[4] who showed that the existence of the low-energy singlet virtual bound state and the triplet bound state implies that the low-energy two-nucleon t matrix is approximately separable. The separable approach has also been considered by Amado,[5] but from the point of view of field theory. The question naturally arises as to whether or not the use of separable potentials leads to physical results which differ from those obtained by using the more conventional potentials such as the Hamada-Johnston[6] or Yale[7] potentials. It has been shown by Tabakin[8] and Mongan[9] that it is possible to fit the two-nucleon data using separable potentials; therefore, it appears that the two-nucleon system cannot be used to distinguish between the two classes of potentials. It is reasonable to inquire into the possibility of using the three-nucleon system to distinguish between the various potentials which have been proposed. Calculations using the rank-2 separable potentials of Tabakin and Mongan are under way by at least one group.[10] If the forces in all but the $^1S_0$ and $^3S_1$-$^3D_1$ two-body states are neglected, such calculations amount to solving six coupled integral equations. Calculations with local, or almost local, potentials appear to be much more difficult. The main purpose of this paper is to present a method which should be of practical value in solving the three-body problem with realistic local potentials.

For simple local-potential models, there has been success in treating the three-body problem. Osborn[11] has carried out a direct numerical solution of the Faddeev equations assuming exponential or Yukawa potentials acting between each pair of particles. Another approach to treating local potentials is to construct a separable expansion of the two-body t matrix in terms of the eigenfunctions of the kernel of the Lippmann-Schwinger equation. Weinberg[12] was the first to suggest expanding the t matrix in terms of these functions, so that in this paper the expansion will be referred to as the Weinberg expansion. The eigenfunctions and related expansions have been studied by several authors.[13] In particular, Ball and Wong[14] have used such an expansion to calculate the low-energy properties of a system of three identical spinless particles interacting via Yukawa potentials. It has also been shown that the Weinberg expansion can be used even if the potential contains a hard

[1] L. D. Faddeev, Zh. Eksperim. i Teor. Fiz. **39**, 1459 (1960) [English transl.: Soviet Phys.–JETP **12**, 1014 (1961)].
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[3] The concept of a separable potential was introduced by Wheeler [J. A. Wheeler, Phys. Rev. **50**, 643 (1936)]. It was first shown that separable potentials lead to a reasonable description of the two-nucleon system by Yamaguchi [Y. Yamaguchi, Phys. Rev. **95**, 1628 (1954)].
[4] C. Lovelace, Phys. Rev. **135**, B1225 (1964).
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[11] T. Osborn, Stanford Linear Accelerator Report No. SLAC-79 (unpublished); J. W. Humberston, R. L. Hull, and T. A. Osborn, Phys. Letters **27B**, 195 (1968).
[12] S. Weinberg, Phys. Rev. **131**, 440 (1963).
[13] K. Meetz, J. Math. Phys. **3**, 690 (1961); M. Rotenberg, Ann. Phys. (N.Y.) **19**, 262 (1962); S. Tani, *ibid.* **37**, 411 (1966); **37**, 451 (1966); Phys. Rev. **174**, 2054 (1968).
[14] J. S. Ball and D. Y. Wong, Phys. Rev. **169**, 1362 (1968).

core.[15,16] The expansion has been used to calculate the ground-state energy of a system of three identical spinless particles interacting via the S-wave part of a square-well potential with a hard core.[16]

At positive energies, the Weinberg expansion has the disadvantage that it does not satisfy the off-shell unitarity relation, the importance of which has been pointed out by Lovelace.[4] A separable approximation which does satisfy the off-shell unitary relation has been proposed by Kowalski[17] and Noyes.[18] This approximation has the further advantage that it is exact half-off the energy shell; i.e., when the initial or final momentum corresponds to the energy of the system. One of the purposes of this paper is to develop a separable expansion for the part of the t matrix neglected in the Kowalski-Noyes approximation (KN). The KN approximation has a serious drawback if the energy at which it is used is in the neighborhood of an energy at which the on-shell scattering amplitude vanishes. When this happens the KN approximation can have an unphysical singularity. A remedy for this situation, when the vanishing of the on-shell amplitude is brought about by the presence of a hard core in the potential, is presented in this paper.

In Sec. II a new separable expansion of the t matrix is given which is of practical value when the relevant energies are not near a point at which the on-shell amplitude vanishes. As pointed out above, the first term of the expansion is the KN approximation. This expansion has the advantage that no matter what order it is truncated at, it has the following properties: It is exact half-off the energy, it exactly satisfies the off-shell unitary relation, and it reproduces the behavior of the exact t matrix when the energy is near a two-body resonance or bound-state energy.

The rate of convergence of the expansion presented in Sec. II is tested in Sec. III, using two examples. In the first example, the expansion is used in a calculation of the ground-state energy of a system of three identical spinless particles interacting via square-well potentials. It is found that a two-term expansion gives a binding energy which is accurate to about 1%. The rate of convergence of the expansion is compared to that of the Weinberg expansion. For the example considered, the Weinberg expansion converges slightly faster. It is also shown in Sec. III that the expansion of Sec. II can be used for the t matrix arising from a pure hard-core potential. The rate of convergence of the hard-core expansion is tested by using it in a calculation of the ground-state energy of a system of three identical spinless particles interacting via the S-wave part of a square-well potential with a hard core. It appears that a one-term separable approximation to the hard-core t matrix gives very accurate results. The two potentials considered in Sec. II (square-well without core and square-well with core whose radius is 0.4 F) have the same scattering length and effective range. The importance of the core is illustrated by the two final results for the three-body binding energy; the potential without the core produces a binding energy of 10.1 MeV, whereas the potential with the core produces a binding energy of 8.00 MeV.

In Sec. IV a modification of the expansion presented in Sec. II is given, which is appropriate for use with hard-core potentials. This expansion has all the desirable features of the Sec. II expansion, but avoids the difficulty brought about by the vanishing of the on-shell t matrix. Discussion and conclusions are given in Sec. V. Throughout this paper a system of units is used in which $\hbar^2$ and the mass of the nucleon are 1.

II. A SEPARABLE EXPANSION FOR THE t MATRIX

In this section, a separable expansion for the t matrix is developed. This expansion is of greatest practical value when the energy range of interest does not include, or lie near, a point at which the on-shell scattering amplitude vanishes. Throughout this section and the following ones, attention will be focused on only a single partial wave; therefore, the subscript l will be omitted whenever possible.

It is convenient to begin by considering the eigenvalue equation

$$G_0(s)V_1(s)\,|\,\Psi_\nu(s)\rangle=|\,\Psi_\nu(s)\rangle\eta_\nu(s), \qquad (2.1)$$

where

$$G_0(s)=(s-H_0)^{-1}$$

and

$$V_1(s)=V-V\,|\,klm\rangle\,\langle klm\,|\,V\,|\,klm\rangle^{-1}\langle klm\,|\,V. \qquad (2.2)$$

H_0 is the kinetic energy operator, V is the two-body potential, and $|\,klm\rangle$ is the eigenstate of H_0 whose coordinate representation is

$$\langle \mathbf{r}\,|\,klm\rangle=(2\pi^2)^{-1/2}j_l(kr)Y_{lm}(\hat{r}). \qquad (2.3)$$

j_l is the usual spherical Bessel function of order l, and Y_{lm} is a spherical harmonic.[19] The eigenvalues η_ν and eigenfunctions, which are the solutions of (2.1), can easily be found if the t matrix arising from the potential λV is known; λ is an arbitrary parameter. The t matrix is obtained by solving either of the equations

$$T(s,\lambda)=\lambda V+\lambda VG_0(s)T(s,\lambda)$$
$$=\lambda V+T(s,\lambda)G_0(s)\lambda V. \qquad (2.4)$$

[15] M. G. Fuda, Phys. Rev. **174**, 1134 (1968).
[16] M. G. Fuda, Phys. Rev. **178**, 1682 (1969).
[17] K. L. Kowalski, Phys. Rev. Letters **15**, 798 (1965).
[18] H. P. Noyes, Phys. Rev. Letters **15**, 538 (1965).

[19] The Bessel functions and Hankel functions used in this paper are normalized as in A. Messiah, *Quantum Mechanics* (John Wiley & Sons, Inc., New York, 1965).

Using the identity

$$[1+G_0(s)T(s,\lambda)][1-G_0(s)\lambda V]=1, \quad (2.5)$$

which follows immediately from (2.4), it is easy to show that the solution of (2.1) is

$$|\Psi_\nu(s)\rangle=-G_0(s)T(s,\lambda_\nu)\,|klm\rangle, \quad (2.6)$$

where

$$\lambda_\nu=\eta_\nu^{-1}, \quad (2.7)$$

and $|\Psi_\nu\rangle$ is normalized so that

$$\langle klm|V|klm\rangle^{-1}\langle klm|V|\Psi_\nu(s)\rangle=1. \quad (2.8)$$

In deriving this result it is necessary to assume that η_ν is not an eigenvalue of the operator $G_0(s)V$, because if it is, $T(s,\lambda_\nu)$ blows up. Weinberg[12] has studied integral equations of the form of (2.1) and discusses this possibility. This possibility will not be considered here. The eigenvalues $\eta_\nu(s)$ can be found from the equation

$$\langle klm|T(s,\lambda_\nu)|klm\rangle=0, \quad (2.9)$$

which follows from (2.4), (2.6), and (2.8), assuming η_ν is not zero. This equation shows that the inverse of the eigenvalues (the λ_ν) can be interpreted as the set of potential strengths which lead to the vanishing of the on-shell scattering amplitude at the energy s. With the normalization implied by (2.3), the on-shell t matrix can be expressed in terms of the phase-shift δ_l by the relation

$$\langle klm|T(s,\lambda)|klm\rangle=-(2\pi^2k)^{-1}\exp[i\delta_l(k,\lambda)]\times\sin[\delta_l(k,\lambda)]. \quad (2.10)$$

Thus, the vanishing of the scattering amplitude implies that δ_l is an integral multiple of π.

In order to further study the properties of the eigenfunctions $|\Psi_\nu\rangle$ and the eigenvalues η_ν, it is convenient to rewrite (2.1) as a differential equation in the coordinate representation with appropriate boundary conditions. The boundary conditions on $|\Psi_\nu\rangle$ are most easily obtained by considering (2.6). Using the well-known expansion for the free-particle Green's function $G_0(s)$,[20] it is easy to show that in the coordinate representation (2.6) becomes

$$\Psi_\nu(\mathbf{r};s)=kY_{lm}(\hat{r})\int j_l(kr_<)h_l^{(+)}(kr_>)Y_{lm}^*(\hat{r}')\times\langle\mathbf{r}'|T(s,\lambda_\nu)|klm\rangle d\mathbf{r}', \quad (2.11)$$

where

$$s=k^2+i\epsilon,$$

$h_l^{(+)}$ is a spherical Hankel function,[19] and $r_<(r_>)$ is the smaller (larger) of r and r'. Only one partial wave appears in (2.11), since the operator $G_0(s)V_1(s)$ commutes with the square and the z component of the orbital-angular-momentum operator. It follows immediately from (2.3), (2.9), and (2.11) that $\Psi_\nu(r;s)$ must vanish at large distances, since

$$\Psi_\nu(r;s)\underset{r\to\infty}{\sim}(2\pi^2)^{1/2}r^{-1}\exp[i(kr-\tfrac{1}{2}l\pi)]Y_{lm}(\hat{r})\times\langle klm|T(s,\lambda_\nu)|klm\rangle. \quad (2.12)$$

Rather than proceed with $|\Psi_\nu\rangle$, it is convenient to introduce

$$|\Omega_\nu(s)\rangle=|klm\rangle-|\Psi_\nu(s)\rangle, \quad (2.13)$$

which, from (2.6), is

$$|\Omega_\nu(s)\rangle=[1+G_0(s)T(s,\lambda_\nu)]\,|klm\rangle. \quad (2.14)$$

From (2.4), (2.14), and the fact that $|klm\rangle$ is an eigenstate of H_0, it follows that

$$[s-H_0-\lambda_\nu V]\,|\Omega_\nu(s)\rangle=0. \quad (2.15)$$

Thus $|\Omega_\nu(s)\rangle$ is a solution of the Schrödinger equation with a potential whose strength λ_ν has been chosen so that at large distances the wave function becomes a free wave; i.e.,

$$\Omega_\nu(\mathbf{r};s)\underset{r\to\infty}{\sim}[(2\pi^2)^{1/2}kr]^{-1}\sin(kr-\tfrac{1}{2}l\pi)Y_{lm}(\hat{r}). \quad (2.16)$$

If we use the differential equation corresponding to (2.15) and the boundary condition (2.16), it is easy to show that

$$\langle\Omega_\nu(s)|V|\Omega_\mu(s)\rangle=\langle\Omega_\nu(s)|V|\Omega_\nu(s)\rangle\delta_{\nu\mu}, \quad (2.17)$$

and, assuming the eigenfunctions form a complete set, that

$$V=\sum_{\nu=0}^{\infty}V|\Omega_\nu(s)\rangle\langle\Omega_\nu(s)|V|\Omega_\nu(s)\rangle^{-1}\langle\Omega_\nu|V. \quad (2.18)$$

It is also easy to show that all the λ_ν are real. It is important to note that one of the eigenfunctions $|\Omega_\nu\rangle$ is a free wave with eigenvalue λ_ν equal to zero. This eigenfunction and its eigenvalue will always be labeled with ν equal to 0; i.e.,

$$|\Omega_0(s)\rangle=|klm\rangle, \qquad \lambda_0=0. \quad (2.19)$$

Combining (2.1), (2.2), (2.13), (2.17), and (2.19), it follows that

$$G_0(s)V|\Omega_\nu(s)\rangle=[|\Omega_\nu(s)\rangle-|klm\rangle]\eta_\nu(s),\ \nu\neq0. \quad (2.20)$$

When ν is zero, the right-hand side of (2.20) is not well defined, since η_0 is infinite.

Corresponding to the separable expansion (2.17) of the potential, there is a separable expansion of the t matrix. Using (2.4), with λ equal to 1, and (2.17)–(2.20), it is a straightforward matter to show that the separable expansion of the t matrix is

$$T(s)=T(s)|klm\rangle\langle klm|T(s)|klm\rangle^{-1}\times\langle klm|T(s)+T_1(s), \quad (2.21)$$

[20] This expansion can be found, for example, in Appendix B §11 of Ref. 19.

where

$$T_1(s)=\sum_{\nu=1}^{\infty} V\mid\Omega_\nu(s)\rangle\frac{\langle\Omega_\nu(s)\mid V\mid\Omega_\nu(s)\rangle^{-1}}{1-\eta_\nu(s)}\langle\Omega_\nu(s)\mid V. \tag{2.22}$$

The first term on the right-hand side of (2.21) is the separable approximation suggested by Kowalski and Noyes.[17,18] The expansion (2.21) and (2.22) has several important properties. First, any truncation of the expansion of $T_1(s)$ leads to a separable approximation which is exact half-off the energy shell, where by half-off the energy shell is meant

$$\langle plm\mid T(k^2+i\epsilon)\mid klm\rangle \quad\text{or}\quad \langle klm\mid T(k^2+i\epsilon)\mid plm\rangle.$$

This property follows immediately from (2.17) and (2.19). The second important feature of the expansion is that it leads to separable approximations which exactly satisfy the off-shell unitary relation.[21] In order to see this, consider the off-shell unitary relation in the form

$$\begin{aligned}T(s)-T^\dagger(s)&=-4\pi^2ikT(s)\mid klm\rangle\langle klm\mid T^\dagger(s)\\&=-4\pi^2ikT^\dagger(s)\mid klm\rangle\langle klm\mid T(s),\end{aligned} \tag{2.23}$$

where $s=k^2+i\epsilon$, and the dagger means adjoint. From (2.23), it is a straightforward matter to derive the relations

$$\begin{aligned}&T(s)\mid klm\rangle\langle klm\mid T(s)\mid klm\rangle^{-1}\\&\quad=T^\dagger(s)\mid klm\rangle[\langle klm\mid T(s)\mid klm\rangle^*]^{-1},\\&\langle klm\mid T(s)\mid klm\rangle^{-1}\langle klm\mid T(s)\\&\quad=[\langle klm\mid T(s)\mid klm\rangle^*]^{-1}\langle klm\mid T^\dagger(s).\end{aligned} \tag{2.24}$$

By introducing

$$\mid F(s)\rangle=T(s)\mid klm\rangle\langle klm\mid T(s)\mid klm\rangle^{-1}, \tag{2.25}$$

and using (2.24), Eq. (2.21) becomes

$$T(s)=\mid F(s)\rangle\langle klm\mid T(s)\mid klm\rangle\langle F(s)\mid+T_1(s). \tag{2.26}$$

With the observation that $T_1(s)$ is Hermitian and vanishes half-off-shell, it follows from (2.25) and (2.26) that any trunction of (2.22) leads to an approximation for $T(s)$ which exactly satisfies the off-shell unitary relation (2.23). It is also easy to show, using the results of Lovelace,[4] that the first term on the right-hand side of (2.26) reproduces the exact t matrix when the energy s is near a bound state or resonance energy. The expansion given here has an obvious drawback if one wishes to use it in the neighborhood of an energy at which the on-shell t matrix vanishes, since when this occurs the first term on the right-hand side of (2.26) can have an unphysical singularity. One easily checks that this is compensated for by an unphysical singularity in $T_1(s)$; however, this situation could make the use of the expansion in numerical work rather awkward. A partial remedy for this problem can be given, but before doing so it is instructive to look at some applications of the expansion as it stands.

[21] The off-shell unitarity relation is in Ref. 4 and is derived in Ref. 19, Chap. XIX.

III. SOME EXAMPLES

A convenient example to consider is the square-well potential given by

$$\begin{aligned}V(r)&=-V_0, &&0\le r<b\\&=0, &&r>b.\end{aligned} \tag{3.1}$$

For this potential it is easy to determine the half-off-shell t matrix; it is also easy to find the solutions of (2.15) with the boundary condition (2.16). The result for the S-wave part of the expansion is given by the following relations:

$$\begin{aligned}&u=pb,\quad v=qb,\quad w=kb,\quad x=(V_0b^2+w^2)^{1/2},\\&\langle koo\mid T(s)\mid koo\rangle=(b/2\pi^2w)\\&\quad\times\frac{x\sin(w)\cos(x)-w\cos(w)\sin(x)}{x\cos(x)-iw\sin(x)}\exp(-iw),\\&\langle poo\mid F(s)\rangle=(w/u)[V_0b^2/(x^2-u^2)]\\&\quad\times\frac{x\sin(u)\cos(x)-u\cos(u)\sin(x)}{x\sin(w)\cos(x)-w\cos(w)\sin(x)},\\&\langle poo\mid T_1(s)\mid qoo\rangle=-(b/2\pi^2)\sum_{\nu=1}^{\infty}g_\nu(u,w)g_\nu(v,w)\\&\quad\times\frac{V_0b^2}{1-\eta_\nu(w)}\frac{4z_\nu(w)\sin^2z_\nu(w)}{2z_\nu(w)-\sin2z_\nu(w)},\\&g_\nu(u,w)=\frac{u\cos(u)-w\cot(w)\sin(u)}{u[z_\nu^2(w)-u^2]},\\&\eta_\nu(w)=V_0b^2/[z_\nu^2(w)-w^2].\end{aligned} \tag{3.2}$$

The z_ν are those solutions of

$$z_\nu(w)\cot z_\nu(w)=w\cot w$$

which are real and positive when the energy is real and negative. It is clear from the remarks made at the end of Sec. II that the expansion should converge rapidly if $T(s)$ is almost half-off the energy shell; however, in the Faddeev equations, one deals with the fully-off-shell t matrix. In order to test the rate of convergence of the expansion of the square-well t matrix in an off-shell situation, it is convenient to consider the problem of finding the ground-state energy of a system of three identical bosons interacting via the S-wave part of the potential (3.1). This problem is solved by inserting the expansion given above into

TABLE I. Binding energy for various separable approximations to the t matrix arising from potential (3.3).

Number of terms in the expansion	Binding energy (MeV) Expansion I[a]	Expansion II[b]
1	8.82	9.87
2	9.96	10.04
3	10.05	10.07

[a] Given by (3.2).
[b] Given by Eqs. (9) and (10) of Ref. 15.

the Faddeev equations. The resulting infinite system of coupled, linear integral equations is then truncated and solved numerically. The potential parameters were chosen to be

$$V_0=27.17 \text{ MeV}, \qquad b=2.114 \text{ F}. \tag{3.3}$$

This well has a scattering length a of 10.85 F, an effective range r_0 of 1.95 F, and produces one two-body bound state at -0.435 MeV. Table I gives the results for the binding energy as a function of the number of terms retained in the expansion (3.2). This expansion is referred to as expansion I in the Table. Also shown are the results obtained using the Weinberg expansion for the t matrix (expansion II in the table). The formulas for this expansion will not be given here since they have been given before.[15] It is seen from Table I that the one-term Weinberg expansion (expansion II) does better than the expansion given by (3.2); however, the two- and three-term expansions yield binding energies which differ from each other by less than 1%. Both two-term expansions yield binding energies which are within 1% of the exact result (using more than three terms produces no significant effect). It appears that, at negative three-body energies, the Weinberg expansion does slightly better than the one given by (3.2); however, at positive energies, the situation could very easily be reversed. Calculations will soon be carried out to test the expansions at positive energies.

Another example of interest is the expansion of the t matrix arising from a pure hard-core potential. This expansion is obtained from (3.2) by letting

$$V_0 \rightarrow -\infty .$$

Since this limit is so easy to take, there is no point in presenting the limiting forms here. It is interesting to note that the first term of the expansion turns out to be the t matrix arising from a hard-shell potential.[22] The rate of convergence of the expansion for the hard-core t matrix has been examined by using it in a calculation of the ground-state energy of three identical bosons interacting via the S-wave part of the potential

$$\begin{aligned} V(r)&=\infty, && r<0.4 \text{ F} \\ &=-63.85 \text{ MeV}, && 0.4 \text{ F}<r<1.737 \text{ F} \\ &=0, && r>1.737 \text{ F}. \end{aligned} \tag{3.4}$$

This potential has the same scattering length and effective range as the potential (3.3). It was obtained from Enge's[23] spin-dependent central potential by the usual averaging procedure,[24] which weights the triplet- and singlet-potential strengths equally. In order to use the hard-core limit of the expansion (3.2) in the calculation described above, it is necessary to divide the potential V into two parts,

$$V=U+W, \tag{3.5}$$

where U is the pure hard-core potential and W is the part of the potential which is outside the core. Corresponding to this separation of the potential, one has the separation of the t matrix

$$T(s)=t(s)+t^{(1)}(s), \tag{3.6}$$

where $t(s)$ is the solution of

$$t(s)=U+UG_0(s)t(s), \tag{3.7}$$

and

$$t^{(1)}(s)=[1+t(s)G_0(s)]\tau(s)[1+G_0(s)t(s)]. \tag{3.8}$$

The t-matrix-like operator $\tau(s)$ is the solution of

$$\tau(s)=W+WR(s)\tau(s), \tag{3.9}$$

where

$$\begin{aligned} R(s)&=(s-H_0-U)^{-1} \\ &=G_0(s)+G_0(s)t(s)G_0(s). \end{aligned} \tag{3.10}$$

Discussions of this separation technique can be found

TABLE II. Binding energy for various separable approximations to the t matrix arising from potential (3.4).

Case	N[a]	N_t[b]	$N_t(1)$[c]	Binding energy (MeV)
1	2	1	1	7.76
2	3	2	1	7.73
3	3	1	2	7.96
4	4	1	3	8.00

[a] N is the total number of terms in the expansion of the t matrix.
[b] N_t is the number of terms in the expansion of the pure hard-core t matrix, i.e., the hard-core limit of (3.2).
[c] $N_t(1)$ is the number of terms in the expansion of the operator $t^{(1)}(s)$, which is given by (3.6)–(3.10). Formulas for $t^{(1)}(s)$ are given by (4.6) of Ref. 16.

[22] As far as this author knows, the hard-shell potential was first used in nuclear physics by Puff [R. D. Puff, Ann. Phys. (N. Y.) **13**, 317 (1961)].

[23] H. Enge, *Introduction to Nuclear Physics* (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1966).
[24] The justification for the averaging procedure is given in J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley & Sons, Inc., New York, 1952).

in Refs. 16 and 25. The hard-core limit of (3.2) was used for the operator $t(s)$ and a Weinberg-like expansion was used for $\tau(s)$. The separable expansion of $t^{(1)}(s)$ which arises from the Weinberg expansion of $\tau(s)$ is given in Ref. 16. The results for the three-body binding energy obtained by truncating the separable expansions for $t(s)$ and $\tau(s)$ in various ways are given in Table II. Comparison of cases 1 and 2 shows that the expansion for the pure hard-core t matrix converges very rapidly. The rapid convergence of the Weinberg series for the operator $t^{(1)}(s)$ is illustrated by cases 1, 3, and 4. In Ref. 16, a Weinberg series was also used for the pure hard-core t matrix. Comparison of Table II of the present paper with Table I of Ref. 16 shows that the Weinberg series for the hard-core t matrix had not fully converged. The expansion (2.22) and (2.26) for the hard-core t matrix is much more rapidly convergent than the Weinberg series. The results of Table II indicate that it should be possible to obtain accurate results for the three-body binding energy for spin-dependent potentials with hard-cores, that is, at least for central spin-dependent potentials.

IV. HARD-CORE POTENTIALS AND SEPARABLE EXPANSIONS

It is well known that a potential consisting of a hard core surrounded by an attractive well produces phase shifts which change sign. At the energy where the sign change occurs, the on-shell t matrix vanishes, and, in general, the first term on the right-hand side of (2.26) [see also (2.25)] will have an unphysical singularity. In this section a modification of the expansion presented in Sec. II is given; the modified expansion has no unphysical singularities in the energy range of interest for nuclear physics.

The starting point for developing the modified expansion is the separation of the t matrix given by (3.6)–(3.10). The basic idea is to use the expansion presented in Sec. III for the pure hard-core t matrix [the operator $t(s)$ in (3.6)], and to make another separable expansion for the operator $\tau(s)$ [see (3.9)], which, in turn, will lead to a separable expansion for the operator $t^{(1)}(s)$ [see (3.8)]. Unphysical singularities do appear in the expansion for the hard-core t matrix when $k^{-1}\sin kc$ vanishes (c is the core radius); however, for c equal to 0.4 F, the singularity of lowest energy is at 2558 MeV, an energy well above the range of interest for conventional nuclear physics calculations.

We now turn our attention to constructing a separable expansion for the t-matrix-like operator $\tau(s)$. This will be done in close analogy to the method presented in Sec. II. Consider the integral equation

$$R(s)W_1(s)\mid\Phi_\nu(s)\rangle=\mid\Phi_\nu(s)\rangle\chi_\nu(s), \tag{4.1}$$

where the resolvent $R(s)$ is given by (3.10), and

$$W_1(s)=W-W\mid klm]\,[klm\mid W\mid klm]^{-1}[klm\mid W. \tag{4.2}$$

$\mid klm]$ is an eigenstate of the Schrödinger equation with a hard-core potential. In operator form, the eigenstates corresponding to outgoing and incoming spherical waves are given by

$$\mid klm]^{(\pm)}=[1+G_0(k^2\pm i\epsilon)t(k^2\pm i\epsilon)]\times\mid klm\rangle\exp[\mp i\delta_l{}^c(k)], \tag{4.3}$$

where $\delta_l{}^c(k)$ is the pure hard-core phase shift. The phase factor is introduced so as to make the incoming and outgoing eigenstates the same. In configuration space the eigenstates are given by

$$\begin{aligned}\langle\mathbf{r}\mid klm]&=(2\pi)^{-1/2}[j_l(kr)n_l(kc)-j_l(kc)n_l(kr)]\\&\quad\times\mid h_l{}^{(\pm)}(kc)\mid^{-1}Y_{lm}(\hat{r}),\qquad r\geq c\\&=0,\qquad\qquad r\leq c.\end{aligned} \tag{4.4}$$

The superscripts $(\pm)$ have been dropped, since they are no longer necessary. By introducing, in analogy to (2.4), the operator

$$\begin{aligned}\tau(s,\lambda)&=\lambda W+\lambda WR(s)\tau(s,\lambda)\\&=\lambda W+\tau(s,\lambda)R(s)\lambda W,\end{aligned} \tag{4.5}$$

it is easy to show that with suitable normalization

$$\mid\Phi_\nu(s)\rangle=-R(s)\tau(s,\lambda_\nu)\mid klm], \tag{4.6}$$

where

$$\lambda_\nu=\chi_\nu{}^{-1}.$$

The inverse eigenvalues λ_ν are the solutions of

$$[klm\mid\tau(s,\lambda_\nu)\mid klm]=0. \tag{4.7}$$

The behavior of $\mid\Phi_\nu(s)\rangle$ in configuration space can be determined by using the expression for the resolvent $R(s)$ in configuration space. This is

$$\begin{aligned}\langle\mathbf{r}\mid R(k^2+i\epsilon)\mid\mathbf{r}'\rangle&=-k\sum_{l,m}\Big(j_l(kr_<)h_l{}^{(+)}(kr_>)\\&\quad-\frac{j_l(kc)}{h_l{}^{(+)}(kc)}h_l{}^{(+)}(kr)h_l{}^{(+)}(kr')\Big)\\&\quad\times Y_{lm}(\hat{r})Y_{lm}{}^*(\hat{r}'),\qquad r\text{ and }r'>c\\&=0,\qquad r\text{ or }r'<c.\end{aligned} \tag{4.8}$$

Using (4.4)–(4.8), it can be shown that

$$\langle\mathbf{r}\mid\Phi_\nu(s)\rangle=0,\qquad 0\leq r\leq c \tag{4.9}$$

and

$$\langle r\mid\Phi_\nu(s)\rangle\underset{r\to\infty}{\longrightarrow}0.$$

Rather than proceed with $\mid\Phi_\nu\rangle$, it is more convenient

[25] M. G. Fuda, Phys. Rev. 166, 1064 (1968).

to deal with

$$|\Sigma_\nu(s)\rangle = |klm] - |\Phi_\nu(s)\rangle = [1+R(s)\tau(s,\lambda_\nu)]\,|klm], \quad (4.10)$$

which from (4.4), (4.5), (4.8), and (4.9), is, in configuration space, the solution of the differential equation

$$[k^2+\nabla^2-\lambda_\nu W(r)]\,\Sigma_\nu(\mathbf{r};s)=0, \qquad r>c \quad (4.11)$$

with the boundary conditions

$$\Sigma_\nu(r=c;s)=0,$$
$$\Sigma_\nu(\mathbf{r};s)\underset{r\to\infty}{\sim}\langle r\,|\,klm]. \quad (4.12)$$

Preceding as in Sec. II, it is then easy to show that

$$W=\sum_{\nu=0}^{\infty} W\,|\,\Sigma_\nu(s)\rangle\,\langle\Sigma_\nu(s)\,|\,W\,|\,\Sigma_\nu(s)\rangle^{-1}\langle\Sigma_\nu(s)\,|\,W \quad (4.13)$$

and

$$\langle\Sigma_\nu(s)\,|\,W\,|\,\Sigma_\mu(s)\rangle=\langle\Sigma_\nu(s)\,|\,W\,|\,\Sigma_\nu(s)\rangle\delta_{\nu\mu}. \quad (4.14)$$

The separable expansion (4.13) for W leads in turn to the following expansion for the t-matrix-like operator $\tau(s)$:

$$\tau(s)=\tau(s)\,|\,klm][klm\,|\,\tau(s)\,|\,klm]^{-1} \times [klm\,|\,\tau(s)+\tau_1(s), \quad (4.15)$$

where

$$\tau_1(s)=\sum_{\nu=1}^{\infty} W\,|\,\Sigma_\nu(s)\rangle\,\frac{\langle\Sigma_\nu(s)\,|\,W\,|\,\Sigma_\nu(s)\rangle^{-1}}{1-\chi_\nu(s)}\,\langle\Sigma_\nu(s)\,|\,W. \quad (4.16)$$

The eigenfunctions and eigenvalues have been labelled so that

$$|\Sigma_0\rangle = |klm], \qquad \lambda_0=0. \quad (4.17)$$

By using (3.6), (3.8), (4.3), and assuming that

$$s=k^2+i\epsilon,$$

it is easy to show that

$$T(s)\,|\,klm\rangle=t(s)\,|\,klm\rangle + [1+t(s)G_0(s)]\tau(s)\,|\,klm]\exp[i\delta_l^c(k)] \quad (4.18)$$

and

$$\langle klm\,|\,T(s)=\langle klm\,|\,t(s) + \exp[i\delta_l^c(k)][klm\,|\,\tau(s)[1+G_0(s)t(s)]. \quad (4.19)$$

From (4.14)–(4.19), it is clear that any truncation of the series (4.16) for $\tau_1(s)$ will lead to an approximation for the total t matrix $T(s)$, which is exact half-off the energy shell. This assumes that the exact expression for $t(s)$ is used in constructing $t^{(1)}(s)$ [see (3.8)], and that an expression for $t(s)$ which is exact half-off the energy shell is used for the first term on the right-hand side of (3.6), i.e., the approximation used for the complete t matrix should be

$$T_a(s)=t_a(s)+[1+t(s)G_0(s)]\tau_a(s)[1+G_0(s)t(s)], \quad (4.20)$$

where $t_a(s)$ is the separable expansion for the hard-core t matrix discussed in Sec. III, $t(s)$ is the exact expression for the hard-core t matrix, and $\tau_a(s)$ is a separable approximation to $\tau(s)$ obtained by truncating the series for $\tau_1(s)$. It will now be shown that the separable approximation (4.20) exactly satisfies the off-shell unitary relation (2.23).

It is a straightforward matter to show that the operator $\tau(s)$, which is the solution of (3.9), can be written in the form

$$\tau(s)=W+WG(s)W, \quad (4.21)$$

where

$$G(s)=(s-H_0-V)^{-1}. \quad (4.22)$$

This result follows immediately from the resolvent identities

$$G(s)=R(s)+R(s)WG(s) =R(s)+G(s)WR(s). \quad (4.23)$$

From (4.21)–(4.23) it follows that $\tau(s)$ satisfies a relation analogous to the off-shell unitarity relation (2.23); i.e.,

$$\tau(s)-\tau^\dagger(s)=-4\pi^2ik\tau(s)\,|\,klm][klm\,|\,\tau^\dagger(s) =-4\pi^2ik\tau^\dagger(s)\,|\,klm][klm\,|\,\tau(s). \quad (4.24)$$

By using (4.24) it is easy to show that $\tau(s)$ satisfies relations analogous to (2.24). These relations, which it is not necessary to write out, allow one to write (4.15) in the form

$$\tau(s)=|\,\Lambda(s)\rangle[klm\,|\,\tau(s)\,|\,klm]\langle\Lambda(s)\,|+\tau_1(s), \quad (4.25)$$

where

$$|\,\Lambda(s)\rangle=\tau(s)\,|\,klm][klm\,|\,\tau(s)\,|\,klm]^{-1}. \quad (4.26)$$

It follows from (4.14), (4.17), (4.25), and the fact that $\tau_1(s)$ is Hermitian, that any truncation of the series (4.16) for $\tau_1(s)$ leads to a separable approximation for $\tau(s)$ which exactly satisfies (4.24). In order to proceed, it is convenient to write (4.24) in the form

$$\tau_a(s)-\tau_a^\dagger(s)=-4\pi^2ik\tau_a(s)[1+G_0(s)t(s)] \times |\,klm\rangle\,\langle klm\,|\,[1+t^\dagger(s)G_0^\dagger(s)]\tau_a^\dagger(s) =-4\pi^2ik\tau_a^\dagger(s)[1+G_0^\dagger(s)t^\dagger(s)] \times |\,klm\rangle\,\langle klm\,|\,[1+t(s)G_0(s)]\tau_a(s). \quad (4.27)$$

This form follows immediately from (4.3); the subscript a indicates that a separable approximation is being used for $\tau(s)$. Let $t_a^{(1)}(s)$ stand for the second term on the right-hand side of (4.20); i.e., let

$$t_a^{(1)}(s)=[1+t(s)G_0(s)]\tau_a(s)[1+G_0(s)t(s)]. \quad (4.28)$$

Now, since $t(s)$ is the exact hard-core t matrix and therefore satisfies (2.23), it is easy to show that

$$G_0(s)t(s)=G_0^\dagger(s)t^\dagger(s) -4\pi^2ik[1+G_0(s)t(s)]\,|\,klm\rangle\langle klm\,|\,t^\dagger(s). \quad (4.29)$$

From (4.27)–(4.29) it follows immediately that

$$t_a^{(1)}(s)-t^{(1)}(s)^\dagger=-4\pi^2ik[t(s)+t_a^{(1)}(s)] \times|\,klm\rangle\langle klm\,|\,[t^\dagger(s)+t_a^{(1)\dagger}(s)] +4\pi^2ikt(s)\,|\,klm\rangle\langle klm\,|\,t^\dagger(s). \quad (4.30)$$

Since the right-hand side of (4.30) involves only half-off-shell operators, $t(s)$ can be replaced by $t_a(s)$, where $t_a(s)$ is a separable approximation for the hard-core t matrix which is exact half-off the energy shell, and satisfies the off-shell unitary relation (2.23). Using (4.20) and (4.28), (4.30) becomes

$$T_a(s)-T_a^\dagger(s)=-4\pi^2ikT_a(s)\,|\,klm\rangle\langle klm\,|\,T_a^\dagger(s). \quad (4.31)$$

Thus, it has been shown that the separable approximation (4.20) is exact half-off the energy shell and exactly satisfies the off-shell unitary relation (2.23). Furthermore, (4.20) should have no unphysical singularities due to the vanishing of the on-shell t matrix.

V. DISCUSSION AND CONCLUSIONS

A new separable expansion for the two-body t matrix has been developed, which has a number of desirable features. The expansion when truncated at any order gives an approximation to the t matrix which is exact half-off the energy shell, exactly satisfies the off-shell unitary relations, and reproduces the behavior of the exact t matrix in the vicinity of two-body bound-state and resonance energies. By its very nature, the expansion is expected to converge rapidly at positive energies, when one of the momenta is not too far from its on-shell value. At negative energies it is expected that the nearness of the three-nucleon binding energy to the deuteron binding energy and the energy of the singlet virtual state will lead to a rapid convergence of the results for the three-nucleon binding energy. The first example [see Table I] of Sec. III bears out the conjecture about the behavior at negative energies, and it appears that two terms give a good approximation for the t matrix arising from a purely attractive potential. The second example of Sec. III [see Table II] indicates that the leading term of the expansion gives an extremely good approximation to the hard-core t matrix. Adding in the second term of the expansion changes the result for the three-body binding energy by less than $\frac{1}{2}\%$. It also appears from the results of Table II that a three-term separable expansion of the t matrix arising from potentials with hard cores can give three-body binding energies which are accurate to about one percent. Of course, one cannot draw firm conclusions from two examples. Other applications of the expansion are now being considered. In particular, calculations at positive energies are being planned.

As mentioned previously, unphysical singularities appear in the expansion if the energy is in the neighborhood of an energy at which the on-shell t matrix vanishes. Of course, this is possible in nuclear physics, since the phase shifts change sign at high energies. The expansion of Sec. II can still be used in principle, since the appearance of an unphysical singularity in one term of the expansion is compensated for by the appearance of an unphysical singularity in another term of the expansion. However, handling this cancellation of singularities in a practical numerical calculation could be awkward. The expansion of Sec. IV avoids this difficulty in the case of hard-core potentials by using one expansion for the part of the total t matrix arising from a pure hard-core potential and another expansion for the rest of the total t matrix. The modified expansion leads to separable approximations which have the same desirable features as the expansion of Sec. II. The modified expansion has not been used yet in numerical calculations; however, it certainly will be used in the near future.

The question naturally arises as to whether or not the expansions presented here will be of practical value in calculations using realistic local-potential models. The reader is reminded that the calculations of Sec. III deal only with spin-*independent* central forces. If the results presented in Table II are at all typical, it appears that calculations with spin-*dependent* central forces are feasible. If three-term expansions are adequate for spin-independent central forces, six-term expansions should be adequate for spin-dependent central forces. Using a six-term expansion in the Faddeev equations leads to a system of six coupled, linear integral equations. Such a system of equations can certainly be handled on present-day computers. It is not yet clear whether or not calculations with tensor forces are practical. It may be necessary to use some form of perturbation theory in order to treat such forces.

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