Energy Straggling of Heavy Charged Particles in Thick Absorbers

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The theory of energy straggling attempts to calculate $F(E,S)$, where $F(E,S)dE$ is the fraction of the heavy charged particles which have an energy between E and $E+dE$ after a path length S has been traversed in absorbing medium. This paper develops a method of calculating $F(E,S)$ for path lengths large enough so that $F(E,S)$ is almost Gaussian. The method remains valid until a large fraction of the particles run out of energy. The theory is applied to calculations of $F(E,S)$ for 50-MeV protons in Be and for 5.3-MeV α particles in air. The calculations for α particles in air are in good agreement with the experimental results of Rotondi and Gieger. The theory is also in agreement with numberical calculations by Tschalär.

I. INTRODUCTION

We consider a parallel beam of heavy charged particles incident on a plane layer of homogeneous absorber. We will calculate the function $F(E,S)$, where $F(E,S)dE$ is the fraction of the particles with energy between E and $E+dE$ after a path length S has been traversed in the absorbing medium. In calculating $F(E,S)$, electron capture will be neglected.

It can be shown that

$$\frac{\partial F(E,S)}{\partial S}=-\int_0^\infty P(E,t)F(E,S)dt+\int_0^\infty P(E+t,t)F(E+t,S)dt, \quad (1)$$

where $P(E,t)dtdS$ is the probability that a heavy charged particle with energy E will lose an amount of energy in the interval t–$t+dt$ in traversing a thickness of absorber dS. Almost all previous work has been based on Eq. (1).[1] We begin by discussing the earlier work on energy straggling. In all of these theories it is assumed that the initial beam is monoenergetic.

A. Thin Absorber Approximation

Vavilov[2] has solved the energy straggling problem for absorbers which are so thin that one can replace $P(E,t)$ and $P(E+t,t)$ in Eq. (1) by $P(E_0,t)$, where E_0 is the initial energy. Once this approximation is made, one can easily solve for the Laplace transform of $F(E,S)$. The numerical evaluation of the inverse transform yields the approximate $F(E,S)$.

When relativistic corrections are small, the Vavilov approximation begins to fail for reductions in average energy greater than 10%. In the case of protons or α particles, $F(E,S)$ becomes almost Gaussian by the time this approximation becomes inaccurate.

The Vavilov theory, the best theory for thin absorbers, replaces more approximate theories by Landau[3] and by Symon.[4]

B. Intermediate Absorber Thicknesses

At small S, the energy spectrum is strongly skewed, with a long tail toward lower energies. As S is increased, the skewness becomes smaller and $F(E,S)$ approaches a Gaussian.[1,2,4] If an absorber is thick enough so that $F(E,S)$ is almost Gaussian, but thin enough so that the full width at half-maximum is small compared with the mean energy, we say that its thickness is intermediate.

The part of Symon's work[4] which deals with absorbers of intermediate thickness has not been published. Judging from the continued use[1,5–7] of the more approximate Bohr theory[8] in this region, it seems that many workers are not aware that Symon's theory is much more complete than that of Bohr. For this reason we will now outline the Symon approximation.

Symon makes use of the fact that a statistical distribution function can be calculated from its central moments.[9] In particular, if the distribution function does not differ greatly from a Gaussian, then only the first few central moments need be known. (The actual construction of a distribution function from its central moments will be discussed at a later point in the present study.)

The equations to be derived in this section are due to Symon, but we will also use them in order to discuss the Bohr theory. We define the central moments $A_n(S)$ by

$$A_n(S)=\int_0^\infty [E-\langle E\rangle]^n F(E,S)dE, \quad (2)$$

where

$$\langle E\rangle=\int_0^\infty EF(E,S)dE. \quad (3)$$

[1] U. Fano, Ann. Rev. Nucl. Sci. **13**, 1 (1963).
[2] P. V. Vavilov, Zh. Eksperim. i Teor. Fiz. **32**, 920 (1957) [English transl.: Soviet Phys.—JETP **5**, 749 (1957)].
[3] L. Landau, J. Phys. USSR **8**, 201 (1944).
[4] K. R. Symon, Ph.D. thesis, Harvard University, 1948 (unpublished).
[5] J. R. Comfort, J. F. Decker, E. T. Lynck, M. O. Scully, and A. R. Quinton, Phys. Rev. **150**, 249 (1966).
[6] D. L. Mason, R. M. Prior, and A. R. Quinton, Nucl. Instr. Methods **45**, 41 (1966).
[7] E. Segrè, *Nuclei and Particles: An Introduction to Nuclear and Subnuclear Physics* (W. A. Benjamin, Inc., New York, 1965).
[8] N. Bohr, Phil. Mag. **30**, 581 (1915).
[9] H. Cramer, *Mathematical Methods of Statistics* (Princeton University Press, Princeton, N. J., 1946).

On differentiating Eq. (2) we find

$$\frac{dA_n}{dS}=-n\frac{d\langle E\rangle}{dS}A_{n-1}+\int_0^\infty [E-\langle E\rangle]^n\frac{\partial F(E,S)dE}{\partial S}.$$

If we use Eq. (1), we find that

$$\frac{dA_n}{dS}=-n\frac{d\langle E\rangle}{dS}A_{n-1}+\sum_{L=1}^{n}\frac{n!(-1)^L}{(n-L)!}\int_0^\infty M_L(E)\times(E-\langle E\rangle)^{n-L}F(E,S)dE, \quad (4)$$

where

$$M_L(E)=\frac{1}{L!}\int_0^\infty P(E,t)t^L dt. \quad (5)$$

Further details concerning the derivation of Eq. (4) may be found in Symon's thesis.

Equation (4) provides a good starting point for understanding all intermediate path length theories. We first consider the Bohr theory which argues from the central limit theorem[9] that $F(E,S)$ will become Gaussian. Once $F(E,S)$ reaches the Gaussian limit,

$$F(E,S)\approx[2\pi A_2(S)]^{-1/2}\exp[-(E-\langle E\rangle)^2/2A_2(S)].$$

Note that only $\langle E\rangle$ and $A_2(S)$ are needed to determine $F(E,S)$. Bohr argues that the Gaussian limit will be reached while the full width at half-maximum is still very small compared with the mean energy. When $n=1$, Eq. (4) yields

$$0=-\frac{d\langle E\rangle}{dS}-\int_0^\infty M_1(E)F(E,S)dE, \quad (6)$$

where we have used

$$A_0(S)\equiv 1, \quad A_1(S)\equiv 0.$$

When $n=2$, Eq. (4) yields

$$\frac{dA_2}{dS}=-2\int_0^\infty M_1(E)(E-\langle E\rangle)F(E,S)dE+2\int_0^\infty M_2(E)F(E,S)dE. \quad (7)$$

The Bohr approximation assumes that $F(E,S)$ is so sharply peaked about $E=\langle E\rangle$ that $M_1(E)$ and $M_2(E)$ are essentially constant over the region where one obtains a contribution to the integrals in Eqs. (6) and (7). Consequently,

$$\frac{d\langle E\rangle}{dS}\approx -M_1(\langle E\rangle), \quad \frac{dA_2}{dS}\approx 2M_2(\langle E\rangle),$$

and

$$A_2(S)=2\int_{\langle E\rangle}^{E_0}\frac{M_2(E)dE}{M_1(E)}, \quad (8)$$

where E_0 is the initial energy. If we use the classical expression for $P(E,t)$, we obtain Bohr's original result.

Symon allows for deviations from a Gaussian form. He also derives the A_n's in a better approximation. We will now derive $A_2(S)$ in Symon's approximation. In evaluating the integrals on the right-hand side of Eqs. (6) and (7) we use

$$M_1(E)\approx M_1(\langle E\rangle)+\frac{dM_1}{dE}(\langle E\rangle)(E-\langle E\rangle)$$

and

$$M_2(E)\approx M_2(\langle E\rangle)+\frac{dM_2}{dE}(\langle E\rangle)(E-\langle E\rangle).$$

We obtain

$$\frac{d\langle E\rangle}{dS}=-M_1(\langle E\rangle),$$

$$\frac{dA_2}{dS}=-2\frac{dM_1}{dE}(\langle E\rangle)A_2+2M_2(\langle E\rangle).$$

On solving the last two equations simultaneously, we obtain

$$A_2(S)=\left[\frac{M_1(\langle E\rangle)}{M_1(\langle E\rangle_0)}\right]^2\times\left\{A_2(0)+2\int_{\langle E\rangle}^{\langle E\rangle_0}\frac{M_2(E)}{M_1(E)}\left[\frac{M_1(\langle E\rangle_0)}{M_1(E)}\right]^2 dE\right\}. \quad (9)$$

In the case of nonrelativistic protons or α particles, Eq. (9) begins to differ appreciably from Eq. (8) by the time the mean energy is reduced by 20%. Once the mean energy is reduced by 50%, Eq. (9) yields an A_2 which is larger by more than a factor of 2.

To understand the difference between the Bohr and Symon approximations, note that $M_1(E)$ is the average rate of energy loss by particles with energy E. When we replace the $M_1(E)$ by $M_1(\langle E\rangle)$, we assume that the average rate of energy loss by all particles is the same. When Symon includes a linear variation of $M_1(E)$, he includes that fact that the width of $F(E,S)$ changes not only because of statistical fluctuations but also because particles with different energies lose energy at different average rates. We shall eventually show that the nonstatistical type of change in $F(E,S)$ completely dominates its evaluation at large S.

C. Thick Absorbers

Once S is made so large that it is not correct to assume that the full width of $F(E,S)$ at half-maximum is small compared with $\langle E\rangle$, we say that the absorber is thick. In this region the Symon approximation becomes poor.

One might think that Eq. (4) could be used in this region. However, the $M_n(E)$ vary so much over the region where $F(E,S)$ is nonzero that the integral in Eq. (4) cannot be expressed in terms of a reasonable

number of A_n's. None of the previous theories holds in this region.

The purpose of this paper is to show how to calculate $F(E,S)$ throughout the intermediate and thick absorber regions.

II. QUALITATIVE DISCUSSION OF THICK ABSORBER LIMIT

At large path lengths, where $\sqrt{A_2}$ is large compared with the maximum possible energy loss in any single collision, the collision integral in Eq. (1) can be expanded as

$$\int_0^\infty [P(E+t,t)F(E+t,S)-P(E,t)F(E,S)]dt = \sum_{k=1}^{\infty} \frac{\partial^k}{\partial E^k}[M_k(E)F(E,S)],$$

by a generalization of the Fokker-Planck expansion. If $P(E,t)$ were analytic, one could arrive at the expansion by simply using a Taylor series for $P(E+t,t)F(E+t,S)$. In any case, the infinite sum converges very rapidly once the full width at half-maximum of $F(E,S)$ becomes much larger than the maximum possible energy loss in any single collision by a particle with the mean energy. Equation (1) becomes

$$\frac{\partial F(E,S)}{\partial S}=\sum_{k=1}^{\infty}\frac{\partial^k}{\partial E^k}[M_k(E)F(E,S)].$$

In the case of α particles or protons at nonrelativistic energies, the energy spectrum becomes essentially Gaussian by the time the mean energy is reduced by 10%. The spectrum remains nearly Gaussian until the mean energy is reduced to less than 35% of its initial value. Hence, to a good approximation only $A_2(S)$ and $\langle E\rangle$ are required for the calculation of $F(E,S)$. Symon's expressions for $\langle E\rangle$ and $A_2(S)$ are still valid when the spectrum becomes nearly Gaussian. Hence, one would obtain the same values for $\langle E\rangle$ and $A_2(S)$ if $M_3(E)$, $M_4(E)$, etc., were all set equal to zero. This suggests that the Fokker-Planck equation

$$\frac{\partial F(E,S)}{\partial S}=\frac{\partial}{\partial E}[M_1(E)F(E,S)]+\frac{\partial^2}{\partial E^2}[M_2(E)F(E,S)],$$

is valid for proton or α particles once the mean energy has been reduced by more than 10%. In the case of lighter particles, $\langle E\rangle$ must be reduced somewhat farther before this equation becomes valid.

We will now investigate the possibility that even the second term will eventually become unimportant, so that

$$\frac{\partial F(E,S)}{\partial S}\approx\frac{\partial}{\partial E}[M_1(E)F(E,S)]. \tag{10}$$

To see that Eq. (10) may become valid for thick absorbers we let

$$P(E,t)=K/Et^2, \quad \text{if } t_{\min}\leqq t\leqq t_{\max}$$
$$=0, \quad \text{otherwise.} \tag{11}$$

Now, $K=\pi NZz^2e^4M/m$, N is the number of absorber atoms per unit volume, Z is the number of electrons per atom, z is the number of electronic charges of the incident particle, e is the electron charge, M is the mass of an incident particle, m is the electron mass, $t_{\max}\approx(4m/M)E$, $t_{\min}=I^2/t_{\max}$, and I is the mean excitation potential of the absorber molecules as defined by Fano.[1] This form of $P(E,t)$ neglects relativistic effects and inner shell corrections.

If we substitute the approximate $P(E,t)$ into Eq. (5), we find

$$M_1(E)=(2K/E)\ln(4mE/MI)$$

and

$$M_2(E)=(2m/M)K.$$

In evaluating $M_2(E)$ we have assumed that $t_{\max}\gg t_{\min}$. The latter condition is necessary for this form of $P(E,t)$ to be accurate. To simplify the mathematics we approximate as follows:

$$M_1(E)\approx\frac{2K}{\langle E\rangle_0}\left(\frac{\langle E\rangle_0}{E}\right)^a\ln\left(\frac{4m\langle E\rangle_0}{MI}\right),$$

where

$$a=1-[\ln(4m\langle E\rangle_0/MI)]^{-1}.$$

If $4m\langle E\rangle_0/MI=20$, the approximate $M_1(E)$ represents the original function with an error smaller than 5% for $0.5\langle E\rangle_0\leqq E\leqq 1.5\langle E\rangle_0$. The approximate $M_1(E)$ is good enough for use in what follows.

If $A_2(0)=0$, Eq. (9) becomes

$$A_2(S)=2\int_{\langle E\rangle}^{E_0}\frac{M_2(E)}{M_1(E)}\left(\frac{M_1(\langle E\rangle)}{M_1(E)}\right)^2dE,$$

where E_0 is the initial energy. We find

$$A_2(S)=\frac{2m}{M}E_0^2\left[(3a+1)\ln\left(\frac{4mE_0}{MI}\right)\right]^{-1}\times\left(\frac{M_1(\langle E\rangle)}{M_1(E_0)}\right)^2\left[1-\left(\frac{\langle E\rangle}{E_0}\right)^{3a+1}\right].$$

Consider Symon's equation for A_2:

$$\frac{dA_2}{dS}=-2\frac{dM_1(\langle E\rangle)}{d\langle E\rangle}A_2+2M_2(\langle E\rangle).$$

We will now determine how much $\langle E\rangle$ must be reduced before the nonstatistical term becomes ten times as

large as the statistical term. We have

$$\frac{2M_2(\langle E\rangle)}{2|dM_1(\langle E\rangle)/d\langle E\rangle|A_2}\leqq\frac{1}{10},$$

or

$$\frac{\langle E\rangle}{E_0}\leqq\left[\frac{1-[\ln(4m\langle E\rangle/MI)]^{-1}}{20-15[\ln(4mE_0/MI)]^{-1}}\right]^{1/4}\times\left[\frac{\ln(4m\langle E\rangle/MI)}{\ln(4mE_0/MI)}\right]^{3/4}\left[1-\left(\frac{\langle E\rangle}{E_0}\right)^{3a+1}\right]^{1/4}.$$

If $4mE_0/MI=50$, one finds $\langle E\rangle M0.35E_0$. When $4mE_0/MI>50$, it is found that $\langle E\rangle$ does not have to be reduced quite so far. We will now check to see if Symon's equation is still valid at this point. Let $4mE_0/MI=50$ and $\langle E\rangle=0.35E_0$. Then $(A_2)^{1/2}/\langle E\rangle\approx2.5(m/M)^{1/2}$. Symon's equation is still valid for protons or α particles. However, it is questionable for pions and muons. We then see that in many cases of interest $A_2(S)$ starts to evolve according to

$$\frac{dA_2}{dS}\approx-2\frac{dM_1(\langle E\rangle)A_2}{d\langle E\rangle}$$

by the time $\langle E\rangle$ is reduced to 35% of its initial value. Hence, at larger path lengths nonstatistical changes dominate the evolution of $A_2(S)$. We can show that the same is true of A_3 and A_4. At larger S, we get the same evolution for the central moments if we set $M_2(E)$, $M_3(E)$, etc., all equal to zero. Thus, at larger S, Eq. (10) is valid.

It is reasonable that Eq. (10) will also become valid at large S in many cases where the simple $P(E,t)$ used in the previous discussion is not sufficiently accurate. All that is required for the validity of Eq. (10) is that $F(E,S)$ should eventually become broad enough so that its rate of change in shape and width due to differences in average rate of energy loss should dominate the rate of change in shape and width due to statistical fluctuations. We will now investigate the possibility of taking advantage of the validity of Eq. (10) at large S.

Equation (10) can be solved by the method of characteristics. We find

$$F(E,S)=G(R(E)+S)/M_1(E), \tag{12}$$

where G is an arbitrary function and

$$R(E)=\int_0^E\frac{dU}{M_1(U)}. \tag{13}$$

$R(E)$ is the total distance that a particle with initial energy E would travel before coming to rest if it lost energy at a continuous rate given exactly by

$$dE/dS=-M_1(E).$$

The mean range, as measured along the actual path of the particle, is given approximately by Eq. (13).[10]

Because of the close relation between $R(E)$ and the mean range of a particle with energy E, we can make use of tables prepared by Whaling[11] in relating R and E.

The validity of Eq. (10) at large S suggests that we should let

$$H(R,S)=M_1(E)F(E,S). \tag{14}$$

Once Eq. (10) becomes valid, $H(R,S)$ becomes equal to $G(R+S)$. Hence, at large path lengths $H(R,S)$ takes on a constant shape and width. In Sec. III, we will develop a method that allows us to calculate $H(R,S)$ until Eq. (10) becomes accurate, and hence determines it for arbitrarily large S.

III. CALCULATION OF $H(R,S)$

We can write

$$H(R,S)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-iPR}g(P,S)dP, \tag{15}$$

where

$$g(P,S)=\int_0^{\infty}e^{iPR}H(R,S)dR. \tag{16}$$

We express $g(P,S)$ as

$$g(P,S)=\exp\left[\sum_{L=0}^{\infty}\frac{K_L(S)(iP)^L}{L!}\right], \tag{17}$$

where the K_L's are the well-known cumulants of the distribution.[12]

We define the central moments of $H(R,S)$ by

$$P_n(S)=\int_0^{\infty}(R-\bar{R})^nH(R,S)dR, \tag{18}$$

where

$$\bar{R}(S)=\int_0^{\infty}RH(R,S)dR. \tag{19}$$

The P_n's are related to the cumulants of $H(R,S)$ as follows[12]:

$$K_0(S)=0, \qquad K_1(S)=\bar{R}(S),$$
$$K_2(S)=P_2(S), \qquad K_3(S)=P_3(S),$$
$$K_4(S)=P_4(S)-3(P_2(S))^2, \quad \text{etc.}$$

A Gaussian distribution function has $K_n(S)=0$ for $n>2$. If the distribution function is close to Gaussian, one finds that the infinite sum in Eq. (15) converges rapidly and only a few terms are needed to represent $g(P,S)$ properly. In the case of nonrelativistic protons or helium ions whose mean energy has been reduced by more than 3% only terms through $n=3$ need be re-

[10] U. Fano, Phys. Rev. **92**, 328 (1953).
[11] W. Whaling, in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1958), Vol. 34, p. 193.
[12] M. G. Kendall and A. H. Stuart, *The Advanced Theory of Statistics* (Hafner Publishing Co., Inc., New York, 1963).

tained. Once the mean energy of the protons or helium ions has been reduced by 10%, only terms through $n=2$ need be retained.

If terms through $n=3$ are sufficient, one can invert the Fourier transform exactly in terms of the Airy function. $H(R,S)$ becomes

$$H(R,S)=\frac{1}{2\pi}\int_0^\infty \exp[-iP(R-\bar{R}) -\tfrac{1}{6}iP_3(S)P^3-\tfrac{1}{2}P_2(S)P^2]dP$$
$$=\frac{e^v\,\mathrm{Ai}(w)}{(3\gamma_1)^{1/3}(2P_2)^{1/2}}, \tag{20}$$

where

$$w=\frac{-y+(48\gamma_1)^{-1}}{(3\gamma_1)^{1/3}},$$
$$v=\frac{-y+(72\gamma_1)^{-1}}{12\gamma_1},$$
$$y=(R-\bar{R})/[2P_2(S)]^{1/2}$$
$$\gamma_1=-P_3(S)/\{3![2P_2(S)]^{3/2}\},$$

and

$$\mathrm{Ai}(w)=\frac{1}{\pi}\int_0^\infty \cos(\tfrac{1}{3}t^3+wt)dt.$$

If $H(R,S)$ deviates so far from a Gaussian that more terms are required, one may, as in the case of the Vavilov theory, be forced to invert the Fourier transform numerically.

We can also express $H(R,S)$ in the form of a Gaussian times an Edgeworth series.[12] Usually, Eq. (20) is almost as accurate as a Gaussian times a six-term Edgeworth series. Symon[4] has used the Edgeworth series to calculate $F(E,S)$, and Lewis[13] has used it to calculate the range distribution function for heavy charged particles.

We have seen that $H(R,S)$ can be determined from its cumulants, and that the cumulants can be calculated from the central moments. We will now show how the central moments can be calculated.

On differentiating Eq. (18) one obtains

$$\frac{dP_n(S)}{dS}=-n\frac{d\bar{R}}{dS}P_{n-1}(S) +\int_0^\infty \frac{\partial H(R,S)}{\partial S}[R-\bar{R}]^n dR. \tag{21}$$

Now,

$$\frac{\partial H(R,S)}{\partial S}=M_1(E)\frac{\partial F(E,S)}{\partial S},$$

and

$$dR=(dR/dE)dE$$
$$=dE/M_1(E).$$

[13] H. W. Lewis, Phys. Rev. **85**, 20 (1952).

Thus, Eq. (21) becomes

$$\frac{dP_n(S)}{dS}=-n\frac{d\bar{R}}{dS}P_{n-1}(S) +\int_0^\infty \frac{\partial F(E,S)}{\partial S}[R(E)-\bar{R}]^n dE. \tag{22}$$

Equation (1) gives

$$\int_0^\infty \frac{\partial F(E,S)}{\partial S}[R(E)-\bar{R}]^n dE$$
$$=\int_0^\infty\int_0^\infty [P(E+t,t)F(E+t,S)-P(E,t)F(E,S)] \times[R(E)-\bar{R}]^n dt dE.$$

Let

$$J_2=\int_0^\infty\int_0^\infty P(E+t,t)F(E+t,S)[R(E)-\bar{R}]^n dt dE,$$

$$J_1=\int_0^\infty F(E,S)M_0(E)[R(E)-\bar{R}]^n dE.$$

Then

$$J_2-J_1=\int_0^\infty \frac{\partial F(E,S)}{\partial S}[R(E)-\bar{R}]^n dE.$$

If J_2-J_1 can be expressed in terms of the $P_n(S)$'s, Eq. (22) will become a set of simultaneous differential equations from which these central moments can be calculated.

In evaluating J_2, we interchange orders of integration and let $E'=E+t$. J_2 becomes

$$J_2=\int_0^\infty dt\int_t^\infty P(E',t)F(E',S)[R(E'-t)-\bar{R}]^n dE'$$
$$\approx\int_0^\infty dt\int_0^\infty P(E',t)F(E',S)[R(E'-t)-\bar{R}]^n dE',$$

where we have assumed that $F(0,S)=0$. Interchanging orders of integration,

$$J_2=\int_0^\infty F(E',S)\int_0^\infty P(E',t)[R(E'-t)-\bar{R}]^n dt dE'.$$

If $[R(E'-t)-\bar{R}]^n$ is expanded in a Taylor series about $t=0$,

$$J_2=\sum_{k=0}^\infty (-1)^k\int_0^\infty F(E',S)\frac{\partial^k}{\partial E'^k}[R(E')-\bar{R}]^n dE' \times\int_0^\infty \frac{P(E',t)t^k dt}{k!}$$
$$=\sum_{k=0}^\infty (-1)^k\int_0^\infty F(E',S)\frac{\partial^k}{\partial E'^k}[R(E')-\bar{R}]^n M_k(E')dE'.$$

Subtracting J_1,

$$J_2-J_1=\sum_{k=1}^{\infty}(-1)^k \times\int_0^{\infty}F(E',S)\frac{\partial^k}{\partial E'^k}[R(E')-\bar{R}]^nM_k(E')dE'.$$

Equation (22) becomes

$$\frac{dP_n(S)}{dS}+n\frac{d\bar{R}}{dS}P_{n-1}(S)=\sum_{k=1}^{\infty}(-1)^k \times\int_0^{\infty}F(E',S)M_k(E')\frac{\partial^k}{\partial E'^k}[R(E')-\bar{R}]^ndE'. \quad (23)$$

We would like to approximate the right-hand side of Eq. (23) in terms of the $P_n(S)$'s. In order to achieve this, we will first need to change variables. We use Eq. (13) in order to express E in terms of R and note that

$$F(E,S)dE=F(E,S)(dE/dR)dR =H(R,S)dR.$$

Also,

$$\frac{\partial}{\partial E}=\frac{\partial R}{\partial E}\frac{\partial}{\partial R} =U(E(R))\frac{\partial}{\partial R},$$

where

$$U(E)=[M_1(E)]^{-1}.$$

Consequently,

$$\frac{dP_n}{dS}+n\frac{d\bar{R}}{dS}P_{n-1}=\sum_{k=1}^{\infty}(-1)^k\int_0^{\infty}H(R,S)M_k(E(R)) \times\left[U(E(R))\frac{\partial}{\partial R}\right]^k(R-\bar{R})^ndR. \quad (24)$$

We will now use Eq. (24) in order to derive expressions for $\bar{R}$, $P_2(S)$, and $P_3(S)$. A study of these cases will point out approximations which will be useful for general n.

n=1. Using $P_0(S)\equiv 1$ and $P_1(S)\equiv 0$, we have

$$\frac{d\bar{R}}{dS}=\sum_{k=1}^{\infty}(-1)^k\int_0^{\infty}H(R,S)M_k(E(R)) \times\left[U(E(R))\frac{\partial}{\partial R}\right]^k(R-\bar{R})dR.$$

Note that

$$\left[U(E(R))\frac{\partial}{\partial R}\right]^k(R-\bar{R})=\left[U(E(R))\frac{\partial}{\partial R}\right]^{k-1}U(E(R)) =\frac{\partial^{k-1}}{\partial E^{k-1}}U(E(R)).$$

Then

$$\frac{d\bar{R}}{dS}=\sum_{k=1}^{\infty}(-1)^k\int_0^{\infty}H(R,S)M_k(E(R))\frac{\partial^{k-1}}{\partial E^{k-1}}U(E(R))dR.$$

In order to see what approximations are plausible, we use the $P(E,t)$ from Sec. II. Then M_n's corresponding to this $P(E,t)$ are

$$M_1(E)=(2K/E)\ln(4ME/MI),$$

$$M_n(E)=\frac{K(4m/M)^{n-1}E^{n-2}}{n!(n-1)}, \quad \text{if } n>1. \quad (25)$$

We have

$$\frac{d\bar{R}}{dS}=-1+\sum_{k=2}^{\infty}\frac{(-1)^k(4m/M)^{k-1}}{2(k-1)k!}\int_0^{\infty}H(R,S)E^{k-2}(R) \times\frac{\partial^{k-1}}{\partial E^{k-1}}\left[\frac{E(R)}{\ln[4mE(R)/MI]}\right]dR.$$

Now,

$$\frac{\partial}{\partial E}E\left(\ln\frac{4mE}{MI}\right)^{-1}=\left(\ln\frac{4mE}{MI}\right)^{-1}\left[1-\left(\ln\frac{4mE}{MI}\right)^{-1}\right],$$

$$\frac{\partial^2}{\partial E^2}E\left(\ln\frac{4mE}{MI}\right)^{-1}=\left(\ln\frac{4mE}{MI}\right)^{-2}\left[1-2\left(\ln\frac{4mE}{MI}\right)^{-1}\right]\Big/E.$$

It is fairly evident that as k increases, each new term is smaller than the previous one by a factor of the order of m/M. Even in the case of muons one has as an excellent approximation:

$$d\bar{R}/dS=-1.$$

To see that the last result is not strongly dependent on the validity of Eq. (25), let $t_m(E)$ be the maximum possible energy loss for a particle of energy E. Then

$$\frac{M_n(E)}{M_{n-1}(E)}=n^{-1}\int_0^{\infty}P(E,t)t^ndt\Big/\int_0^{\infty}P(E,t)t^{n-1}dt \leqq n^{-1}t_m(E).$$

Thus, if the maximum possible energy loss is very small compared with E, we expect

$$\left|M_k\frac{\partial^{k-1}U}{\partial E^{k-1}}\Big/M_{k-1}\frac{\partial^{k-2}U}{\partial E^{k-2}}\right|\leqq\left|\frac{\partial^{k-1}U}{\partial E^{k-1}}t_m(E)\Big/k\frac{\partial^{k-2}U}{\partial E^{k-2}}\right| \ll 1, \quad (26)$$

where the arguments of the functions have been suppressed for brevity. We will assume throughout this study that the last inequality holds. Again, the terms in the sum are found to be small compared with unity

and $\bar{R}(S)\approx\bar{R}(0)-S$. We define $\bar{E}=\bar{E}(S)$ so that

$$\bar{R}(0)-S=\int_0^{\bar{E}} dU/M_1(U). \tag{27}$$

In the approximation $\bar{R}=\bar{R}(0)-S$, we have $\bar{E}=E(\bar{R})$. Now

$$d\bar{E}/dS=-M_1(\bar{E}).$$

To improve our first approximation let

$$\frac{d\bar{R}}{dS}=-1+\int_0^{\infty} H(R,S)M_2(E(R))\frac{\partial U(E(R))dR}{\partial E}.$$

We assume that $H(R,S)$ remains fairly sharply peaked about $\bar{R}$ until Eq. (10) becomes valid. If so, we can use

$$M_2(E(R))\frac{\partial U(E(R))}{\partial E}\approx M_2(\bar{E})\frac{\partial U(\bar{E})}{\partial\bar{E}}+\frac{\partial}{\partial\bar{E}}\left[M_2(\bar{E})\frac{\partial U(\bar{E})}{\partial\bar{E}}\right]\frac{R-\bar{R}}{U(\bar{E})}.$$

This is the first two terms in a Taylor series about $\bar{R}$. We then find

$$\frac{d\bar{R}}{dS}=-1+M_2(\bar{E})\frac{\partial U(\bar{E})}{\partial\bar{E}}.$$

Integrating,

$$\bar{R}(S)=\bar{R}(0)-S+\int_{\bar{E}(S)}^{\bar{E}(0)} M_2(E)U(E)\frac{\partial U(E)}{\partial E}dE. \tag{28}$$

Equation (28) includes a linear variation of $M_2(E)$ and variations of $M_1(E)$ through quadratic terms.

The reader may have worried about the fact that $U(E)$ becomes infinite when $4mE/MI=1$. The difficulty is only apparent since the approximate M_n's are only valid if $4mE/MI\gg1$. The problem would not arise with a correct set of M_n's.

n=2. In this case, Eq. (24) yields

$$\frac{dP_2}{dS}=\sum_{k=2}^{\infty}(-1)^k\int_0^{\infty} H(R,S)M_k(E(R))\times\left[U(E(R))\frac{\partial}{\partial R}\right]^k(R-\bar{R})^2dR.$$

If the M_n's given by Eq. (25) bear any similarity to the correct ones, the $k=3$ term is smaller than the $k=2$ term by a factor of the order of m/M. More generally, if Eq. (26) is valid, the terms with $k>2$ are small compared with the $k=2$ term and

$$\frac{dP_2}{dS}\approx2\int_0^{\infty} H(R,S)M_2(E(R))\times\left[U^2(E(R))+\frac{\partial U}{\partial E}(E(R))(R-\bar{R})\right]dR.$$

We assume, once again, that $H(R,S)$ is fairly sharply peaked about $\bar{R}$. Let

$$U^2(E(R))\approx U^2(\bar{E})+2\frac{\partial U(\bar{E})(R-\bar{R})}{\partial\bar{E}}+\frac{\partial^2U(\bar{E})}{\partial\bar{E}^2}\frac{(R-\bar{R})^2}{U(\bar{E})}$$

and

$$\frac{\partial U(E(R))}{\partial E}\approx\frac{\partial U(\bar{E})}{\partial\bar{E}}+\frac{\partial^2U(\bar{E})}{\partial\bar{E}^2}\frac{R-\bar{R}}{U(\bar{E})},$$

with

$$M_2(E(R))=M_2(\bar{E})+\frac{\partial M_2(\bar{E})}{\partial\bar{E}}\frac{R-\bar{R}}{U(\bar{E})}.$$

These approximations lead to

$$\frac{dP_2}{d\bar{E}}=-4M_2(\bar{E})\frac{\partial^2U(\bar{E})}{\partial\bar{E}^2}P_2-6\frac{\partial U(\bar{E})}{\partial\bar{E}}\frac{\partial M_2(\bar{E})}{\partial\bar{E}}P_2-2M_2(\bar{E})U^3(\bar{E}).$$

$M_2(E)$ is usually a very slowly varying function. In fact, the $M_2(E)$ given by Eq. (25) is constant. To get some idea about the importance of the terms involving P_2 treat $M_2(E)$ as a constant. The integrating factor for the resulting differential equation is

$$\exp[4M_2(\bar{E})\partial U(\bar{E})/\partial\bar{E}].$$

Using the M_n's from Eq. (25),

$$4M_2(\bar{E})\frac{\partial U(\bar{E})}{\partial\bar{E}}=\frac{4m}{M}\left(\ln\frac{4m\bar{E}}{MI}\right)^{-1}\left[1-\left(\ln\frac{4m\bar{E}}{MI}\right)^{-1}\right].$$

The latter quantity is always very small compared with unity in any case where the $P(\bar{E},t)$ used in Eq. (25) is valid. Because of the inequality in Eq. (26) we expect the quantity to be small in almost any case. We will replace the integrating factor by unity. $P_2(S)$ becomes

$$P_2(S)=P_2(0)+2\int_{\bar{E}(S)}^{\bar{E}(0)} M_2(E)U^3(E)dE. \tag{29}$$

n=3. In this case only the first three terms on the right-hand side of Eq. (24) are important and

$$\frac{dP_3}{dS}+3\frac{d\bar{R}}{dS}P_2(S)=-3P_2+\int_0^{\infty} H(R,S)M_2(E(R))\left[U(E(R))\frac{\partial}{\partial R}\right]^2(R-\bar{R})^2dR-\int_0^{\infty} H(R,S)M_3(E(R))\left[U(E(R))\frac{\partial}{\partial R}\right]^3(R-\bar{R})^3dR.$$

If we use our previous approximation for $d\bar{R}/dS$, we find

$$\frac{dP_3}{dS}=-3M_2(\bar{E})\frac{\partial U(\bar{E})}{\partial \bar{E}}P_2+\int_0^\infty H(R,S)M_2(E(R))$$
$$\times\left[3\frac{\partial U(E(R))}{\partial E}(R-\bar{R})^2+6U^2(E(R))(R-\bar{R})\right]dR$$
$$-\int_0^\infty H(R,S)M_3(E(R))\left[3\frac{\partial^2 U(E(R))}{\partial E^2}(R-\bar{R})^2\right.$$
$$\left.+18U(E(R))\frac{\partial U(E(R))}{\partial E}(R-\bar{R})+6U^3(E(R))\right]dR.$$

Because of Eq. (26)

$$M_2\frac{\partial U}{\partial E}(R-\bar{R})^2\gg M_3\frac{\partial^2 U}{\partial E^2}(R-\bar{R})^2$$

and

$$6U^2(R-\bar{R})M_2\gg 18U\frac{\partial U}{\partial E}M_3(R-\bar{R}).$$

Thus, it is quite accurate to use

$$\int_0^\infty H(R,S)M_3(E(R))\left[U(E(R))\frac{\partial}{\partial R}\right]^3(R-\bar{R})^3dR$$
$$\approx 6M_3(\bar{E})U^3(\bar{E}).$$

In evaluating the other integral, the same approximations are used for $\partial U/\partial E$ and U^2 as in deriving Eq. (29). We find

$$\frac{dP_3}{d\bar{E}}+3\left[5\frac{\partial U(\bar{E})}{\partial \bar{E}}\frac{\partial M_2(\bar{E})}{\partial \bar{E}}+3\frac{\partial^2 U(\bar{E})}{\partial \bar{E}^2}M_2(\bar{E})\right]P_3$$
$$=-6P_2(S)\frac{d}{d\bar{E}}[U^2(\bar{E})M_2(\bar{E})]+6M_3(\bar{E})U^4(\bar{E}).$$

Once again the integrating factor is approximately unity, and

$$P_3(S)=P_3(0)+6\int_{\bar{E}(S)}^{\bar{E}(0)}P_2\frac{d}{d\bar{E}}[U^2(\bar{E})M_2(\bar{E})]d\bar{E}$$
$$-6\int_{\bar{E}(S)}^{\bar{E}(0)}M_3(\bar{E})U^4(\bar{E})d\bar{E}.$$

Consider the integral involving P_2. On integrating by parts and using

$$dP_2/d\bar{E}\approx -2M_2(\bar{E})U^3(\bar{E}),$$

we find

$$\int_{\bar{E}(S)}^{\bar{E}(0)}P_2\frac{d}{d\bar{E}}[U^2(\bar{E})M_2(\bar{E})]d\bar{E}$$
$$=[U^2(\bar{E}(0))M_2(\bar{E}(0))P_2(0)$$
$$-U^2(\bar{E}(S))M_2(\bar{E}(S))P_2(S)]$$
$$+2\int_{\bar{E}(S)}^{\bar{E}(0)}[M_2(\bar{E})]^2U^5(\bar{E})d\bar{E}.$$

Hence,

$$P_3(S)=P_3(0)+6[U^2(\bar{E}(0))M_2(\bar{E}(0))P_2(0)$$
$$-U^2(\bar{E}(S))M_2(\bar{E}(S))P_2(S)]$$
$$+12\int_{\bar{E}(S)}^{\bar{E}(0)}[M_2(E)]^2U^5(E)dE$$
$$-6\int_{\bar{E}(S)}^{\bar{E}(0)}M_3(E)U^4(E)dE. \quad (30)$$

The fourth cumulant can be calculated by solving

$$\frac{d}{d\bar{E}}[P_4-3(P_2)^2]=-12P_3\frac{d}{d\bar{E}}[U^2(\bar{E})M_2(\bar{E})]$$
$$+36U^2(\bar{E})M_3(\bar{E})P_2\frac{\partial U(\bar{E})}{\partial \bar{E}}$$
$$+24\frac{d}{d\bar{E}}[U^3(\bar{E})M_3(\bar{E})]P_2-24U^5(\bar{E})M_4(\bar{E}). \quad (31)$$

In the case of originally monoenergetic α particles, protons, or K mesons, three cumulants suffice, except at short path lengths where the Vavilov theory holds. Three cumulants also suffice for nonrelativistic pions and muons if they were originally monoenergetic, and their mean energy has been reduced by at least 20%. Thus, in most intermediate or large path length situations we can combine Eqs. (20) and (28)–(30) in order to calculate $H(R,S)$ and hence $F(E,S)$ from Eqs. (13) and (14).

IV. SOME SPECIAL CASES

In this section we will demonstrate how the theory is used to calculate $F(E,S)$ in two special cases.

A. Straggling of 50-MeV Protons in Be

In this situation, relativistic effects can be neglected, and inner shell corrections are small. The $M_n(E)$'s given by Eq. (25) can be used.

In the case under consideration, one has $4mE_0/MI \approx 1700$, where E_0 is the initial energy of the protons. We will now demonstrate how one can derive simple but accurate formulas for $\bar{R}$, P_2, and P_3 if $4mE_0/MI$

$\geqq 100$ and Eq. (25) holds. From Eq. (25)

$$U(E)=(E/2K)(\ln 4mE/MI)^{-1}.$$

We will now investigate the possibility of approximating this $U(E)$ by a simpler function $U_a(E)$, where

$$U_a(E)=\frac{E_0}{2K}\left(\ln\frac{4mE_0}{MI}\right)^{-1}\left(\frac{E}{E_0}\right)^a,$$

with

$$a=1-(\ln 4mE_0/MI)^{-1}.$$

The reader will recognize this approximation as being identical to the one used in calculating $A_2(S)$ in Sec. II. $U(E)$ and $U_a(E)$ are equal and have the same first derivative at E_0. This leads us to think that the approximation will be very good for E near E_0. In order to see how good the approximation is for smaller E, let $X=E/E_0$, $U_0=4mE_0/MI$, $b=\ln U_0$, and

$$R(X)=U(E)/U_a(E)$$
$$=\exp\{(\ln X)/b-\ln[1+(\ln X)/b]\}.$$

Let $y(X)=-(\ln X)/b$. Then

$$R(X)=e^{-y(X)}/[1-y(X)].$$

$R(X)$ is graphed as a function of y in Fig. 1. From Fig. 1 we see that when $y(X)<0.19$, the error made by replacing $U(E)$ by $U_a(E)$ is less than 2%. If $E/E_0>\exp(-0.19b)$, $U_a(E)$ represents $U(E)$ with an error less than 2%. If $4mE_0/MI\geqq 100$, E can be reduced to $0.42E_0$ before the error becomes greater than 2%. In a similar way, we find that E can be reduced at least as far as $0.28E_0$ before the error made by using $U_a(E)$ becomes greater than 5%.

The integrals involved in Eqs. (28)–(30) have the property that the integrand decreases as E is decreased. In fact, our qualitative discussion in Sec. II suggests that most of the contribution to these integrals should occur before the mean energy is reduced to 35% of its initial value, that is, from the region where $U_a(E)$ approximates $U(E)$ very accurately. We will replace $U(E)$ by $U_a(E)$ in evaluating P_2, P_3, and $\bar{R}$.

From Eq. (28)

$$\bar{R}(S)=\bar{R}(0)-S+\frac{m\bar{E}(0)}{2M}U(\bar{E}(0))\left(\ln\frac{4m\bar{E}(0)}{MI}\right)^{-1}\times\left[1-\left(\frac{\bar{E}(S)}{\bar{E}(0)}\right)^{2a}\right], \quad (32)$$

where one must use $\bar{E}(0)$ instead of E_0 in calculating a (one does not have to make this distinction if the initial beam is monoenergetic). From Eq. (29)

$$P_2(S)=P_2(0)+\frac{2m}{M}[U(\bar{E}(0))\bar{E}(0)]^2(3a+1)\times\left(\ln\frac{4m\bar{E}(0)}{MI}\right)^{-1}(1-r^{3a+1}), \quad (33)$$

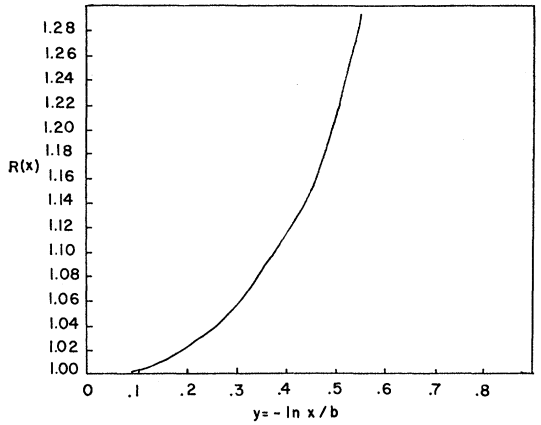

FIG. 1. $U(E)/U_a(E)$ versus $y(X)$. $M_1(E)$ is the average stopping power of a particle with energy E, $U(E)=[M_1(E)]^{-1}$, $U_a(E)$ is the approximate $U(E)$, $b=\ln(4mE_0/MI)$, and $X=E/E_0$. From this graph one can determine the error made by approximating the stopping power by an expression of the form $M_1(E)\approx M_1(E_0)(E_0/E)^a$, with $a=1-b^{-1}$.

with $r=\bar{E}(S)/\bar{E}(0)$. From Eq. (30) we have

$$P_3(S)=P_3(0)+\frac{m}{M}[\bar{E}(0)U(\bar{E}(0))]\times\left(\ln\frac{4m\bar{E}(0)}{MI}\right)^{-1}P_2(0)[1-r^{2a}]+4(m/M)^2[U(\bar{E}(0))\bar{E}(0)]^3(1-a)^2g(r,a), \quad (34)$$

where

$$g(r,a)=\frac{3(1-r^{5a+1})}{5a+1}-\frac{3r^{2a}(1-r^{3a+1})}{3a+1}-\frac{1-r^{4a+2}}{(1-a)(4a+2)}.$$

If the initial beam is monoenergetic, we can express γ_1 as

$$\gamma_1=-\tfrac{1}{12}(m/M)^{1/2}(3a+1)^{3/2}(1-a)^{1/2}\times[1-r^{3a+1}]^{-3/2}g(r,a). \quad (35)$$

For 50-MeV protons incident of Be, we use $I=64$ eV and find $a=0.866$ and $U(E_0)=105$ mg/MeV cm^2. Equations (33) and (35) yield the values of $P_2(S)$ and γ_1 tabulated in Table I. Equation (27) was used to relate the r values in Eqs. (33) and (35) to the corresponding absorber thicknesses.

In Figs. 2 and 3, we have graphed $H(R,S)$, as calculated from Eqs. (20), (33), and (35), for $r=0.97$ and for $r=0.90$. In order to show that $H(R,S)$ evolves toward a Gaussian as S is increased we have also graphed

$$H_G(R,S)=[2\pi P_2(S)]^{-1/2}\exp[-(R-\bar{R})^2/2P_2(S)].$$

Note that $H(R,S)$ is almost Gaussian by the time r is reduced to 0.9.

In the above case, or in any other where the full width at half-maximum of $F(E,S)$ is very small com-

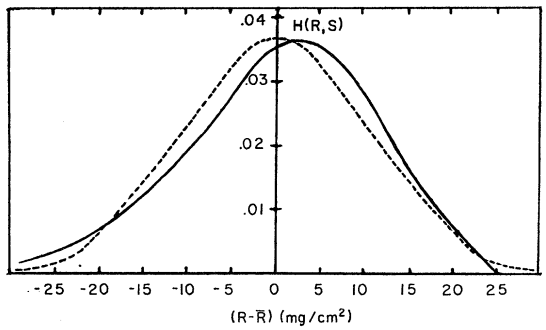

FIG. 2. $H(R,S)$ and $H_G(R,S)$ versus $R-\bar{R}$ for 50-MeV protons in Be. $H_G(R,S)$, the Gaussian approximation to $H(R,S)$, is given by the dashed line. The mean energy has been reduced to 48.5 MeV.

pared with the mean energy, $F(E,S)$ can be calculated from $H(R,S)$ quite easily. Note that

$$R-\bar{R}=\int_{E(\bar{R})}^{E} U(E')dE' \approx U(E(\bar{R}))[E-E(\bar{R})], \quad (36)$$

for any E which is very near $E(\bar{R})$. If S corresponds to a thin or intermediate absorber thickness, Eq. (36) is accurate at any E for which $F(E,S)$ is appreciably nonzero. In the above situation

$$F(E,S)\approx U(E(\bar{R}))H(R(E),S). \quad (37)$$

When Eq. (36) is valid, the shape of $F(E,S)$ is almost identical to that of $H(R,S)$.

When Eqs. (36) and (37) are valid, $F(E,S)$ can be calculated directly. From Eqs. (20), (36), and (37).

$$F(E,S)=\frac{U(E(R))e^{v}\,\mathrm{Ai}(w)}{(3\gamma_1)^{1/3}(2P_2)^{1/2}}, \quad (38)$$

where γ_1, P_2, v, w, and $\mathrm{Ai}(w)$ are the same as in Eq. (20), except the y's in w and v must be replaced by

$$y=U(E(\bar{R}))[E-E(\bar{R})]/(2P_2(S)).$$

Equation (38) holds when all of Eqs. (36), (37), and (20) are valid. However, the three cumulant approximation used in deriving Eq. (20) is not valid at small S where $\gamma_1\geqq 0.03$, and Eqs. (36) and (37) are not valid in the thick absorber region. Thus, we should remember that Eq. (38) fails both at small and large S.

If γ_1 becomes smaller than 0.01 before Eq. (38) becomes inaccurate, $F(E,S)$ can be approximated at larger S by

$$F(E,S)=[2\pi A_2(S)]^{-1/2}\times\exp\{-[E-E(\bar{R})]^2/2A_2(S)\}, \quad (39a)$$

where

$$A_2(S)=P_2(S)/[U(E(\bar{R}))]^2. \quad (39b)$$

At large S, the energy spectrum becomes fairly broad and we cannot continue to use a linear relation between R and E. However, from Eqs. (11), (27), and (32)

$$R-\bar{R}=\int_{\bar{E}}^{E} U(E')dE'-mE_0U(E_0)\times\left[\left(\ln\frac{4mE_0}{MI}\right)(2M)\right]^{-1}\left[1-\left(\frac{\bar{E}}{E_0}\right)^{2a}\right]. \quad (40)$$

If E is near $\bar{E}$, $U(E')$ can be approximated by

$$U_a(E')=U(\bar{E})(E'/\bar{E})^{c},$$

where

$$c=1-(\ln 4m\bar{E}/MI)^{-1}.$$

As was pointed out previously, this type of approximation is quite good if $4m\bar{E}/MI$ is large compared with unity and E' is near $\bar{E}$. In particular, if $4m\bar{E}/MI\geqq 100$, we can use $U_a(E')$ in the range $0.42\bar{E}\leqq E'\leqq 2.3\bar{E}$ and be sure that the error made by using the approximate $U(E')$ is less than 2%. Even when $4m\bar{E}/MI=20$, the error is less than 2% in the interval $0.58\bar{E}\leqq E'\leqq 1.75\bar{E}$. If the full width of $F(E,S)$ at half-maximum is less than $0.6\bar{E}$, we will usually be able to relate R and E by

$$R-\bar{R}=U(\bar{E})\bar{E}(1+c)^{-1}[(E/\bar{E})^{1+c}-1]. \quad (41)$$

With 50-MeV protons incident on Be, Eq. (36) can be used until $H(R,S)$ becomes Gaussian. In fact, Eq. (36) is valid until $\bar{E}$ is reduced by more than 50%. At larger S, we note that $P_2(S)$ approaches a constant value, and $H(R,S)$ is a Gaussian with the width $(2P_2)^{1/2}=47.1$ mg/cm². In the $U(E')$ given by Eq. (25) is still valid at large S (and small $\bar{E}$),

$$F(E,S)=U(\bar{E})(E/\bar{E})^{c}(2\pi P_2)^{-1/2}\times\exp\{-[R(E)-\bar{E}]^2/2P_2\}, \quad (42)$$

where $R(E)-\bar{R}$ is given by Eq. (41). Figure 4 is a graph of Eq. (42) for the case $\bar{E}=8$ MeV or $r=0.16$. The reader may also be interested in using Whaling's[11] range-energy tables in order to relate E to R. In this approach, we write

$$F(E,S)=U(E)(2\pi P_2)^{-1/2}\exp[-(R-\bar{R})^2/2P_2], \quad (43)$$

and use $\bar{R}=96.7$ mg/cm². R can be related to E by use of the range tables. $U(E)$ is obtained by the numerical

TABLE I. $\bar{R}$, P_2, and γ_1 for 50-MeV protons in Be.

r	S (mg/cm²)	$\bar{R}$ (mg/cm²)	$P_2(S)$ (mg²/cm⁴)	γ_1
0.98	105	2770	78	0.0375
0.97	157	2718	116	0.0307
0.96	206	2669	152	0.0266
0.90	501	2374	366	0.017
0.70	1330	1545	806	0.0077
0.50	2080	795	1020	0.0057
0.30	2567	308	1100	0.0045
0.16	2778	97	1112	0.004
0.00	2875	0	1116	0.0039

evaluation of dR/dE in terms of differences from the tables. For example,

$$U(8\text{ MeV}) \approx \frac{R(8.5\text{ MeV}) - R(7.5\text{ MeV})}{8.5\text{ MeV} - 7.5\text{ MeV}}$$

$$\approx 20.5\text{ mg/MeV cm}^2.$$

Equation (43) and Whaling's tables lead to values of $F(E,S)$ which are quite close to those obtained from Eq. (42). The use of empirical range-energy tables is particularly useful in cases where the theoretical $U(E)$ becomes incorrect at small E.

In the preceding calculations, we have assumed the strict validity of $U(E)$, $M_2(E)$, and $M_3(E)$ as given by Eq. (25). Actually, the $M_2(E)$ given by Eq. (25) is too small because of the neglect of inner shell corrections. This results in our $P_2(S)$ being too small by 2–3%. We base this estimate on calculations by Sternheimer.[14] $U(E')$, as calculated from Eq. (25), differs from what would be estimated from Whaling's range-energy tables by 5–15% in the interval $2 < E' < 10$ MeV.

At large S, statistical fluctuations become unimportant. If further statistical fluctuations can be neglected after a thickness S_1, a particle with energy in the interval E to $E+dE$ will end up with a range in the interval $S_1+R(E)$ to $S_1+R(E+dE)$. Let $G(\rho,E_0)d\rho$ be the fraction of particles with initial energy E_0 which have a range between ρ and $\rho+d\rho$. Choose E and dE so that $\rho=S_1+R(E)$ and $d\rho=R(E+dE)-R(E)$. Then

$$G(\rho,E_0)d\rho = F(E,S_1)dE = H(R(E),S_1)d\rho,$$

or

$$G(\rho,E_0) = H(\rho-S_1,S_1).$$

It has been assumed that S_1 is large enough so that P_2 and P_3 have reached their limiting values. We then see that the limiting values of P_2 and P_3 should be identical to the corresponding central moments of the range distribution function. The reader may have already noticed that Eq. (29) reduces to the well-known expression for the second central moment of $G(\rho,E_0)$ when $P_2(0)=0$, $\bar{E}(0)=E_0$, and $\bar{E}(S)\approx 0$.

B. Energy Straggling of 5.3-MeV α Particles in Air

In this situation, $H(R,S)$ becomes Gaussian by the time the mean energy is reduced by 5%. This prediction is based on Eq. (25), but there is probably considerable similarity between the correct and approximate $M_n(E)$'s. Hence, we will proceed on the assumption that $H(R,S)$ is Gaussian for $E \leqq 5.0$ MeV. We also expect that Eq. (36) will remain valid until the mean energy is lowered considerably. Equations (39a) and (39b) can be used until Eq. (36) fails.

[14] R. M. Sternheimer, Phys. Rev. **117**, 485 (1960).

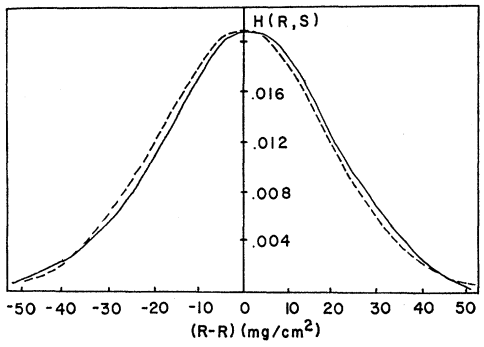

FIG. 3. $H(R,S)$ and $H_G(R,S)$ versus $R-\bar{R}$ for 50-MeV protons in Be. $H_G(R,S)$, the Gaussian approximation to $H(R,S)$, is given by the dashed line. The mean energy has been reduced to 45 MeV.

This situation has been studied experimentally by Rotondi and Geiger.[15] They find that $F(E,S)$ becomes Gaussian as expected. Since a Gaussian is completely specified by $\langle E\rangle$ and $A_2(S)$, we will not compare every detail of the theoretical and experimental $F(E,S)$'s. Instead, it is sufficient to calculate $\alpha=[2A_2(S)]^{1/2}$ and compare it with the experimental values.

In this situation, inner shell corrections are important; therefore the $M_n(E)$'s given by Eq. (25) cannot be used. Livingston and Bethe[16] have used the first-order Born approximation to derive an expression for $M_2(E)$ which includes inner shell corrections. They find

$$M_2(E) = M_{2B}\left[\frac{Z'}{Z} + \sum_n \frac{k_n I_n Z_n}{ZmV^2} \ln\frac{2mV^2}{I_n}\right], \qquad (44)$$

where M_{2B} is the $M_2(E)$ as calculated from Eq. (25), Z_n is the number of electrons in the nth shell, I_n is the effective ionization potential of the nth shell, k_n is a constant which will be taken as $\frac{4}{3}$ for all shells, V is the speed of the incident particle, the sum extends over all

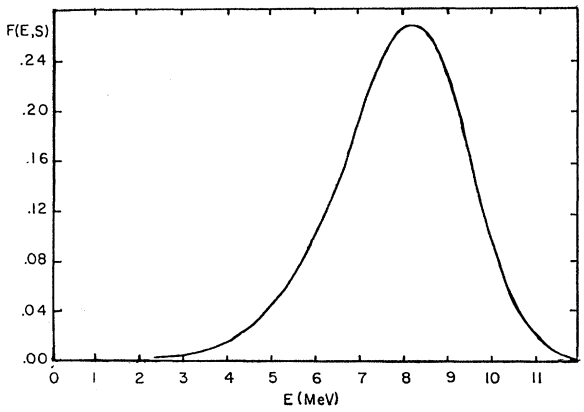

FIG. 4. Energy distribution function for 50-MeV protons in Be. The mean energy has been reduced to 8 MeV.

[15] E. Rotondi and K. W. Geiger, Nucl. Instr. Methods **40**, 192 (1966).

[16] M. S. Livingston and H. A. Bethe, Rev. Mod. Phys. **9**, 261 (1937).

TABLE II. Results of theory and experiment for the energy straggling of 5.3-MeV α particles in air at 760-mm Hg and 15°C. $\bar{E}$, S, and the experimental values of α were taken from Rotondi and Geiger (Ref. 15). The values of $U(\bar{E})$ were extracted from the $\bar{E}$-versus-S data of Rotondi and Geiger (Ref. 15) and from the range-energy tables of Whaling (Ref. 11) (the latter values are marked with an asterisk). $U_a(\bar{E})=1.075(\bar{E}/5.3)^{0.57}$ cm/MeV, with $\bar{E}$ in MeV.

S (cm)	$\bar{E}$ (MeV)	$U_a(\bar{E})$ (cm/MeV)	$U(\bar{E})$ (cm/MeV)	$P_2(S)$ (cm²)	α (keV) Expt.	Theory
0.000	5.300	1.075	1.075	0.000	13	0
	5.000	1.04	1.05*			
0.833	4.500	0.98	1.01, 0.99*	0.00046	32	31.0
	4.000	0.92	0.91*			
1.388	3.930	0.91	0.97	0.00071	43	41.4
	3.500	0.85	0.84*			
1.943	3.280	0.82	0.833	0.000933	54	51.9
2.554	2.450	0.69	0.690	0.00112	70	68.6
2.998	1.705	0.56	0.556	0.00122	92	89.0
3.276	1.140	0.44	0.445	0.00126	109	113.0
3.526	0.545	0.30	0.40	0.00128	122	127
3.590	0.385		0.44	0.00128	115	115
3.640	0.275		0.484	0.00128	105	105
3.664	0.225		0.528	0.00128	94	95.5

shells for which $I_n<2mV^2$ and Z' is the effective number of participating electrons as defined by Livingston and Bethe. We will use Eq. (44) in calculating $M_2(E)$ for air.

We first consider $M_2(E)$ for pure nitrogen. We use[14,17] $I_1=495$ eV, $Z_1=2$, $I_2=57$ eV, $Z_2=2$, $I_3=38$ eV, and $Z_3=3$. We find that M_2 is $1.35M_{2B}$ at $E=5.3$ MeV, but it increases to $1.46M_{2B}$ by the time E is decreased to 3.0 MeV. For pure oxygen $I_1=710$ eV, $Z_1=2$, $I_2=68$ eV, $Z_2=2$, $I_3=49$ eV, and $Z_3=4$. It is found that M_2 is $1.35M_{2B}$ at $E=5.3$ MeV, but increases to $1.41M_{2B}$ by the time E is reduced to 3.0 MeV. By the time $\bar{E}$ is reduced to 3.0 MeV, P_2 grows to about 75% of its limiting value. The heavy contribution to P_2 from $\bar{E}$ near $\bar{E}(0)$ suggests that we can replace $M_2(E)$ for air by a constant value 1.38 times as large as that calculated from Eq. (25). This approximation should lead to values which are within 3% of what would be obtained if we included the E dependence of $M_2(E)$. If the air is at a pressure of 760-mm Hg and a temperature of 15°C, we have

$$M_2(E)\approx 264 \text{ keV}^2/\text{cm}. \tag{45}$$

$U(E)$ cannot be calculated from Eq. (25), but when $E>1.1$ MeV, we can use

$$U(E)=1.075\times10^{-3}(E/5300)^{0.57} \text{ cm/keV}, \tag{46}$$

where E is in keV. Equation (46) is an empirical relation which reproduces range-energy data fairly accurately in the interval $1.1\leqq E\leqq 5.3$ MeV. Table II includes both values of $U(E)$ calculated from Eq. (46) and values determined from the experimental data of Rotondi and Geiger. When $E<1.7$ MeV, $U(E)$ will be taken from experimental data.

From Eqs. (29), (45), and (46)

$$P_2(S)=1.28\times10^{-3}[1-r^{2.71}] \text{ cm}^2, \tag{47}$$

where $r=\bar{E}/E_0$, and we have taken $P_2(0)=0$. Actually, $P_2(0)$ is not zero in the Rotondi-Geiger experiment, but it is small and can be neglected at intermediate and large absorber thicknesses. Equation (39b) and (47) can be combined to calculate $\alpha=[2A_2(S)]^{1/2}$. The calculated α is compared with experiment in Table II.

The reader may wonder about the continued use of the theory for $\bar{E}<1.7$ MeV. After all, the present theory neglects electron capture and loss, and these effects are important for $\bar{E}<1.7$ MeV. However, the solid-state detector used by Rotondi and Geiger detects the particles and classifies them according to their energy, without regard for their charge state at the instant they enter the detector. Also, $F(E,S)$ may be broad enough by the time electron capture and loss becomes important so that nonstatistical changes will dominate changes in the energy spectrum due to statistical fluctuations of any kind. In this event, each particle can be treated as if it loses energy at a definite rate. This rate of energy loss is the weighted mean of the average rates of energy loss for the different charge states, the weight factors being the fractions of the distance traveled while in each charge state. If we determine $M_1(E)$ from the experimental data and interpret $F(E,S)dE$ as the fraction of the particles (without regard for charge) with energy between E and $E+dE$, the theory could continue to hold at average energies below 1. MeV. This is observed to be the case.

The excellent agreement between theory and experiment for $\bar{E}<1.7$ MeV may involve some luck. We say this because when $\bar{E}<1.7$ MeV we use $\alpha=(2P_2)^{1/2}/U_2(\bar{E})$. It is difficult to believe that $U_2(E)$, the value for $U(\bar{E})$ extracted from the Rotondi-Geiger $\bar{E}$-versus-S data, is as accurate as the theoretical values of α seem to be.

Comfort, Decker, Lynck, Scully, and Quinton[5] have studied the energy straggling of α particles ($E_0=8.78$ MeV) in various metal foils. They compare the measured full widths at half-maximum (FWHM) with the Bohr theory. Even though several different forms of $M_2(E)$ are used, they find that the calculated FWHM is always much smaller than the experimentally determined value at large S. We have found that the same type of theory used here for air gives good agreement with experiment[5] in the case of nickel foils. It is also found that the experimental results for thin aluminum foils can be calculated fairly accurately if one assumes that each individual foil has a Gaussian distribution of thicknesses with a standard deviation of 0.12 times the average thickness. The inclusion of nonstatistical spreading of the energy spectrum (and multiple scat-

[17] R. M. Sternheimer, Phys. Rev. 88, 851 (1952).

tering) may also bring the theoretical calculations for the other types of foils into much better agreement with experiment.

V. CONCLUDING REMARKS

At the time the preceding pages were written there was no alternative theory at large path lengths. However, Tschalär[18] has recently extended the calculation of $F(E,S)$ into the thick absorber region. This was achieved by carrying out the numerical solution of a simplified form of Eq. (1). In these calculations Tschalär uses a $P(E,t)$ which leads to the $M_1(E)$ and $M_2(E)$ given by Eq. (25). Hence, these calculations neglect both inner shell corrections and relativistic effects. We will now compare the results derived in the present paper to those of Tschalär.

Tschalär has considered 49-MeV protons incident on an aluminum absorber. In particular, he works out the case where the absorber thickness is such that the peak of $F(E,S)$ is at $E=5$ MeV. It is found that the calculated spectrum is in good agreement with the experimental results of Raju[19] for $E>2$ MeV. To calculate this spectrum from the present theory, note that γ_1 is small so that $H(R,S)$ is almost Gaussian. We can use Eqs. (41) and (42) in calculating $F(E,S)$. However, before using these equations we must determine $\bar{E}$. If we let

$$U(E)\approx U(E_m)(E/E_m)^d,$$

with $d=1-(\ln V_m)^{-1}$ and $V_m=4mE_m/MI$, we can show that

$$\bar{E}/E_m\approx[1-(2m/M)(d)(d+1)(3a+1)^{-1}\times(1-a)^3(1-d)^{-2}(E_0/E_m)^4]^{1/(d+1)}.$$

With $E_m=5$ MeV, $M/m=1840$, $E_0=49$ MeV, $I=163$ eV, $a=0.846$, and $d=0.762$, we find $\bar{E}=4.26$ MeV. At large S (or small r),

$$2P_2[U(\bar{E})\bar{E}]^{-2}\approx\frac{4m}{M}\left(\frac{E_0}{\bar{E}}\right)^4\left(\frac{1-a}{1-d}\right)^2\frac{1-a}{1+3a}.$$

[18] C. Tschalär, Nucl. Instr. Methods **64**, 237 (1968).
[19] M. R. Raju, University of California Lawrence Radiation Laboratory Report No. UCRL-16613, 1965 (unpublished).

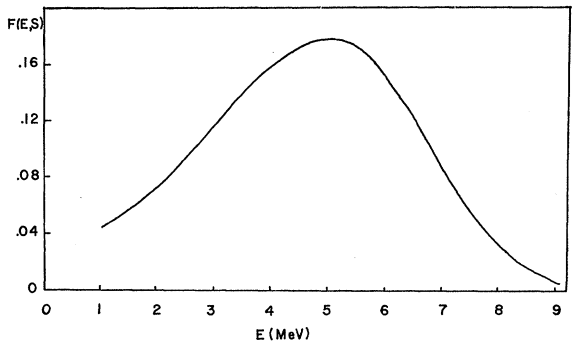

FIG. 5. Energy distribution function for 49-MeV protons in Al. The most probable energy has been reduced to 5 MeV.

If we use the last relation, Eq. (42) becomes

$$F(E,S)=0.167(E/4.26)^{0.752}\times\exp\{-0.514[(E/4.26)^{1.752}-1]^2\},$$

where E is in MeV. The $F(E,S)$ calculated from the last equation is graphed in Fig. 5. The $F(E,S)$ calculated here is in excellent agreement with the data of Raju and the calculation of Tschalär. We have compared the present theory with that of Tschalär for cases where the heavy charged particles were pions and helium ions. In every case that was considered the agreement between these calculations was excellent. This work achieves to a great extent analytically what Tschalär has done numerically.

The theory that has been developed here is not restricted to cases where the $M_n(E)$ given by Eq. (25) are accurate. This is important since inner shell corrections to $M_2(E)$ are important in many situations of interest. Sternheimer[14] has estimated that these corrections are about 16% for 10-MeV protons in aluminum. These corrections are larger at lower energies and are about 6% even at $E=50$ MeV. At a given energy, these corrections are even more important for α particles.

ACKNOWLEDGMENT

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