Cheng's result leads to values which differ from unity by less than 10%.

[33]This method has previously been employed by P. G. Klemens, Australian J. Phys. 7, 64 (1954), in calculations of the transport coefficients of the electron-phonon system.

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Attenuation of Zero Sound in a Normal Fermi Liquid*

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The attenuation of zero sound in the collisionless regime ($\omega\tau \gg 1$) is calculated using perturbation theory to treat the effects of collisions. Detailed calculations are performed taking into account Landau parameters with $l \leq 2$; the results are applied to the experimental data for liquid He^3.

1. INTRODUCTION

In the second of his classic papers on normal Fermi liquids, Landau predicted that a new type of sound, which he called zero sound, could propagate in such systems.[1] One of the necessary conditions for the existence of zero sound is that a typical quasiparticle collision frequency be small compared with the frequency of the wave; in other words, zero sound exists in the collisionless region ($\omega\tau \gg 1$). Ordinary sound (first sound), on the other hand, exists in the hydrodynamic regime ($\omega\tau \ll 1$), where the liquid is in local thermodynamic equilibrium.

The velocity and attenuation coefficient of zero sound in a normal Fermi liquid may be determined from the eigenvalues of the quasiparticle transport equation.[1,2] The collision integral in the transport equation is rather difficult to handle, and in previous calculations[3,4] it has been common to replace it by an approximate expression which involves a single relaxation time and which conserves quasiparticle number and total quasiparticle momentum. The first such calculation was that of Khalatnikov and Abrikosov,[3] who took into account only the Landau parameters[5] F_0^S and F_1^S. The calculations were extended by Brooker,[4] who included the Landau parameter F_2^S as well. A microscopic calculation of the attenuation of zero sound has been performed by Eliashberg[6]; in this calculation a number of approximations were made and the results were very similar to those obtained using the quasiparticle transport equation and the relaxation-time approximation.

The work described here was stimulated by the fact that the calculations of Khalatnikov and Abrikosov[3] apparently do not give a consistent account of the observed attenuation of zero sound and first sound in liquid He^3 – the relaxation time required to account for the zero-sound data is somewhat shorter than that required to account for the first-sound data.[7] Here, we derive expressions for the attenuation of zero sound without making the single relaxation-time approximation and show that one can give a consistent account of the data. In the collisionless regime, the properties of the sound wave are little affected by collisions; one may, therefore, use perturbation theory to calculate the effect of collisions on the zero-sound wave. This situation should be contrasted with that in the hydrodynamic regime where it is important to take into account multiple scattering effects. Detailed calculations are performed taking into account Landau parameters with $l \leq 2$ and the results are compared with the experimental data for liquid He^3. By comparing the observed attenuation of zero sound with that of first sound a rough estimate for the Landau parameter F_2^S is obtained, but the uncertainty in its value is rather large as a result of uncertainties in the experimental data. The value of F_2^S is consistent with, but somewhat more uncertain than, the value obtained from measurements of the velocity of zero sound.[4]

In Sec. 2, we describe the perturbation-theory calculation of the attenuation coefficient, and give limiting forms of the result when the zero-sound velocity is very much greater than the Fermi velocity. The method used is modeled closely on the standard calculation of the attenuation of sound in the hydrodynamic regime,[8] and is close in spirit to Gavoret's[9] calculation of the acoustic impedance of liquid He^3. The calculation is applied to liquid He^3 in Sec. 3. In Sec. 4, the influence of higher Landau parameters is shown to be small. Finally, in Sec. 5, we give a brief discussion of the results.

2. CALCULATION OF THE ATTENUATION COEFFICIENT

Consider first of all a sound wave which is weakly damped; the sound may be either first sound or zero sound. The amplitude attenuation coefficient may be written[8]

$$\alpha = |\dot{E}_{\mathrm{mech}}|/(2c\bar{E}) \ , \tag{1}$$

where $\dot{E}_{\mathrm{mech}}$ is the rate at which mechanical energy is dissipated, c is the velocity of the wave, and $\bar{E}$ is the mean energy density in the wave. $\dot{E}_{\mathrm{mech}}$ is given in terms of the rate of entropy production, $\dot{S}$, by the relation

$$\dot{E}_{\mathrm{mech}} = - T\dot{S}. \tag{2}$$

In the case of first sound, the rate of entropy production is related to the transport coefficients; for a normal Fermi liquid at low temperatures only the viscous attenuation is important and one finds the amplitude attenuation coefficient[2]

$$\alpha_1 = \frac{2}{15}\frac{m^*}{m}\frac{v_F^{\ 2}}{c_1^{\ 3}}\,\omega^2\tau_\eta \tag{3}$$

where the velocity of first sound c_1 is given by

$$c_1^{\ 2} = \tfrac{1}{3}\, v_F^{\ 2}(m^*/m)(1+F_0^{\ s}) \ . \tag{4}$$

Here, the $F_l^{\ s}$ are the spin-symmetric Landau parameters,[5] v_F is the Fermi velocity, m^* is the fermion effective mass, and m is the bare fermion mass. The relaxation time τ_η is defined in terms of the shear viscosity η by the equation

$$\eta = \tfrac{1}{5}\, np_F v_F \tau_\eta \ , \tag{5}$$

where n is the number density of particles in the liquid, and p_F is the Fermi momentum.

In the collisionless regime, a sound wave is again weakly damped, but the rate of entropy production cannot be expressed in terms of the transport coefficients; however, it can be determined from microscopic considerations. The standard expression for the rate of entropy production due to collisions may be written[10]

$$\dot{S} = \frac{1}{4k_B T^2}\sum_{\vec{p}_1,\vec{p}_2,\vec{p}_3,\vec{p}_4} W(\theta,\phi)\, n_{p_1}^{\ o} n_{p_2}^{\ o}(1-n_{p_2}^{\ o})(1-n_{p_4}^{\ o})\,\delta(\mathcal{E}_{p_1}+\mathcal{E}_{p_2}-\mathcal{E}_{p_3}-\mathcal{E}_{p_4})$$
$$\times\, \delta(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)(\bar{\nu}_{\vec{p}_1}+\bar{\nu}_{\vec{p}_2}-\bar{\nu}_{\vec{p}_3}-\bar{\nu}_{\vec{p}_4})^2 \ . \tag{6}$$

$\mathcal{E}_p$ is the energy of a quasiparticle of momentum $\vec{p}$, $n_p^{\ o} = \{\exp[(\mathcal{E}_p-\mu)/k_BT]+1\}^{-1}$ is the Fermi function and the deviation from local equilibrium of the number of quasiparticles of momentum $\vec{p}$ is given by $\bar{\nu}_{\vec{p}}(-\partial n_p^{\ o}/\partial\mathcal{E}_p)$. $W(\theta,\phi)$ is the spin-averaged scattering rate for a two-quasiparticle collision process; θ is the angle between the initial momenta $\vec{p}_1$ and $\vec{p}_2$, and ϕ is the angle between the $\vec{p}_1\vec{p}_2$ plane and the $\vec{p}_3\vec{p}_4$ plane, where $\vec{p}_3$ and $\vec{p}_4$ are the final momenta. For simplicity spin indices have been suppressed since here we deal only with spin-symmetric phenomena. Also we consider the case of a wave whose angular frequency, ω, is small compared with $k_BT/\hbar$. When $\hbar\omega \gtrsim k_BT$ the collision integral must be modified.[1,3]

When the wave is only weakly damped the rate of entropy production may be calculated by substituting into (6) the expression for $\bar{\nu}_{\vec{p}}$ for the zero-sound wave in the absence of collisions. Under these conditions $\bar{\nu}_{\vec{p}}$ is independent of the magnitude of $\vec{p}$; for a longitudinal wave, such as zero sound, $\bar{\nu}_{\vec{p}}$ will depend only on $\hat{p}\cdot\hat{q}$, where $\vec{q}$ is the wave vector of the disturbance. We may, therefore, expand $\bar{\nu}_{\vec{p}}$ in terms of Legendre polynomials

$$\bar{\nu}_{\vec{p}} = \sum_l \bar{\nu}_l P_l(\hat{p}\cdot\hat{q}) \ .$$

Substituting this expression into (6) one finds

$$T\dot{S} = \frac{\nu(0)}{\tau}\sum_{l=2}^{\infty}\frac{\xi_l}{2l+1}\,\bar{\nu}_l^{\ 2} \ . \tag{7}$$

Here $\nu(0) = m^* p_F/\pi^2\hbar^3$ is the density of quasiparticle states at the Fermi surface and τ is defined by the equation

$$1/\tau = (m^{*3}/12\pi^2\hbar^6)(k_BT)^2\langle W\rangle \ , \tag{8}$$

where $\langle W\rangle = \int(\sin\theta\, d\theta\, d\phi/4\pi)[W(\theta,\phi)/\cos\frac{1}{2}\theta]$,

and ξ_l is given by the relation

$$\xi_l\langle W\rangle = \int(\sin\theta\, d\theta\, d\phi/4\pi)[W(\theta,\phi)/\cos\tfrac{1}{2}\theta][1+P_l(\hat{p}_1\cdot\hat{p}_2)-P_l(\hat{p}_1\cdot\hat{p}_3)-P_l(\hat{p}_1\cdot\hat{p}_4)] \ .$$

ξ_0 and ξ_1 vanish identically because collisions conserve quasiparticle number and total momentum. We also note that τ/ξ_2 is the characteristic time for viscosity one obtains using the simplest variational approximation[11]; from the variational principle it follows that $\tau_\eta \geq \tau/\xi_2$.

The mean energy density in the wave is given by the usual expression[2]

$$\bar{E} = \frac{\nu(0)}{2}\sum_{l=0}^{\infty}\left(1+\frac{F_l^s}{2l+1}\right)\frac{\nu_l^2}{2l+1} \ . \tag{9}$$

The ν_l are related to the deviation from equilibrium of the quasiparticle distribution function in the same way as the $\bar{\nu}_l$ are related to the deviation from local equilibrium. The relationship between ν_l and $\bar{\nu}_l$ is[2]

$$\bar{\nu}_l = \nu_l[1+F_l^s/(2l+1)] \ . \tag{10}$$

The attenuation coefficient for zero sound α_0 may be found by combining Eqs. (1), (2), (7), and (9); the result is

$$\alpha_0 = \frac{1}{c_0\tau}\sum_{l=2}^{\infty}\xi_l\left(1+\frac{F_l^s}{2l+1}\right)^2\frac{\nu_l^2}{2l+1}\Bigg/\sum_{l=0}^{\infty}\left(1+\frac{F_l^s}{2l+1}\right)\frac{\nu_l^2}{2l+1} \ , \tag{11}$$

where c_0 is the velocity of zero sound.

To obtain the ν_l one solves the quasiparticle transport equation neglecting the collision term; some of the details of this calculation are described in Appendix A. If one takes into account only Landau parameters with $l \leq 2$, one finds

$$\alpha_0 = \frac{1}{6}\frac{m^*}{m}\frac{v_F^2}{c_0^3\tau}\frac{(1+\frac{1}{5}F_2^s)^2[\sum_{l=2}^{\infty}|\Omega_{0l}(s_0)|^2(2l+1)\xi_l]}{[(1+\frac{1}{5}F_2^s)\chi(s_0)-\frac{1}{2}F_2^s\Omega_{02}(s_0)]^2\{1-\frac{2}{15}(m^*/m)(1+\frac{1}{5}F_2^s)(s_0)^{-1}[\partial\Lambda(s_0)/\partial s_0]\}} \ , \tag{12}$$

where $s_0 = c_0/v_F$,

$$\chi(s) = 1+\tfrac{1}{2}s\ln\left(\frac{s-1}{s+1}\right) \approx -\frac{1}{3s^2}-\frac{1}{5s^4}-\frac{1}{7s^6}+O\left(\frac{1}{s^8}\right), \quad (s\gg 1) \ , \tag{13}$$

$$\Omega_{02}(s) = \tfrac{1}{2}+P_2(s)\chi(s) \approx -\frac{2}{15s^2}-\frac{4}{35s^4}-\frac{2}{21s^6}+O\left(\frac{1}{s^8}\right), \quad (s\gg 1) \ , \tag{14}$$

and

$$\Lambda(s) = \tfrac{5}{2}\Omega_{02}(s)/[(1+\tfrac{1}{5}F_2^s)\chi(s)-\tfrac{1}{2}F_2^s\Omega_{02}(s)$$

$$\approx 1+\frac{9}{35s^2}(1+\tfrac{1}{5}F_2^s)+\frac{1}{s^4}\left[\frac{23}{175}+\frac{242}{6125}F_2^s+\left(\frac{9}{175}\right)^2(F_2^s)^2\right]+O\left(\frac{1}{s^6}\right), \quad (s\gg 1) \ . \tag{15}$$

Expressions for the Ω_{0l} are given in Appendix B. The form in which the result (12) has been written is particularly convenient for studying its behavior for $s_0^2 \gg 1$, which is well satisfied for liquid He^3. (For liquid He^3, $s_0^2 \approx 12$ at low pressure and $s_0^2 \approx 140$ at 28 atm.) At high values of s_0, the attenuation is given by

$$\alpha_0 = \frac{2}{15}\frac{m^*}{m}\frac{v_F^2}{c_0^3}\frac{\xi_2}{\tau}(1+\tfrac{1}{5}F_2^s)^2\left[1+O\left(\frac{1}{s_0^2}\right)\right] \ . \tag{16}$$

When the velocity of zero sound is very much greater than the Fermi velocity, the dominant contribution to the attenuation comes from the second harmonic distortion of the Fermi surface. Mathematically the reason for this is that $\Omega_{0l} \sim 1/s_0^l$, $(l \geq 1,\ s_0 \gg 1)$. Physically this corresponds to the fact that the amplitudes of the higher harmonic distortions of the Fermi surface fall off rapidly as a function of l. Equation (16) also exhibits clearly the possibly important influence of the Landau parameter F_2^S on the attenuation.

Another simple result is obtained if one puts $\xi_l = \xi_2(1+\frac{1}{5}F_2^S)$, $(l > 2)$; this corresponds to the single relaxation-time approximation. The sum over l in (12) may then be evaluated easily by using the completeness relation for Legendre polynomials, and one finds

$$\alpha_0 = \frac{2}{15}\frac{m^*}{m}\frac{v_F^2}{c_0^3}\frac{\xi_2}{\tau}(1+\tfrac{1}{5}F_2^S)[\Lambda - s_0(\partial\Lambda/\partial s_0)]\Big/[1-\tfrac{2}{15}(m^*/m)(1+\tfrac{1}{5}F_2^S)(s_0)^{-1}(\partial\Lambda/\partial s_0)] \ , \tag{17}$$

a result which agrees with that of Brooker,[4] provided one identifies his τ with our $\tau/[\xi_2(1+\frac{1}{5}F_2^S)]$. The Fermi-liquid factor $1+\frac{1}{5}F_2^S$ occurs in the relationship between the two relaxation times because it is conventional when using the single relaxation-time approximation to write the collision integral in terms of the deviation of the quasiparticle distribution from equilibrium, whereas in the microscopic expression for the collision integral, the quantity which occurs naturally is the deviation of the quasiparticle distribution from local equilibrium. [See Eqs. (6) and (10).]

The results of this section may also be derived by calculating the shifts in the eigenvalues of the quasiparticle transport equation; this procedure is closely related to earlier work on zero sound and is described in Appendix A.

3. APPLICATION TO LIQUID He^3

We now apply the results to the experimental data for liquid He^3 at low pressure.[7] For this system $s_0^2 \approx 12$ and, therefore, expansions in inverse powers of s_0^2 are very convenient for calculations. Making such an expansion of (12) one finds

$$\alpha_0 = \frac{2}{15}\frac{m^*}{m}\frac{v_F^2}{c_0^3\tau}(1+\tfrac{1}{5}F_2^S)^2\left[\xi_2 + \frac{9}{35s_0^2}[2\xi_2(1+\tfrac{1}{5}F_2^S)+\xi_3] + O\left(\frac{1}{s_0^4}\right)\right] . \tag{18}$$

For liquid He^3 at low pressure, the s_0^{-4} terms are only of the order of 0.1% of the total, and may therefore be neglected.

In analyzing the experimental data, we first of all extract a value of τ/ξ_2 from the first-sound attenuation measurements. As one can see from Eq. (18) the time τ/ξ_2 also determines the dominant contribution to the attenuation of zero sound, and by using Eq. (18) and the observed value of α_0 we then obtain an estimate of the Landau parameter F_2^S.

According to the work of Brooker and Sykes,[12] and Jensen, Smith, and Wilkins,[13] the characteristic relaxation time which determines the viscosity is given by

$$\frac{\xi_2\tau_\eta}{\tau} = \frac{\xi_2}{3}\sum_{n=0}^{\infty}\frac{(4n+3)}{(n+1)(2n+1)[n(2n+3)+\xi_2]} . \tag{19}$$

ξ_2 cannot be determined directly from experiment, but if one uses for $W(\theta, \phi)$ the approximations in terms of Landau parameters described by Dy and Pethick,[14] one finds values of $\xi_2\tau_\eta/\tau$ which lie between 1.035 and 1.055; the actual value depends both on the value of the Landau parameter F_1^a used and also on how the scattering amplitude is approximated. In the calculations we use the value 1.045 for $\xi_2\tau_\eta/\tau$; this number is uncertain by a few percent. The observed value of α_1 is $2.68\times 10^{-18}\ \omega^2(T^*)^{-2}\ \text{cm}^{-1}$, where T^* is the magnetic temperature in °K. Using for m^*/m the value[15,16] 3.0, and for c_1 the value[17] 187.9 m/sec, one finds from Eq. (3) that $\tau(T^*)^2/\xi_2 = 1.36\times 10^{-12}$ sec (°K)^2.

The Landau parameter F_2^S may be estimated by using Eq. (18) if ξ_3 is known. ξ_3 cannot be determined experimentally, but one can show that it must lie between zero and $\frac{20}{9}$. If one replaces $W(\theta, \phi)$ by a constant, one finds $\xi_3 = \frac{4}{7}$, which is the value used in the calculation. Using the approximations for the scattering amplitude given in Ref. 14 one finds values of ξ_3 which are very close to this value. Fortunately, the value of ξ_3 is relatively unimportant, since even if ξ_3 took on the extreme value $\frac{20}{9}$, the third harmonic would give rise to only about 10% of the total attenuation. The observed value of α_0 is $1.57\times 10^6(T^*)^2\ \text{cm}^{-1}$, which leads to a value of 1.08 for $1+\frac{1}{5}F_2^S$, or, in other words, $F_2^S \approx 0.4$.

The uncertainty in the value of F_2^S is rather large. If Eqs. (3) and (16) are multiplied together, and terms of order s_0^{-2} are neglected for simplicity, one finds

$$1+\frac{1}{5}F_2{}^S \approx \frac{15}{2}\,\frac{m}{m^*}\left[\left(\frac{\alpha_0}{T^2}\right)\left(\frac{\alpha_1 T^2}{\omega^2}\right)\left(\frac{c_0}{v_F{}^0}\right)^3\left(\frac{c_1}{v_F{}^0}\right)^3\left(\frac{\tau}{\xi_2\tau_\eta}\right)\right]^{1/2}, \tag{20}$$

where $v_F{}^0$ is the Fermi velocity of a free Fermi gas. The major sources of uncertainty are the measurements of α_0/T^2 and $\alpha_1 T^2/\omega^2$. The error in these measurements is difficult to estimate accurately, but according to Wheatley[15] the error in the value of $\alpha_1 T^2/\omega^2$ is unlikely to be greater than 15%. In the measurements of α_0/T^2 there is an additional source of error due to uncertainties in the temperature scale. We assumed $T^* = T$, although recent work by Abel and Wheatley[18] indicates that if one writes the relation between T and T^* in the form $T = T^* + \Delta$, the value of Δ is $\lesssim 0.3m\,°\mathrm{K}$. Uncertainties of this order of magnitude will have an appreciable effect on the evaluation of $\alpha_0(T)$, since all the measurements on zero sound were carried out at temperatures below $10m°\mathrm{K}$. The first-sound results, which were obtained at somewhat higher temperatures, will be little affected by changes in the temperature scale. Anderson[19] estimates that the uncertainty from all sources in the value of α_0/T^2 amounts to less than 15%. The other sources of error in the evaluation of $1+\frac{1}{5}F_2{}^S$ are all very much smaller. m^*/m was determined from measurements of the specific heat C_V at low temperatures. Besides the problem of the temperature scale, the main difficulty here is the extrapolation of the data to $T=0$. The value of m^*/m we used $(m^*/m = 3.0)$ is the average of the two values obtained by Mota, Platzeck, Rapp, and Wheatley[16] who fitted C_V/T^* to functions of the form $\gamma - \beta T^*$ and $\gamma + \Gamma T^{*2}\ln(T^*/\theta_c)$, where γ, β, Γ, and θ_c are constants. The two values of m^*/m differ by only 2%, which is a measure of the uncertainty in the extrapolation. Adjustment of the temperature scale could also lead to a downward shift of m^*/m by a percent or so, and the total uncertainty in the value of m^*/m is therefore of the order of a few percent. The uncertainties in the values of c_0, c_1, and $v_F{}^0$ are all negligible compared with the other uncertainties. Taking into account the uncertainties in $\tau/\xi_2\tau_\eta$ and ξ_3 we conclude that a generous estimate of the uncertainty in the value of $1+\frac{1}{5}F_2{}^S$ is 20%. Thus, from the attenuation data we find $F_2{}^S = 0.4 \pm 1$. We also note that the calculations of Abrikosov and Khalatnikov[3] do give a consistent account of the data if one takes into account the experimental uncertainties.

$F_2{}^S$ may also be estimated from the velocity of zero sound.[4] Rearranging the Eq. (A4) for $c_0\,(=s_0 v_F)$, and making use of Eq. (4) for c_1 one finds

$$\frac{c_0{}^2 - c_1{}^2}{c_1{}^2} = \frac{4}{5}\,\frac{(1+\frac{1}{5}F_2{}^S)}{(1+F_0{}^S)}\,\Lambda(s_0) = \frac{4}{5}\,\frac{(1+\frac{1}{5}F_2{}^S)}{(1+F_0{}^S)}\left[1+\frac{9}{35 s_0{}^2}(1+\frac{1}{5}F_2{}^S) + 0\left(\frac{1}{s_0{}^4}\right)\right] \qquad (s_0 \gg 1)\ . \tag{21}$$

From the measurements of Abel, Anderson, and Wheatley,[7] $(c_0 - c_1)/c_1 = 0.035 \pm 0.003$. Substituting this value into Eq. (21) and using the value of $F_0{}^S$ obtained from the measured first-sound velocity, one finds[20] $F_2{}^S = 0 \pm 0.4$. Uncertainties other than that in $(c_0 - c_1)/c_1$ are small and may be neglected in estimating the uncertainty in the value of $F_2{}^S$.

The two estimates of $F_2{}^S$ we have obtained are consistent with each other, but the uncertainty in the value of $F_2{}^S$ obtained from measurements of the velocity is somewhat less than the uncertainty in the value obtained from the attenuation.

4. EFFECTS OF HIGHER-ORDER LANDAU PARAMETERS

The effects of higher-order Landau parameters on the properties of zero sound may easily be investigated by extending the analysis described in Sec. 2. These parameters are relatively unimportant for the case of liquid He^3 because the zero-sound wave contains rather small admixtures of the higher-harmonic distortions of the Fermi surface. The leading contribution to c_0 as a result of $F_l{}^S$ being nonzero is of the order of $(F_l{}^S/s_0{}^{2(l-1)})c_0$, and the corresponding contribution to α_0 is of order $F_l{}^S/s_0{}^{2(l-2)}$ relative to the result for the strong-coupling limit [Eq. (16)]. One expects that for liquid He^3, the most important higher-order Landau parameter will be $F_3{}^S$. If one puts[21] $F_3{}^S = 1$, the change in the velocity of zero sound for liquid He^3 at low pressure is of the order of one part in 10^4, and the corresponding change in the attenuation is of the order of one part in 10^2; the effects of Landau parameters with $l > 2$ are therefore expected to be negligible.

5. DISCUSSION

Using exact expressions for the attenuation of zero sound and first sound, we have given an account of the experimental data for liquid He^3. These data and those on the velocities of zero sound and first sound are consistent with $F_2{}^S = 0$.

In the calculations described above, we have considered only the collisionless regime, where the attenuation may be evaluated rather easily. To discuss the transition from zero sound to first sound, it is necessary to expand $\nu_{\vec{p}}$ in terms of complete sets of functions of $\mathcal{E}_p$ and angles. The calculations involve rather large determinants which will generally have to be solved numerically.

For the case of liquid He^3, the single relaxation-time approximation works remarkably well: Our calculations indicate that the effective relaxation times in the hydrodynamic and collisionless regimes differ by only about 5%. The essential reason for this is that in both the zero-sound and first-sound regimes the dominant contribution to the attenuation comes from the quadrupolar ($l=2$) distortion of the Fermi surface. In a weak-coupling Fermi system the single relaxation-time approximation works less well because the harmonic content of the sound wave alters greatly in the transition from zero sound to first sound. The single relaxation-time approximation will also be poor for "weak-coupling" modes such as the transverse zero-sound mode in liquid He^3.[1-3,22]

The method we have described may be used to calculate the damping of other collective modes in the collisionless regime. One such problem to which the method has been applied is that of calculating the attenuation of phonons in dilute solutions of He^3 in liquid He^4; this calculation is of interest in evaluating the thermal conductivity of the solutions.[23] Results of these calculations will be reported elsewhere.

Note added in proof. Calculations of the attenuation of zero sound using somewhat more restrictive assumptions than the ones made here have been carried out by L. R. Corruccini, J. S. Clarke, N. D. Mermin, and J. W. Wilkins [Phys. Rev. 180, 225 (1969)] and by B. S. Lukyanchouk [Zh. Eksperim. i Teor. Fiz. 56, 1338 (1969)]. Dr. G. A. Brooker has informed me that he and J. Sykes (unpublished) have studied the propagation of sound in a normal Fermi liquid and arrive at results which apparently agree with our own.

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APPENDIX A: QUASIPARTICLE TRANSPORT EQUATION

The standard quasiparticle transport equation may be written in the form[1-3]

$$(\omega-\vec{q}\cdot\vec{v}_{\vec{p}})\nu_{\vec{p}}+\vec{q}\cdot\vec{v}_{\vec{p}}\sum_{\vec{p}'}f_{\vec{p}\vec{p}'}\left(\frac{\partial n_{p'}^0}{\partial\mathcal{E}_{p'}}\right)\nu_{\vec{p}}=-iI[\nu_{\vec{p}}]/(\partial n_p^0/\partial\mathcal{E}_p)\ . \tag{A1}$$

ω is the angular frequency of the wave, $\vec{v}_{\vec{p}}$ is the velocity of a quasiparticle of momentum $\vec{p}$, $f_{\vec{p}\vec{p}'}$ is Landau's quasiparticle interaction function and $I[\nu_{\vec{p}}]$ is the collision integral. The velocity and attenuation of sound may be found from the eigenvalues of Eq. (A1).

In the absence of collisions, the eigenfunctions depend only on the direction of $\vec{p}$, and may be expanded in Legendre polynomials. Equation (A1) then reduces to[3]

$$\frac{\nu_l}{2l+1}+\sum_{l'=0}^{\infty}\Omega_{ll'}F_{l'}^s\frac{\nu_{l'}}{2l'+1}=0\ , \tag{A2}$$

where $$\Omega_{ll'}=\int_{-1}^{1}\frac{d\mu}{2}P_l(\mu)\frac{\mu}{\mu-s}P_{l'}(\mu)\ . \tag{A3}$$

Some useful properties of the $\Omega_{ll'}$ are discussed in Appendix B. If Landau parameters with $l>2$ are neglected, the eigenvalue problem is

$$1+F_2^s\Omega_{22}(s_0)+(F_0^s+A_1^s s_0^2)\left[\left(1+\frac{F_2^s}{5}\right)\chi(s_0)-\tfrac{1}{2}F_2^s\Omega_{02}(s_0)\right]=0\ , \tag{A4}$$

where $A_1^s=F_1^s/[1+\frac{1}{3}F_1^s]$. The corresponding eigenvector is

$$\bar{\nu}_1/3=s_0\nu_0\ , \tag{A5}$$

$$\frac{\bar{\nu}_l}{2l+1} = -\Omega_{l0}\frac{(F_0{}^s + A_1{}^s s_0{}^2)}{(1+F_2{}^s\Omega_{22})}(1+\tfrac{1}{5}F_2{}^s)\nu_0 \qquad (l \geq 2)\ . \tag{A6}$$

Equation (A5) follows directly from the equation of continuity. Substituting Eqs. (A5) and (A6) into (11), and making use of Eq. (A4) and the completeness relation for Legendre polynomials one may obtain Eq. (12).

An alternative way of deriving an expression for the attenuation in the collisionless regime is to evaluate directly the shift in the eigenvalue of Eq. (A1) due to the collision term. In the absence of collisions $\nu_{\vec{p}}$ depends only on $\mu = \hat{p}\cdot\hat{q}$, and, therefore, to lowest order in $(\omega\tau)^{-1}$ one may evaluate the collision term in (A1) neglecting the dependence of $\nu_{\vec{p}} = \nu(p,\mu)$ on p. After multiplying the equation by $-\partial n_p{}^0/\partial\mathcal{E}_p$ and integrating over the magnitude of $\vec{p}$ one finds

$$(s-\mu)\sum_{l=0}^{\infty}\nu_l P_l(\mu) - \mu\sum_{l=0}^{\infty}\frac{F_l{}^s}{(2l+1)}\nu_l P_l(\mu) = -\frac{i}{v_F q\tau}\sum_{l=2}^{\infty}\xi_l\left(1+\frac{F_l{}^s}{2l+1}\right)\nu_l P_l(\mu)\ , \tag{A7}$$

where $\mu = \hat{p}\cdot\hat{q}$, and

$$\sum_{l=0}^{\infty}\nu_l P_l(\mu) = \int d\mathcal{E}_p\left(-\frac{\partial n_p{}^0}{\partial\mathcal{E}_p}\right)\nu(p,\mu)\ . \tag{A8}$$

Equation (A7) is identical in form with the equation one obtains using a generalized relaxation-time approximation in which the relaxation time is allowed to depend on l. However, we stress the fact that (A7) is correct only to lowest order in $(\omega\tau)^{-1}$.

Dividing Eq. (A7) by $s-\mu$ and expanding in terms of Legendre polynomials one finds

$$\frac{\nu_l}{2l+1} + \sum_{l'=0}^{\infty}\Omega_{ll'}F_{l'}{}^s\frac{\nu_{l'}}{2l'+1} = \frac{i}{\omega\tau}\sum_{l'=2}^{\infty}\left(\Omega_{ll'} - \frac{\delta_{ll'}}{2l+1}\right)\left(1+\frac{F_{l'}{}^s}{2l'+1}\right)\nu_{l'}\ . \tag{A9}$$

If the eigenvalue of this equation is $s = s_0 - i\Delta s$, α_0 is given by $q\Delta s/s_0$.

The eigenvalue problem may be simplified if one assumes that all but a finite number of Landau parameters may be neglected. In particular, if one assumes that $F_l{}^s = 0$ for $l > l_0$, one may substitute for ν_l, $(l > l_0)$, in the collision term its value in the absence of collisions. One then finds

$$\frac{\nu_l}{2l+1} + \sum_{l'=0}^{\infty}\left[\Omega_{ll'}F_{l'}{}^s\frac{\nu_{l'}}{2l'+1} - \frac{i}{\omega\tau}\left(\Omega_{ll'} - \frac{\delta_{ll'}}{2l+1}\right)\left(1+\frac{F_{l'}{}^s}{2l'+1}\right)\xi_{l'}\nu_{l'} + \frac{i}{\omega\tau}\sum_{l''=l_0+1}^{\infty}\Omega_{ll''}(2l''+1)\xi_{l''}\Omega_{l''l'}F_{l'}{}^s\frac{\nu_{l'}}{2l+1}\right] = 0\ . \tag{A10}$$

The results obtained by this method are completely equivalent to those obtained using the method described in Sec. 2.

APPENDIX B: PROPERTIES OF $\Omega_{ll'}$

$\Omega_{ll'}$ [Eq. (A3)] may be expressed directly in terms of Legendre functions of the second kind, $Q_l(s)$[24]:

$$\Omega_{l'l} = \Omega_{ll'} = \frac{\delta_{ll'}}{(2l+1)} - sP_{l'}(s)Q_l(s) \qquad (l' \leq l)\ . \tag{B1}$$

$sQ_l(s)$ may be written in the form[24]

$$sQ_l(s) = -P_l(s)\chi(s) + P_l(s) - sW_{l-1}(s)\ , \tag{B2}$$

where

$$W_{l-1}(s) = \sum_{k=1}^{2E[\frac{1}{2}(l-1)]}\frac{2(l-2k)-1}{(2k+1)(l-k)}P_{l-2k-1}(s)\ , \quad (l \geq 1) \tag{B3}$$

$$W_{l-1}(s)=\sum_{k=1}^{l}\frac{1}{k}P_{k-1}(s)P_{l-k}(s)\ ,\quad (l\geq 1) \tag{B4}$$

$$W_{-1}=0\ .$$

Here $E[x]$ denotes the integer part of x. The first few $\Omega_{ll'}$ are given by

$$\Omega_{00}=\chi\ ,$$

$$\Omega_{10}=s\chi\ ,\qquad \Omega_{11}=\tfrac{1}{3}+s^2\chi\ ,$$

$$\Omega_{20}=\tfrac{1}{2}+P_2(s)\chi\ ,\ \Omega_{21}=s[\tfrac{1}{2}+P_2(s)\chi\,]\ ,\ \Omega_{22}=\tfrac{3}{4}s^2-\tfrac{1}{20}+P_2(s)\chi\ .$$

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COMMENTS AND ADDENDA

The Comments and Addenda section is for short communications which are not of such urgency as to justify publication in Physical Review Letters and are not appropriate for regular Articles. It includes only the following types of communications: (1) comments on papers previously published in The Physical Review or Physical Review Letters; (2) addenda to papers previously published in The Physical Review or Physical Review Letters, in which the additional information can be presented without the need for writing a complete article. Manuscripts intended for this section may be accompanied by a brief abstract for information-retrieval purposes. Accepted manuscripts will follow the same publication schedule as articles in this journal, and galleys will be sent to authors.

Fermi-Liquid Transport Coefficients of Dilute Solutions of He^3 in He^4: An Addendum

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Using recent exact solutions of the quasiparticle transport equation, we have reexamined the consistency between Bardeen, Baym, and Pines's phenomenological theory of dilute solutions of He^3 in He^4 and measurements of the spin-diffusion and thermal-conductivity coefficients of two solutions in the degenerate Fermi-liquid regime. Previously, Baym and the author had used lowest-order variational solutions for this purpose. Discrepancies of 10–15% persist which, beyond experimental uncertainty, must be attributed to oversimplification in treating the He^3 scattering amplitudes as being independent of spin, velocity, and concentration.

This paper is intended to bring up to date a recent paper by Baym and Ebner[1] (BE) in which it was demonstrated that the phenomenological theory of dilute solutions of He^3 in He^4 as given by Bardeen, Baym, and Pines,[2] is consistent with experimental determinations of the thermal-conductivity[3] and spin-diffusion[4] coefficients of 1.3 and 5.0% systems if simple variational solutions of the transport equation are used to relate the He^3-He^3 effective interaction to the transport coefficients. Previously, the approximate solutions of Abrikosov and Khalatnikov[5] and of Hone[6] had been used for this purpose.

More recently, exact analytical solutions of the transport equation have been determined by Brooker and Sykes[7] and independently by Jensen and co-workers.[8] Using the exact results of Brooker and Sykes, we have repeated the calculations reported in (BE). That is, we begin by assuming that $V(k)$, the Fourier transformed He^3-He^3 effective interaction, may be expanded in powers of k^2 with undetermined coefficients which are chosen by attempting to fit the experimental transport coefficients. In so doing, $V(k)$ is allowed to be only reasonably rapidly varying. A typical result of this procedure is the interaction

$$V(k) = V_0(1 - 3.389y + 6.353y^2 - 9.576y^3 + 5.402y^4), \quad (1)$$

where $y = (k/2k_0)^2$; k_0 is the Fermi momentum of a 5.0% solution of He^3 in He^4, $k_0/\hbar = 0.318$ Å^{-1}; and $V_0 = -0.078\ m_4 s^2/n_4$. The mass of a He^4 atom is m_4 and the speed of first sound and the number density in pure He^4 at $T=0$ are s and n_4, respectively. Figure 1 shows this $V(k)$ and also the interaction found in BE as well as the original interaction of Bardeen, Baym, and Pines.[2] As can be seen, $V(0)$ is close to the original value of Ref. 2; it is also very close to the value deduced by Baym[9] from thermodynamic arguments, which is $V(0) = \alpha^2 m_4 s^2/n_4 \cong -0.077 m_4 s^2/n_4$; α is the fractional excess molar volume of He^3 in He^4.

The present $V(k)$ shows different behavior from