Unitary Pole Approximation and Binding Energy of the Trinucleon*

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The unitary pole approximation (UPA) uses the two-body binding energy and wave function to determine the form factor for the UPA separable t matrix. We develop the UPA for Tabakin's 1965 spin-independent model potential for the trinucleon, and obtain a trinucleon energy within 0.2 MeV of his result. We then develop the UPA for Tabakin's 1964 spin-singlet potential, and for the Schrenk-Mitra singlet. We combine these with Yamaguchi shapes and also a modified Hulthén shape for the spin-triplet central and tensor potentials. These choices give trinucleon energies within 0.3 MeV of the experimental value, provided that we fit the deuteron with 4% D state. We further study the dependence of trinucleon energy on the percent D state in the range $0.78\% \leq P_D \leq 7\%$. We also use Tabakin's recent rank-1 separable fit to singlet phase shifts. This separable potential gives a trinucleon energy 1.5 MeV higher than the singlet choices above because of its relatively weak attraction in off-shell t-matrix elements.

I. INTRODUCTION

In the past several years many physicists have calculated the binding energy of H^3 using separable two-body forces that agree with properties of the two-nucleon system at low energies.[1–6] The use of a tensor rank-1 separable potential[7] reduces the Faddeev equations for the H^3 ground state to three coupled one-dimensional integral equations. Table I presents a sample of the results found for the energy; we see that many workers obtain values within an MeV of the experimental value of -8.48 MeV. (See Noyes[4] for further calculations, references, and discussions.) Of course, extensive calculations have also been made with local potentials.[4,5]

These calculations suggest five different sorts of questions.[4] (i) How good is the separable approximation? Should we use rank-2 or even higher-rank separable potentials? (ii) How sensitive is the three body energy to numerical values we use for the singlet effective range r_0 or for the percent of D state in the deuteron P_D? (iii) What shape should we use for the separable potentials? (iv) How large are various corrections to the energy (e.g., those due to interactions in other states than the ^1S and coupled ^3S-^3D treated above, or due to relativistic corrections, or to many-body forces)? (v) How well do other results of the separable approximation agree with experiment? (The Coulomb energy and form factors of the trinucleon[8]; the absence of particle-stable excited states of the trinucleon[9]; the scattering lengths for the n-d system[2,3] and scattering results at higher energies.[9,10])

In this paper we shall ignore the last two questions. We give a tentative answer to the first question on the validity of a rank-1 separable approximation by reference to recent work[11] on separable approximations to a local central square well. A particular separable approximation which we call the unitary pole approximation (UPA) gives off-shell values of the two-body t matrix in remarkably close agreement with those for the assumed local potential, while a two-term Weinberg-Rotenberg series[12] is still more accurate. (We shall not discuss here the philosophical problems[11] involved in attempting to give meaning to the subjective word "good" in our original question.) Similarly, Fuda[13] finds rapid convergence of the binding energy of a model trinucleon for a central local square-

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TABLE I. Some published results on triton binding energy. All calculations use Yamaguchi triplet rank-1 separable potential including tensor force (see Yamaguchi, Ref. 7), but they vary in the percent D state P_D; r_0 is the singlet effective range. See Eq. (1) for an explanation of the shape $n=3$ of Karchenko *et al.*; r_c is the radius of Dabrowski and Dworzecka's core. See Noyes, Ref. 4, for further calculations.

Workers	Singlet potential form	r_0 (F)	P_D (%)	Triton energy (MeV)
Bhakar and Mitra[a]	rank-1 Yamaguchi	2.15	4	−10.4
Phillips[b]	rank-1 Yamaguchi	2.7	4	−9.2
Phillips[b]	rank-1 Yamaguchi	2.7	5.5	−8.6
Phillips[b]	rank-1 Yamaguchi	2.7	7	−8.05
Phillips[b]	rank-1 Yamaguchi	2.5	4	−9.7
Phillips[b]	rank-1 Yamaguchi	2.85	4	−8.9
Kharchenko[c]	rank-1 Yamaguchi	2.7	4	−9.18
Kharchenko[c]	rank-1, $n=3$	2.7	4	−8.59
Schrenk and Mitra[d]	rank-2	2.7	4	−9.21
Dabrowski[e]	rank-2, $r_c=0.25$	2.5	4	−8.81
Experiment		2.7±0.1[f]		−8.48

[a] Reference 1.
[b] Reference 3.
[c] Reference 2.
[d] Potential G1, Ref. 19.
[e] Reference 20.
[f] Houk and Wilson, Ref. 25; Noyes, Ref. 4.

well potential, as the two-body t matrix is expanded in a Weinberg-Rotenberg series.

A one-term or two-term separable approximation is less satisfactory for a local potential of Yukawa or Hulthén shape.[4] See Ball and Wong[14] for the speed of convergence of the Weinberg-Rotenberg series results for Yukawa shape, and Sitenko *et al.*[15] and Lu[15] for Hulthén shape. One of us (E.H.) has recently developed a more rapidly converging and more convenient series, called the unitary pole expansion[16]; its first term is the UPA (Yamaguchi shape for a Hulthén potential) which gives a three-body energy about $1\frac{1}{2}$ MeV greater[17] than that for a local Hulthén potential. The unitary pole expansion[16] converges particularly rapidly if the local potential has a soft core: e.g., the Malfielt-Tjon[17] spin-independent potential, or the Reid[17] singlet potential. Here the difference between the UPA and the exact energy is an order of magnitude smaller than the $1\frac{1}{2}$ MeV difference for the Hulthén potential.

Kok *et al.*[18] find that a separable and local potential with the same two-body phase shifts differ by some $1\frac{1}{2}$ MeV in the three-body energy; in this case the separable potential gives the lower value.

As shown in Table I, Schrenk and Mitra[19] and also Dabrowski and Dworzecka[20] have used rank-2 separable potentials for the ^1S state, together with single-rank Yamaguchi shape for the triplet system. They solve four coupled one-dimensional integral equations. Stagat[21] has worked out the $3N$ coupled integral equations for the general case of rank-N separable potentials. One of us (T.J.B.) has recently been able to solve[22] these six coupled equations for $N=2$ using the rank-2 separable tensor potentials of Tabakin[23] and Mongan.[24] We note that a rank-2 separable potential permits fitting S phase shifts that change sign at high energy, and also permits fitting the three triplet-phase parameters.

Of course numerical methods are used to solve the three (or more) coupled integral equations. We study our accuracy by varying the size of mesh used in our numerical work.

Phillips[3] pays special attention to the dependence of the trinucleon binding energy on the values assumed for the singlet effective range r_0 (2.5, 2.70, and 2.85 F) and for the percent D state P_D (4, 5.5, and 7%). His steps for the singlet effective range r_0 are somewhat larger than its experimental error,[4,25] and each step in r_0 changes the binding energy by about 0.4 MeV. Each step in P_D changes the binding energy by about 0.5 MeV; it is difficult to estimate the experimental value and its experimental error ΔP_D. The value of the deuteron's quadrupole moment demands[26] $P_D \geq 0.45\%$.

In this paper we limit ourselves to a singlet effective range value of 2.70 F; we treat values of P_D from 0.78 to 7%, thus extending Phillips's range.

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Noyes[4] discusses the uncertainty in calculations of the trinucleon energy due to (i) use of the separable approximation, and (ii) values used for the parameters r_0 and P_D. We agree that the uncertainty in P_D is a major difficulty in accurate calculation of the two-nucleon t matrix, and therefore of the three-nucleon energy. The error due to our use of the UPA is hard to assess quantitatively. In any case, the UPA represents a first term in the t matrix for the unitary pole expansion as a means of calculation of the three-body energy for any assumed two-nucleon potential.

Previous workers with rank-1 separable potentials have generally chosen the Yamaguchi shapes.[7] However, Kharchenko *et al.*[2] has treated singlet potentials with form factor

$$g(k)=(k^2+\beta^2)^{-n}. \tag{1}$$

Of course $n=1$ is the Yamaguchi singlet shape. The binding energy changes by 0.6 MeV when n changes from 1 to 3, keeping the singlet effective range parameters fixed.

We determine the shape of $g(k)$ by the use of the UPA.[27,28] In the UPA we start with knowledge of the two-body binding energy and ground-state wave function. We then construct a (rank-1) separable potential with identical binding energy and wave function and use this separable potential to find off-shell values of the two-body t matrix, and, finally, the binding energy of the trinucleon. By use of the UPA we can limit ourselves to rank-1 separable potentials, and still include much of the effects of a strong short-range repulsion in the two-body interaction.

In Sec. II we develop the UPA for Tabakin's model[29] of a central rank-2 separable potential with a short-range repulsion. We also develop the UPA for the ^1S unbound state, using Tabakin's[23] and Schrenk-Mitra's[19] rank-2 separable potentials, and for a modified Hulthén potential with centrifugal repulsion. In Sec. III, we give our results for the trinucleon binding for several different rank-1 separable tensor potentials. We include Tabakin's recent[30] rank-1 separable potential for the ^1S state, even though we believe that this potential gives a poor approximation to the off-shell t matrix. In the Sec. IV we discuss our results and compare them with those of other workers. The Appendix summarizes our Gauss-Gegenbauer method of numerical integration.

II. UNITARY POLE APPROXIMATION

We first apply the UPA[27,28] to a model central potential with a two-body bound state. We choose Tabakin's[29] model spin-independent central potential, for which he has determined the trinucleon energy, so that by comparison we can find the accuracy of the UPA. Also, Tabakin has compared the energy for his potential, which includes a strong short-range repulsion, with that for a Yamaguchi potential with the same effective range parameters. We determine what fraction of the effect of Tabakin's short-range repulsion is included when we replace the Yamaguchi shape by the UPA for Tabakin's potential.

Suppose the energy B and the two-body momentum-space wave function $\langle p \mid B\rangle$ are known. The UPA separable potential has the form [with unknown form factor $f_u(p)$ and strength λ_u]

$$\langle p \mid V \mid k\rangle=-\lambda_u f_u(p) f_u(k). \tag{2}$$

We use units $\hbar=M=1$. We substitute (2) in the momentum-space Schrödinger equation, and solve[31] for the form factor $f_u(p)$ and strength λ_u in terms of the two-body binding energy B and the momentum-space wave function of the bound state $\langle p \mid B\rangle$. The t matrix corresponding to Eq. (2) is

$$t(p,k;s)=-f_u(p)f_u(k)/D_u(s), \tag{3}$$

$$f_u(p)=-N(p^2+B)\langle p \mid B\rangle, \tag{4}$$

$$D_u(s)=\lambda_u^{-1}+4\pi\int\frac{f_u^2(p)\,p^2dp}{s-p^2}. \tag{5}$$

The normalization N in (4) is arbitrary, and the strength λ_u is determined by the relation $D_u(-B)=0$.

If the ground-state wave function $\langle p \mid B\rangle$ has the Hulthén form, we recover the Yamaguchi form for $f_u(p)$; this merely amounts to taking Yamaguchi's original calculation[31] and reading it backwards from the conclusion to the first equation.

We obtain something new for $f_u(p)$ if we choose another form for $\langle p \mid B\rangle$, e.g., that for a local square well[11] or that for Tabakin's[29] rank-2 separable potential, which includes an attractive part plus a hard-shell repulsion:

$$\langle p \mid V \mid k\rangle=-\lambda g(p)g(k)+\lambda_c h(p)h(k), \tag{6}$$

where

$$g(p)=(p^2+\beta^2)^{-1}, \qquad h(p)=p^{-1}\sin pc.$$

This potential gave Tabakin the following t matrix:

$$t(p,k;s)=N(p,k;s)/D(s), \tag{7}$$

where

$$N(p,k;s)=-[\lambda_c^{-1}-M(s)]g(p)g(k) -L(s)[g(p)h(k)+g(k)h(p)]+[\lambda^{-1}+J(s)]h(p)h(k) \tag{8}$$

and

$$D(s)=[\lambda^{-1}+J(s)][\lambda_c^{-1}-M(s)]+L^2(s). \tag{9}$$

[27] C. Lovelace, Phys. Rev. **135**, B1225 (1964).
[28] M. G. Fuda, Nucl. Phys. **A116**, 83 (1968).
[29] F. Tabakin, Phys. Rev. **137**, B75 (1965).
[30] F. Tabakin, Phys. Rev. **174**, 1208 (1968).
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TABLE II. t-matrix parameters for Tabakin's model potential.

t-matrix	λ (F^{-3})	β (F^{-1})	λ_c (F^{-1})	c (F)
Tabakin[a]	0.4332	1.324	9.780	0.182
Yamaguchi	0.1822	1.149	0	0

[a] See Ref. 29 and Eq. (6) for notation. The Yamaguchi potential has the same effective range parameters as Tabakin's potential.

The functions $J(s)$, $L(s)$, and $M(s)$ are given by the integrals

$$J(s)=\int_0^\infty \frac{g^2(p)4\pi p^2dp}{s-p^2}, \tag{10}$$

$$L(s)=\int_0^\infty \frac{g(p)h(p)4\pi p^2dp}{s-p^2}, \tag{11}$$

$$M(s)=\int_0^\infty \frac{h^2(p)4\pi p^2dp}{s-p^2}. \tag{12}$$

The parameters in the potential (6) were adjusted by Tabakin to fit a set of phase shifts, which were obtained by averaging the $^1S_0$ and $^3S_1$ phase shifts. He obtained the Yamaguchi-form t matrix by letting $\lambda_c\to 0$ and readjusting λ and β so as to fit the parameters $B=0.43$ MeV and effective range 2.34 F. We construct the UPA to the Tabakin t matrix by following the theory outlined previously. Our t matrix, therefore, has the same residue at the bound-state pole as Tabakin's. It is given by

$$t_u(p,k;s)=-f_u(p)f_u(k)/[\lambda_u^{-1}+J_u(s)], \tag{13}$$

where

$$f_u(p)=g(p)+\frac{L(-B)}{\lambda_c^{-1}-M(-B)}h(p)$$
$$=g(p)-0.612h(p), \tag{14}$$

$$J_u(s)=J(s)+\frac{2L(-B)L(s)}{\lambda_c^{-1}-M(-B)}$$
$$+\frac{L(-B)}{[\lambda_c^{-1}-M(-B)]^2}M(s)$$
$$=J(s)-1.224L(s)+0.374M(s), \tag{15}$$

and

$$\lambda_u^{-1}=\lambda^{-1}-\lambda_c^{-1}\frac{L^2(-B)}{[\lambda_c^{-1}-M(-B)]^2}=2.270. \tag{16}$$

The parameters appearing in the three t matrices (Tabakin, Yamaguchi, and UPA) are given in Table II. The phase shifts δ are calculated from the standard relation

$$(-ik+k\cot\delta)^{-1}=-2\pi^2t(k,k;k^2+i\epsilon) \tag{17}$$

and are presented in Table III. By construction, Tabakin's t matrix and the UPA to it yield the same deuteron wave function. This implies that these two t matrices have the same low-energy off-shell matrix elements. It is seen from Table III that they have the same on-shell matrix elements up to about 80 Mev (lab). (Note that the UPA phase shift does not go through zero.) All of this indicates that the UPA is a reasonable one. The Yamaguchi t matrix agrees with the other two on the energy shell only up to approximately 20 MeV (lab).

We now compare the three-body binding energies that arise from these three t matrices. Our numerical method for solving the Faddeev equation is discussed in the Appendix. Tabakin[29] has obtained a numerical solution for the ground state for his t matrix and for the Yamaguchi t matrix. We have checked his result for the Yamaguchi t matrix, and have obtained a numerical solution for the UPA, Eq. (13). The three-body energies resulting from the three t matrices are -9.36 MeV for Yamaguchi's, -8.60 MeV for the UPA, and -8.40 MeV for Tabakin's. Assuming Tabakin's t matrix is the most realistic, we see that the Yamaguchi t matrix, which contains no repulsion and fits only the low-energy data, overbinds the Triton by 0.96 MeV, or about 11%. On the other hand, the UPA yields a three-body binding energy which differs from Tabakin's result by only 0.2 MeV, or $2\frac{1}{2}$%. All of this indicates that the UPA appears to be a reasonable one, whereas a t matrix which is adjusted to fit only the effective range parameters cannot be expected to give satisfactory results for the three-body binding energy.

We now consider the singlet rank-2 separable potentials of Tabakin[23] and Schrenk and Mitra.[19] The ^1S state has an antibound state instead of the bound state for which we developed the UPA. If the functions $J(s)$, $L(s)$, and $M(s)$ [Eqs. (10)–(12)] are known in analytical form, we may use the same

TABLE III. Phase shifts (in radians), calculated from Eqs. (6)–(18) using the parameters given in Table II.

Energy (lab) (MeV)	Tabakin	UPA	Yamaguchi
0.0	3.142	3.142	3.142
20	1.157	1.160	1.170
40	0.8766	0.8842	0.9111
60	0.6991	0.7129	0.7564
80	0.5686	0.5894	0.6483
100	0.4657	0.4944	0.5672
120	0.3813	0.4185	0.5035
140	0.3101	0.3564	0.4521
160	0.2488	0.3046	0.4095
180	0.1951	0.2609	0.3737
200	0.1474	0.2236	0.3432
220	0.1048	0.1916	0.3169
240	0.06617	0.1640	0.2939
260	0.03099	0.1400	0.2737
280	−0.0013	0.1191	0.2558
300	−0.03108	0.1010	0.2399
320	−0.05873	0.08512	0.2256
340	−0.08451	0.07131	0.2127

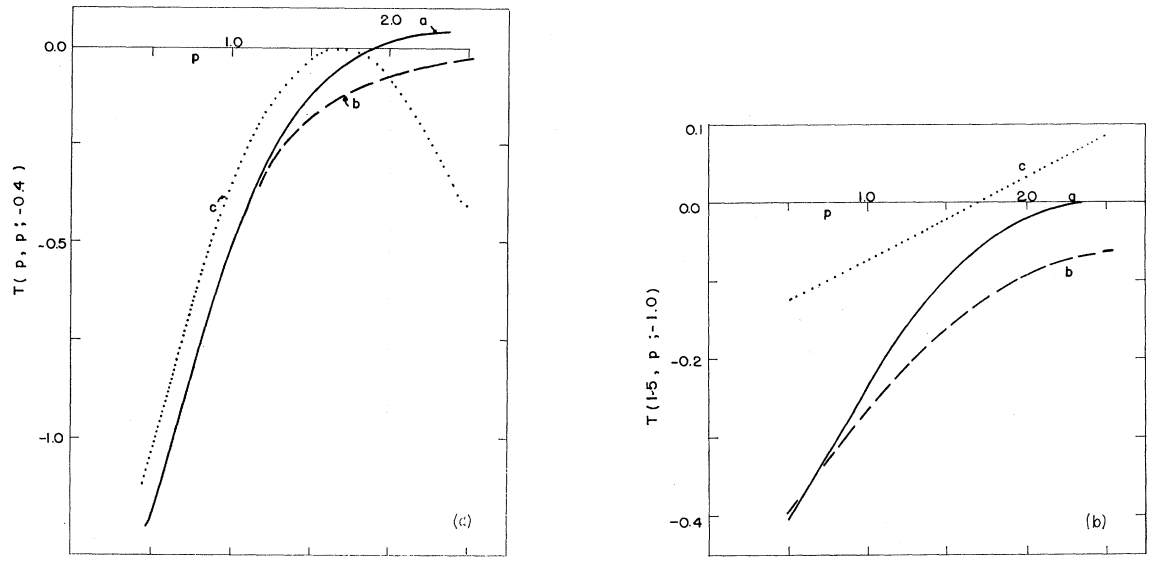

FIG. 1. (a) Diagonal elements of the two-body t matrices $t(p, p; s)$ for energy $s=-0.4\hbar^2/M$ versus momentum p in F^{-1}. See Table IV. (b) Values of $t(1.5, p; -1.0)$ versus momentum p. The solid curve a presents values for Tabakin's (Ref. 23) rank-2 separable fit to singlet phase shifts, the dashed curve b shows the UPA, Eq. (19), and the dotted curve c shows the recent (Ref. 30) Tabakin equation (21).

equations for a bound state and for an antibound state provided that we are careful concerning certain signs in the latter case. We have already made this extension of the UPA in our work on the square well,[11] changing $B^{1/2}$ for a bound state to read $-B^{1/2}$ for an antibound state.

We consider first Tabakin's[23] fit to the ^1S phase shifts: He used a rank-2 separable potential of the form (6) with the same $g(p)$, but with $h(p)$ of (6) replaced by

$$h_T(p)=p^2[(p-d)^2+b^2]^{-1}[(p+d)^2+b^2]^{-1}, \quad (18)$$

with $\beta^{-1}=0.834$ F, $b^{-1}=0.801$ F, $d^{-1}=0.694$ F. The strengths λ and λ_c in (6) are given by Tabakin.[23]

As in Eq. (14), the UPA form factor $f_u(p)$ is a linear combination of $g(p)$ and $h_T(p)$:

$$f_u(p)=g(p)-0.2863h_T(p). \quad (19)$$

The number -0.2863 comes from an evaluation of the coefficient of $h_T(p)$ in (14), at the pole on the second sheet. Also, $1/\lambda_u=5.66$.

We illustrate the accuracy of the UPA for this case in Table IV and Fig. 1 where we compare several examples of Tabakin's[23] singlet t matrix for negative energy with the UPA value based on (19). While Tabakin's t matrix changes sign [corresponding to the short-range repulsion $h_T(p)$] our UPA values do not. We note that our UPA fit to Tabakin's[29] model t matrix did not fit his on-shell values at high energy, but did fit the t matrix well in regions relevant to the three-body energy. We hope that the UPA will work as well in the present case.

The UPA for the Schrenk-Mitra[19] singlet t matrix is found in an analogous manner. The second term $h_s(p)$ in their rank-2 separable potential (their set $G1$) has the form $p^2(p^2+b^2)^{-2}$. The UPA expression replacing (19) is

$$f_u(p)=g(p)-1.308p^2(p^2+b^2)^{-2}, \quad (20)$$

with $b=2.317$ F^{-1}. The strength is $1/\lambda_u=4.49$.

Another method of fitting ^1S phase shifts by a one-term separable potential was recently suggested by Tabakin.[30] He uses a form factor $S(p)$ which goes through zero at $p=k_c$:

$$S(p)=\alpha(k_c^2-p^2)(p^2+d^2)(p^2+b^2)^{-1}(p^4+a^4)^{-1}, \quad (21)$$

with $\alpha^2=400$ F^{-3}, $a=4.05$ F^{-1}, $b=1.09$ F^{-1}, $k_c=1.7$ F^{-1}, and $d=1.68$ F^{-1}. The denominator function in the expression for $\tan\delta(k)$ is adjusted so that it also goes through zero at k_c. Tabakin argues that the phase shift $\delta(k)$ then has a node at $k=k_c$, as desired.

We use Tabakin's[30] singlet form factor, despite two objections to his proposal. First, since the denominator function goes through zero at energy $s=\hbar^2k_c^2/M$, his fit involves a resonance in the continuum around 280-MeV lab energy in disagreement with experiment. Tabakin's choice $S(k_c)=0$ means that the resonance does not couple directly to the continuum. However, Bolsterli[32] argues that other processes, such as coupling to the electromagnetic field, would give the resonance a small but nonzero width, so that the resonance should show up in phase-shift measurements; that is, $\delta(k)$ would rapidly change from zero through $\frac{1}{2}\pi$ to π (instead of going through zero). We suggest

[32] M. Bolsterli, Phys. Rev. 182, 1095 (1969).

TABLE IV. Comparison of singlet two-body t matrices.

s	0.0	−0.2	−0.4	−1.0	−2.0	−3.0
$t^{a}(0.5, 0.5; s)$	−9.74	−1.41	−1.19	−1.00	−0.91	−0.87
$t^{b}(0.5, 0.5; s)$	−9.74	−1.41	−1.19	−1.00	−0.91	−0.88
$t^{c}(0.5, 0.5; s)$	−10.1	−1.28	−1.04	−0.78	−0.61	−0.52
$t^{a}(0.5, 1.5; s)$	−3.82	−0.55	−0.47	−0.40	−0.37	−0.36
$t^{b}(0.5, 1.5; s)$	−3.83	−0.55	−0.47	−0.40	−0.36	−0.34
$t^{c}(0.5, 1.5; s)$	−1.66	−0.21	−0.17	−0.13	−0.10	−0.08
$t^{a}(0.5, 2.5; s)$	−1.50	−0.22	−0.19	−0.16	−0.15	−0.15
$t^{b}(0.5, 2.5; s)$	−1.51	−0.22	−0.19	−0.16	−0.14	−0.14
$t^{c}(0.5, 2.5; s)$	6.39	0.81	0.66	0.49	0.38	0.33
$t^{a}(1.5, 1.5; s)$	−1.43	−0.15	−0.12	−0.094	−0.082	−0.078
$t^{b}(1.5, 1.5; s)$	−1.51	−0.22	−0.18	−0.16	−0.14	−0.14
$t^{c}(1.5, 1.5; s)$	−0.271	−0.034	−0.028	−0.021	−0.016	−0.014
$t^{a}(1.5, 2.5; s)$	−0.522	−0.019	−0.007	0.002	0.006	0.008
$t^{b}(1.5, 2.5; s)$	−0.594	−0.086	−0.073	−0.061	−0.056	−0.053
$t^{c}(1.5, 2.5; s)$	1.04	0.132	0.107	0.080	0.065	0.053
$t^{a}(2.5, 2.5; s)$	−0.17	0.033	0.037	0.041	0.043	0.044
$t^{b}(2.5, 2.5; s)$	−0.23	−0.033	−0.029	−0.024	−0.022	−0.021
$t^{c}(2.5, 2.5; s)$	−4.02	−0.51	−0.41	−0.31	−0.25	−0.21

[a] Rank-2 separable, Tabakin, Ref. 23.
[b] Rank-1 separable UPA, Eq. (19).
[c] Rank-1 separable, Tabakin, Ref. 30; $t(p, k, s)$ has momenta p and k in F^{-1} and energy s in units of $\hbar^2/M$.

that the resonance should also show up in other processes involving off-shell t-matrix elements, such as p-p bremsstrahlung with the resonance as a final state, or magnetic dipole photodisintegration of the deuteron leading to the resonant ^1S state.

Our second objection is that Tabakin's[30] form factor corresponds to a spatial wave function with a node at low energy (in the ^1S case, an antibound state). This $2s$ state *cannot* be the lowest state for a local, or approximately local, potential.[33] Thus Tabakin's fit will give off-shell t-matrix elements near the pole on the second sheet very different from those for an approximately local potential. This difference is illustrated in Fig. 1 and Table IV, where we compare Tabakin's[30] off-shell values of $t(p, k; s)$ with those for Tabakin's[23] rank-2 separable potential, and for the corresponding UPA, Eq. (19).

While the UPA does not give the node in the phase shift $\delta(k)$ at $k=k_c$, given by Tabakin[23] and Eq. (21), the UPA agrees much better with Tabakin's[23] $t(p, k; s)$ at the negative s relevant to our problem.

Mongan[34] has recently given a similar objection to Tabakin's[30] recent prescription. Mongan compares the Noyes $f(p, k)$ function for Tabakin's[30] t matrix with that for Reid's soft-core singlet potential.[18]

For the central term $C(p)$ in the spin-triplet potential, we attempt to take account of possible soft-core effects in the nucleon-nucleon interaction by constructing the UPA to a Hulthén type of potential modified to include a centrifugal potential which suppresses the configuration-space wave function near the origin. The form factor is

$$C_M(p) = (p^2+\beta_M^2)^{-1} - C_1(p^2+\alpha^2)(p^2+\beta_M^2)^{-2}, \quad (22)$$

where α is related to B in the usual manner, and $C_1 = 2\beta_t(\alpha+\beta_t)^{-1}$. β_M is determined from the effective range. The second term in the form factor causes the reduced wave function $u(r)$ to behave as r^2 near the origin, whereas the Hulthén potential gives a behavior linear in r.

III. RESULTS FOR TRINUCLEON ENERGY

Following Yamaguchi's notation[7] we write the spin-triplet rank-1 separable potential as

$$\langle \mathbf{p} \mid V \mid \mathbf{k} \rangle = -\lambda g_t(\mathbf{p}) g_t(\mathbf{k}),$$
$$g_t(\mathbf{p}) = C(p) + (8)^{-1/2} S_{12}(\hat{p}) T(p), \quad (23)$$

where S_{12} is the tensor operator, and $\hat{p}$ is a unit vector. We denote the singlet form factor as $S(p)$.

We calculate the trinucleon energy E for three different cases: (i) the Yamaguchi shapes for $S(p)$, $C(p)$, and $T(p)$, with parameters already used by Kharchenko[2] and Phillips,[3] (ii) changes of values of Yamaguchi triplet parameters to study variation of E with deuteron percent D state P_D, and (iii) changes of the singlet shape $S(p)$ and the central-triplet shape $C(p)$ from Sec. II. We also use Tabakin's[30] rank-1 singlet.

The purpose of the first calculation is merely to determine the accuracy of our numerical procedures for finding the energy eigenvalue for three coupled integral equations as outlined in the Appendix. We treat the case of singlet effective range $r_0 = 2.70$ F and $P_D = 4\%$, for which Kharchenko *et al.*[2] and Phillips[3] give $E = -9.18$ and -9.2 MeV, respectively, as shown in Table I.

We first use a 10-point Gauss-Gegenbauer calculation for each integral equation, and obtain $E = -9.01$ MeV. Since our number is not in good agreement with Kharchenko *et al.* and Phillips, we repeat the procedure using 16-point Gauss-Gegenbauer, and again find $E = -9.01$ MeV. The spectator functions also

[33] A. Messiah, *Quantum Mechanics* (Wiley-Interscience, Inc., New York, 1961), Vol. 1, Chap. III, Sec. 12.
[34] T. Mongan, Phys. Rev. (to be published).

agree very well for the coarser and finer mesh. We believe that our numerical accuracy for 10-point integration is much better than 0.1 MeV in the energy. (Kharchenko *et al.* and Phillips do not quote an estimated calculational error). We use 10-point integration for results quoted below.

Note added in proof. V. F. Kharchenko has pointed out that there are small differences between the Phillips parameters and the Kharchenko parameters used to give the results quoted in our Table I. Our value $E=-9.01$ MeV uses Phillips's parameters. S. Bhatt and L. Laroze at Rensselaer have followed Kharchenko's suggestion and have calculated E, using 16-point Gauss-Gegenbauer quadrature, and Kharchenko's parameters: They obtain $E=-9.18$ MeV, in precise agreement with his published result. We thank V. F. Kharchenko for his help.

As discussed in the Introduction, and shown in Table I, Phillips[3] has calculated $E(P_D)$ for $P_D=4$, 5.5, and 7%, i.e., the "accepted range" for the percent D state. However, Mongan's recent fits[24] to phase parameters and deuteron properties (binding energy and quadrupole moment) use values of P_D of 1% or even less, in agreement with the minimum value[26] of 0.45%.

TABLE V. Dependence on percent D state. The parameters t, β_0, and β_t [see Eq. (24)] are chosen to fit the deuteron binding energy, triplet effective range, deuteron quadrupole moment, and percent D state P_D.

Parameters used t (F^{-1})	β_0 (F^{+1})	β_t (F^{-1})	P_D (%)	Three-body energy E (MeV)
0.055	1.400	0.5130	0.78	−10.60
0.1130	1.396	0.6715	1.0	−10.45
0.4698	1.370	1.0694	2.0	−9.97
1.689	1.313	1.528	4.0	−9.01
4.54	1.239	1.952	7.0	−7.94
Experimental value				−8.48

There are two different types of arguments giving values of P_D: (i) other measurements on deuteron properties, such as its magnetic moment, or the form factor for elastic electron-deuteron scattering, and (ii) calculations based on the assumption that the nucleon-nucleon potential is given by one-pion exchange at reasonable distances.[35] Both arguments indicate $P_D\simeq$ 4%, but since the determination of P_D is still subject to controversy it seems worthwhile to explore the region of low values for P_D. We fit the deuteron binding energy and quadrupole moment, and also triplet effective range, by varying λ in (23) and β_0 in $C(p)$ and t and β_t in $T(p)$ below, using Yamaguchi's shapes[7]:

$$C(p)=(p^2+\beta_0^2)^{-1},$$
$$T(p)=-tp^2(p^2+\beta_t)^{-2}. \quad (24)$$

The values of the parameters in (24), and the resulting three-body energy, are given in Table V. (All calculations here and below use a singlet effective range $r_0=2.70$ F.)

In Table VI we return to the usual 4 or 7% D state and use Yamaguchi's tensor shape $T(p)$ from (24). We examine the variation of the three-body binding energy with the shapes of the singlet form factor $S(p)$ and the triplet central $C(p)$. [Our use of Yamaguchi shapes for both $S(p)$ and $C(p)$ duplicates the results of Table V, but are shown again for comparison with other shapes.]

The first four rows of Table VI treat different singlet shapes: (i) Yamaguchi, (ii) the UPA for Tabakin's[23] singlet, from Eq. (19), (iii) the UPA for the Schrenk-Mitra[19] singlet, Eq. (20), and (iv) Tabakin's[30] recent singlet, with a node in $S(p)$, Eq. (21).

The remaining rows use all combinations of two different singlet shapes (Yamaguchi and UPA for Tabakin[23]), two different triplet central shapes [Yamaguchi, and modified Hulthén, from Eq. (22)], and two different values for P_D (4 or 7%). The values

TABLE VI. Three-body energy E for different form factors. The Yamaguchi shapes use Eq. (24) with triplet parameters from Table V. The UPA's for Tabakin (Ref. 23) and Schrenk and Mitra (Ref. 19) are given in Eqs. (19) and (20), respectively. The modified Hulthén $C(p)$ is given in Eq. (22) and Table VII.

Singlet $S(p)$	Triplet central $C(p)$	P_D	E
Yamaguchi	Yamaguchi	4	−9.01
UPA for Tabakin[a]	Yamaguchi	4	−8.76
UPA for Schrenk-Mitra[b]	Yamaguchi	4	−8.65
New Tabakin[c]	Yamaguchi	4	−7.07
Yamaguchi	Modified Hulthén	4	−8.69
UPA, Tabakin[a]	Modified Hulthén	4	−8.55
Yamaguchi	Yamaguchi	7	−7.94
UPA, Tabakin[a]	Yamaguchi	7	−7.83
Yamaguchi	Modified Hulthén	7	−7.67
UPA, Tabakin[a]	Modified Hulthén	7	−7.65
Experiment			−8.48

[a] See Ref. 23.
[b] See Ref. 19.
[c] See Ref. 30.

TABLE VII. Parameters for modified Hulthén potential. β_M is used for the modified Hulthén central shape $C_M(p)$. See Eqs. (22) and (23). The parameters t and β_t are used in the Yamaguchi tensor shape $T(p)$, Eq. (24).

P_D	β_M	t	β_t
4%	2.410	0.504	1.542
7%	2.272	1.36	1.976

[35] D. Y. Wong, Phys. Rev. Letters 2, 406 (1959); A. C. Phillips (private communication).

of the parameter β_M for the modified Hulthén $C_M(p)$ and the corresponding tensor $T(p)$ are given in Table VII.

IV. DISCUSSION

In Tables V and VI we give values of the three-body energy ranging from -10.60 to -7.07 MeV as we vary the shapes and parameters in the two-body form factors $S(p)$, $C(p)$, and $T(p)$. It is clear that we can fit the experimental value of -8.48 MeV by suitable interpolations, but it is equally clear that adjusting several parameters to fit one experimental number is not a satisfactory procedure.

We have so far given incomplete answers to the five types of questions we raised in the Introduction. First, how good is the separable approximation? It seems fairly good, since we are always within 2 MeV of the experimental value of E confirming earlier separable calculations shown in Table I. Second, how sensitive is the three-body energy to values used for r_0 or P_D? Phillips showed that $\Delta r_0 \simeq 0.1$ F gives $\Delta E \simeq 0.2$ MeV. We find (Table V) that $\Delta P_D = 3\%$ gives $\Delta E \simeq 1.2$ MeV.

Third, what shape should we use for the separable potentials? We find that the UPA allows a simple method of including much of the effect of a strong short-range repulsion in the singlet and central triplet potentials, giving approximately $\frac{1}{4}$ MeV for each or $\frac{1}{2}$ MeV altogether. Our values $E=-8.76$ MeV and $E=-8.65$ MeV (Table VI, rows 2 and 3, for UPA expressions for $S(p)$ agree well with Dabrowski and Dworzecka's[20] -8.81 MeV (they used a slightly smaller singlet effective range), but disagree with Schrenk and Mitra's[19] -9.21 MeV for a similar rank-2 separable potential. [A calculation[22] by one of us (T.J.B.) using the Schrenk-Mitra $G1$ potential gives $E=-8.51$ MeV, in reasonable agreement with the UPA result -8.65 MeV of Table VI.] The large change in E (from -9.01 to -7.07 MeV) resulting from the replacement of Yamaguchi's $S(p)$ by Tabakin's[30] recent $S(p)$, and the disagreement of -7.07 MeV with other rank-1 or rank-2 separable potentials, supports our argument (Table IV and Fig. 1) that Tabakin's recent $S(p)$ provides a poor approximation to relevant[33] off-shell t-matrix elements.[36] Table IV and Fig. 1 show that Tabakin[30] gives less two-body attraction than the UPA, and hence less three-body binding.

Though agreement with experiment cannot be taken seriously at this stage, it also cannot be completely ignored. We therefore note that, if one believes that $P_D \simeq 4\%$, then the most realistic separable potential treated is the singlet UPA for Tabakin[23] and a central triplet modified Hulthén; the calculated -8.55 MeV is within 1% of the experimental -8.48 MeV.

Our use of a separable approximation to the true t matrix introduces poorly known errors into our calculations of the three-body energy. It is unclear whether these errors are of order[4,17] 1 MeV, or an order of magnitude smaller.[16] Our lack of knowledge of the value of P_D introduces[4] an error of order 1 MeV in the three-nucleon energy.

It is also of interest to compare our calculations for the 7% D state with the recent variational results of Delves *et al.*[37] for the Hamada-Johnston potential. The last row in Table VI, for UPA Tabakin[23] and modified Hulthén gives $E=-7.65$ MeV, while Delves finds $E=-6.7\pm0.7$ MeV. We have included much, but not all, of the effects of the hard core in the HJ potential, so that this extra repulsion might account for much of the difference of 1.0 ± 0.7 MeV between the two calculations.

We do not present results bearing on questions (iv) and (v) of the Introduction. The present results show that some rank-1 separable potentials that are consistent with known properties of two-nucleon systems are also not inconsistent with the three-body energy. But we have not yet established the validity of the separable approximation.

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APPENDIX: NUMERICAL METHODS

The Faddeev equations for the one-term separable tensor potentials used reduce to a three-coupled one-dimensional integral equation[28]:

$$\chi_i(p) = \mu \sum_j \int K_{ij}(p, k, E)\chi_j(k)\,dk. \qquad \text{(A1)}$$

Here $\mu=1$ and the kernels $K_{ij}(p, k, E)$ depend on the three-body energy E. The form of the kernels K_{ij} depends on the choice of form-factor functions $S(p)$, $C(p)$, and $T(p)$ in (23). Equation (A1) gives us integrals of the form

$$I = \int_0^\infty f(k)\,dk, \qquad \text{(A2)}$$

where $f(k)$ has the asymptotic behavior

$$f(k) \propto k^2, \qquad \text{as } k \to 0$$

$$f(k) \propto k^{-4}, \qquad \text{as } k \to \infty. \qquad \text{(A3)}$$

[36] R. D. Purrington and R. W. Peacock [Bull. Am. Phys. Soc. **14**, 512 (1969)] use Tabakin's recent $S(p)$ and also his triplet potential (Ref. 30). They find a very low triton binding energy. [*Note added in proof.* Also see V. A. Alessandrini and C. A. Garcia Canal, Nucl. Phys. **A133**, 590 (1969); J. E. Beam, Phys. Letters **30B**, 67 (1969).]

[37] L. M. Delves, J. M. Blatt, C. Pask, and B. Davies, Phys. Letters **28B**, 472 (1969).

TABLE VIII. Positions and weights for n-point Gegenbauer quadrature. See Eqs. (A2), (A3), and (A7). The n positions are given by k_i and the n weights by w_i for the infinite interval: $\int_0^\infty f(k)\,dk \simeq \sum_{i=1} w_i f(k_i)$.

	$n=6$		$n=10$		$n=16$	
i.	k_i	w_i	k_i	w_i	k_i	w_i
1.	0.09899409345	0.1386181340	0.04378098847	0.05922128689	0.01958871105	0.02608494414
2.	0.2981916005	0.2720070362	0.1236163405	0.1024149443	0.05379905793	0.04270701293
3.	0.6825422627	0.5280526456	0.2539473905	0.1618227592	0.1059194898	0.06210103142
4.	1.465110741	1.133491133	0.4570761906	0.2510869418	0.1794099499	0.08574346994
5.	3.353548518	3.059069375	0.7748434802	0.3975126219	0.2795360297	0.1158108085
6.	10.10161278	14.14495215	1.290583228	0.6620990380	0.4142529218	0.1556044983
7.	...	...	2.187819056	1.201840759	0.5957004078	0.2103760850
8.	...	...	3.937823493	2.509297149	0.8428531032	0.2889024229
9.	...	...	8.089545409	6.702110243	1.186446364	0.4066749330
10.	...	...	22.84096442	30.89631720	1.678696182	0.5928441983
11.	...	...	...	...	2.413984180	0.9067571465
12.	...	...	...	...	3.577356383	1.482086354
13.	...	...	...	...	5.573826874	2.663839198
14.	...	...	...	...	9.441133094	5.535375069
15.	...	...	...	...	18.58768608	14.75536153
16.	...	...	...	...	51.04981117	67.97953523

This behavior suggests

$$f(k)=k^2g(k)/(1+k)^6, \tag{A4}$$

so that $g(k)$ should be a much smoother function than $f(k)$ and hence could be integrated numerically with greater accuracy. Transforming the integral (A2) to the interval $-1\leq x\leq 1$, by using $k=(1+x)/(1-x)$, we find

$$I=(32)^{-1}\int_{-1}^{1}(1-x^2)^2 g[(1+x)/(1-x)]dx. \tag{A5}$$

Standard numerical analysis then tells us to look for the polynomials which are orthogonal on $-1\leq x\leq 1$ with weight functions

$$\rho(x)=(1-x^2)^2. \tag{A6}$$

These functions are the Gegenbauer polynomials[38] of order $\frac{5}{2}$. The integration points are then the zeros of these polynomials and the weights are related to their derivatives. That is, we approximate the integral (A2) by the finite sum

$$I\simeq\sum_{i=1}^{n} w_i f(k_i). \tag{A7}$$

The positions k_i and weights w_i are tabulated in Table VIII for $n=6$, 10, and 16.

Gegenbauer integration constitutes an improvement[39] over the usual Gaussian integration formula by a factor of from 2 to 10, depending on the type of function considered. For the calculations in this paper we have generally used 10-point Gegenbauer integration for the momentum integrations. In one calculation we repeated our work with 16-point Gegenbauer. [The angular integrations to find the kernels in (A1) were done by 10-point Gaussian integration.]

We now replace the three-coupled integral equations by matrix equations of dimensionality $3N$. The kernels of the integral equation (A1) contain the three-body energy E as a parameter; we must determine E so that the integral equation holds. We guess an E, and solve for $\mu(E)$ in Eq. (A1). If $\mu(E)$ is not unity (to better than 0.01%), we guess another E and repeat as needed. The value of $\mu(E)$ is found either by Jacobi diagonalization of the matrix equation; or by the faster process of successive matrix multiplications to find the largest eigenvalue μ. We also tabulate[39] the spectator functions $\chi_i(p)$.

The numerical procedure is quite fast if we use the Rensselaer IBM 360–50 computer with 16-digit accuracy. We can make four successive determinations of $\mu(E)$ in 6 min, using 10-point Gauss-Gegenbauer integration; i.e., the matrix replacing (A1) is 30×30.

[38] U. W. Hochstrasser, in *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (Dover Publications, Inc., New York, 1965), Chap. 22.

[39] R. Stagat, Ph.D. thesis, Rensselaer Polytechnic Institute, 1969 (unpublished).