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Transport Coefficients of a Normal Fermi Liquid: Application to Liquid He^3 †

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The transport coefficients of a normal Fermi liquid in the extreme low-temperature limit and the leading finite-temperature contributions to the transport coefficients are studied using the quasiparticle transport equation. The scattering amplitude for quasiparticles on the Fermi surface is expressed in terms of Landau parameters, using an approximation which takes into account s- and p-wave scattering; the calculated values of the transport coefficients in the extreme low-temperature limit for liquid He^3 are shown to be in good agreement with the experimental values. Finite-temperature contributions to the transport coefficients are expressed in terms of the solution of the transport equation in the extreme low-temperature limit, and explicit calculations are performed for the case of liquid He^3. Assuming Landau parameters with $l \geq 2$ vanish, an estimate of the Landau parameter F_1^a for liquid He^3 is obtained by comparing the theoretical and experimental values of the leading finite-temperature contributions to the transport coefficients.

I. INTRODUCTION

Recent experimental work on the transport properties of liquid He^3 and dilute mixtures of He^3 in He^4 has led to renewed interest in the problem of solving the quasiparticle transport equation and of determining the quasiparticle scattering amplitude. Historically, the first method of solving the transport equation was that of Abrikosov and Khalatnikov,[1] who calculated the limiting low-temperature behavior of the coefficient of thermal conductivity K and the coefficient of viscosity η.

This method was applied by Hone[2] in a calculation of the spin-diffusion coefficient D in the low-temperature limit, and by Emery[3] in calculations of finite-temperature contributions to the transport coefficients. A second approximate solution to the transport equation is based on the variational method[4]; using this method, Baym and Ebner[5] were able to give a more consistent account of the transport properties of dilute mixture of liquid He^3 in He^4 than was possible using Abrikosov and Khalantnikov's and Hone's solutions of the transport equation. Rice's[6] calculations of the transport coefficients of almost ferromagnetic Fermi liquids and Pethick's[7,8] calculations of finite-temperature contributions to the transport coefficients of a normal Fermi liquid are also based on the variational method. Recently, exact analytical solutions of the transport equation in the limit $T \to 0$ have been obtained by Brooker and Sykes[9] and by Jensen, Smith, and Wilkins.[10] Emery and Cheng[11] have described an approximate method for solving the transport equation which in the limit $T \to 0$ gives results little different from the exact solution; these authors have also used this method to calculate finite-temperature contributions to the transport coefficients.

In this paper, we show that the leading finite-temperature contributions to the transport coefficients may be calculated using exact solutions of the transport equation in the limit $T \to 0$.[12] These solutions depend on the scattering amplitude for quasiparticles on the Fermi surface, which also determines the values of the transport coefficients in the extreme low-temperature limit. In previous calculations,[9,13,14] the scattering amplitude has often been replaced by the forward-scattering amplitude; the latter may be expressed in terms of Landau parameters,[15] some of which may be determined from measurements of the equilibrium properties of the liquid. However, recent calculations using exact solutions of the transport equation and the forward-scattering approximation for the scattering amplitude lead to values for the transport coefficients[9] which are consistently higher than the experimental values, the difference between theory and experiment being as large as a factor of 3 in the case of the spin-diffusion coefficient of liquid He^3. Replacing scattering amplitudes by forward-scattering amplitudes is equivalent to assuming that all scattering is s wave; this is clearly inconsistent with the Pauli principle, which requires that the triplet scattering amplitude contain no contributions from even partial waves.[16] We shall show that good agreement between theory and experiment may be obtained if one assumes that the singlet scattering amplitude contains only an s-wave contribution and that the triplet scattering amplitude contains only a p-wave contribution. This approximation is more consistent with the requirements of the Pauli principle than is the forward-scattering approximation, and, moreover, it introduces no new parameters into the calculation. The net effect of including the p-wave amplitude is to increase the amount of scattering. The increase is particularly large for collisions between quasiparticles of opposite spin in which the relative momentum of the two quasiparticles is turned through an angle $\approx \pi$. The transport coefficient most affected by the change in the approximation for the quasiparticle scattering amplitude is the spin-diffusion coefficient, which depends strongly on the amplitude for these processes.

Besides the solution of the transport equation in the limit $T \to 0$, the other essential ingredient in the calculation of the finite-temperature contributions to the transport coefficients is the frequency-dependent quasiparticle scattering amplitude for small momentum transfer processes. This frequency-dependent interaction contains the effects which are attributed to scattering by paramagnons in other calculations.[6] By comparing the calculations with the experimental data for liquid He^3, we are able to obtain an estimate for the Landau parameter F_1^a. The calculations show that to within the limits of accuracy of the experimental data and of the theoretical calculations, Landau theory provides a consistent account of both equilibrium and nonequilibrium properties of liquid He^3 if one neglects Landau parameters with $l \geq 2$.

In Sec. II, we discuss the quasiparticle transport equation and separate out finite-temperature contributions to the collision integral. The solutions of the transport equation are described in Sec. III, where we give expressions for the leading finite-temperature contributions to the transport coefficients. The form of the quasiparticle scattering amplitude for quasiparticles on the Fermi surface is discussed in Sec. IV, and the s- and p-wave approximation for this scattering amplitude in terms of Landau parameters is described. The results are applied to the particular case of liquid He^3 in Sec. V, and Sec. VI is a discussion of approximations made in the calculations and extensions of the work to other problems.

II. QUASIPARTICLE TRANSPORT EQUATION

In this section, we consider the quasiparticle transport equation and separate out the terms in the collision integral which are due to the frequency dependence of the effective interaction between quasiparticles; these terms give rise to the leading finite-temperature contributions to the transport coefficients. For definiteness, let us consider the thermal conductivity; the standard linearized quasiparticle transport equation[1,15] with the usual binary collision term may be writ-

ten in a convenient vector notation as

$$|X\rangle = I\,|\Phi\rangle\ . \tag{1}$$

The components of $|X\rangle$ are given by

$$X_i = n_i(1-n_i)[(\mathcal{E}_i + TS)/k_B T]\vec{v}_i \cdot \hat{u}\,, \tag{2}$$

where $\mathcal{E}_i$ is the energy, measured with respect to the chemical potential, of a quasiparticle of momentum $\vec{p}_i$, velocity $\vec{v}_i$, and spin σ_i. n_i is the Fermi distribution function and $\hat{u}$ is a unit vector in the direction of the temperature gradient. S is the entropy per particle. The deviation from local equilibrium of the number of quasiparticles in the state i, δn_i is given in terms of the components of the vector $|\Phi\rangle$ by the expression

$$\delta n_i = -n_i(1-n_i)(\Phi_i\,|\nabla T|/k_B{}^2T^3)\,, \tag{3}$$

where ∇T is the temperature gradient. The product operation in (1) corresponds to summation over the momentum and spin of a quasiparticle, and multiplication by $1/k_BT$. The components of the vector $I\,|\Phi\rangle$ are given by

$$\begin{aligned}(I\,|\Phi\rangle)_1 = (k_BT)^{-2} \sum_{2,3,4} & W_{12}{}^{34} n_1 n_2 \\ & \times (1-n_3)(1-n_4)\delta_{\vec{p}_1+\vec{p}_2,\,\vec{p}_3+\vec{p}_4}\delta_{\sigma_1+\sigma_2,\,\sigma_3+\sigma_4} \\ & \times \delta(\mathcal{E}_1+\mathcal{E}_2-\mathcal{E}_3-\mathcal{E}_4)(\Phi_1+\Phi_2-\Phi_3-\Phi_4).\end{aligned} \tag{4}$$

Here, $W_{12}{}^{34}$ is the transition probability for a collision in which two quasiparticles are scattered from the states 1 and 2 into the states 3 and 4. The sum over states 3 and 4 in Eq. (4) is to be taken only over distinct states of the pair of quasiparticles. In terms of this vector notation, the thermal conductivity K is given by

$$KT = \langle X\,|\Phi\rangle\ . \tag{5}$$

The energies of most thermal excitations differ from the chemical potential only by quantities of the order of k_BT, and therefore, in the extreme low-temperature limit, the dependence of $W_{12}{}^{34}$ on the magnitudes of the quasiparticle momenta may be neglected. $W_{12}{}^{34}$ then depends only on θ, the angle between $\vec{p}_1$ and $\vec{p}_2$, and ϕ, the angle between the $\vec{p}_1\vec{p}_2$ plane and the $\vec{p}_3\vec{p}_4$ plane; the transition probabilities are denoted by $W_{\uparrow\uparrow}(\theta,\phi)$ and $W_{\uparrow\downarrow}(\theta,\phi)$ for collisions between quasiparticles of like spin and unlike spin, respectively. (For definiteness the spin of quasiparticle 3 is here assumed to be the same as that of quasiparticle 1.) At low temperatures, further simplifications of the transport equation are possible. Firstly, for the purpose of calculating both the value of the transport coefficients in the extreme low-temperature limit and the leading finite-temperature contributions, the density of quasiparticle states may be put equal to its value at the Fermi surface. Secondly, the TS term in (2) and the temperature dependence of the chemical potential may be neglected, since both these terms give rise to corrections to the transport coefficients which are of order T^2 relative to their behavior in the extreme low-temperature limit.

With the above simplifications and after transformation of the summations in (4) into integrations over θ, ϕ and the quasiparticle energies[1,9,15,17] the transport equation in the extreme low-temperature limit may be written

$$|X\rangle = I^{(0)}\,|\Phi^{(0)}\rangle, \tag{6}$$

where $|\Phi^{(0)}\rangle$ is the solution of the transport equation in the extreme low-temperature limit, and $I^{(0)}\,|\Phi\rangle$ has components

$$\begin{aligned}\frac{m^{*3}}{8\pi^4\hbar^6(k_BT)^2}\int & \frac{\sin\theta\,d\theta d\phi}{4\pi}\,\frac{W(\theta,\phi)}{\cos\frac{1}{2}\theta} \\ & \times d\mathcal{E}_2 d\mathcal{E}_3 d\mathcal{E}_4 n_1 n_2(1-n_3)(1-n_4) \\ & \times \delta(\mathcal{E}_1+\mathcal{E}_2-\mathcal{E}_3-\mathcal{E}_4)(\Phi_1+\Phi_2-\Phi_3-\Phi_4),\end{aligned} \tag{7}$$

where $2W(\theta,\phi) = W_{\uparrow\downarrow}(\theta,\phi) + \frac{1}{2}W_{\uparrow\uparrow}(\theta,\phi)$, and m^* is the effective mass. From the invariance of W under rotation of the quasiparticle momenta and rotation of the spins, it follows that Φ must have the same angular dependence as X and must be spin-independent. Φ may, therefore, be written in the form

$$\Phi_i = (\vec{v}_i \cdot \hat{u})(8\pi^4\hbar^6/m^{*3}\langle W\rangle)\,c(x_i), \tag{8}$$

where $x_i = \mathcal{E}_i/k_BT$, and $c(x_i)$ is a dimensionless function which in the extreme low-temperature limit depends on the temperature only through its dependence on x_i. $\langle W\rangle$ is defined by the relation

$$\langle W\rangle = \int [d(\cos\theta)\,d\phi/4\pi\cos\tfrac{1}{2}\theta]\,W(\theta,\phi). \tag{9}$$

On combining (1), (2), (7), and (8), the transport equation in the extreme low-temperature limit reduces to[1,9,10]

$$\begin{aligned}x = \tfrac{1}{2}(x^2+\pi^2)\,c^{(0)}(x) - \lambda_K & \\ \times \int_{-\infty}^{\infty} dt\,\frac{t}{(e^t-1)}\,\frac{(1+e^{-x})}{(1+e^{-(t+x)})}\,c^{(0)}(x+t), & \end{aligned} \tag{10}$$

where $(1-\frac{1}{3}\lambda_K)\langle W\rangle = \frac{2}{3}\langle W(1-\cos\theta)\rangle\,.$ (11)

The transport equations for spin diffusion and viscosity may be reduced to equations similar to (10) provided one makes the appropriate modifications in the angular and spin dependence of Φ [Eq. (8)]; the equations for the corresponding functions $c^{(0)}(x)$ are the same as Eq. (10) but with the factor x absent on the left-hand side and λ_K replaced by λ_D, in the case of spin diffusion, or by λ_η, in the case of viscosity. λ_D and λ_η are defined by the equations

$$(1-\lambda_D)\langle W\rangle = \tfrac{1}{2}\langle W_{\uparrow\downarrow}(\theta,\phi)(1-\cos\theta)(1-\cos\phi)\rangle , \quad (12)$$

$$(1-\lambda_\eta)\langle W\rangle = \tfrac{3}{4}\langle W(\theta,\phi)(1-\cos\theta)^2\sin^2\phi\rangle . \quad (13)$$

Only in the extreme low-temperature limit may $W_{12}{}^{34}$ be taken to be independent of variables other than θ and ϕ; at finite temperatures the dependence of $W_{12}{}^{34}$ on the quasiparticle energies and momenta must be taken into account.[3,6,7,11] In liquid He^3 and almost ferromagnetic Fermi liquids, the strongly frequency-dependent screening of the quasiparticle scattering amplitude for small momentum transfer processes gives the major finite-temperature contributions to the transport coefficients. Physically, the strong dynamical screening of the quasiparticle interaction can be understood as follows: Because of the large static magnetic susceptibility of liquid He^3, a localized quasiparticle of a given spin is surrounded by a large dressing cloud consisting predominantly of quasiparticles of the same spin. This dressing cloud greatly enhances the static interaction between quasiparticles. However, the long-wavelength response of the dressing cloud falls off rapidly as the frequency increases, and as a result of this, the enhancement of the interaction also decreases.[17] The full expression for the long-wavelength effective interaction may be determined using Landau's form of the Bethe-Salpeter equation and is given by[18]

$$A^S(\mu_1,\mu_2,\chi;s)\delta_{13}\delta_{24}+A^a(\mu_1,\mu_2,\chi;s)\vec{\sigma}_{13}\cdot\vec{\sigma}_{24}, \quad (14)$$

where for definiteness we take $\vec{p}_1-\vec{p}_3=\vec{q}$ to be the small momentum transfer. Here $\mu_i=\hat{p}_i\cdot\hat{q}$ $(i=1,2)$, χ is the angle between the $\vec{p}_1\vec{q}$ and $\vec{p}_2\vec{q}$ planes, and $s=\omega/v_Fq$, where ω is the energy transfer corresponding to the momentum transfer $\vec{q}$, and v_F is the Fermi velocity. δ is the unit matrix in spin space and σ's are the Pauli matrices. For convenience, we make the scattering amplitude dimensionless by multiplying it by the density of quasiparticle states at the Fermi surface, $\nu(0)=m^*p_F/\pi^2\hbar^3$. ($p_F$ is the Fermi momentum.)

In liquid He^3 and almost ferromagnetic liquids, finite-temperature contributions to the transport coefficients coming from small momentum transfer processes are greatly enhanced by the dynamical screening of the effective interaction. Finite-temperature contributions arising from other sources are not expected to be enhanced,[8] and should be small compared with the contribution from small momentum transfer processes. Thus, here we shall consider only the contributions to $I|\Phi\rangle$ [Eq. (4)] coming from processes in which the momentum transfer q is less than some cutoff value q_c ($\ll p_F$). Since the Fermi functions in (4) restrict all quasiparticle momenta to lie close to the Fermi surface, the summations over momenta in (4) may be replaced by integrals as follows:

$$\sum_{\substack{\vec{p}_2\vec{p}_4\\ \vec{q}(q<q_c)}} \delta_{\vec{p}_1+\vec{p}_2,\,\vec{p}_3+\vec{p}_4}\,\delta(\mathcal{E}_1+\mathcal{E}_2-\mathcal{E}_3-\mathcal{E}_4)$$

$$\to \frac{m^*p_F}{4\pi^4\hbar^6}\int\frac{d\Omega_2}{4\pi}\int\frac{d\Omega_q}{4\pi}\int_{-\infty}^{\infty}d\omega\int_0^{q_c}q^2dq$$

$$\times\int_{-\infty}^{\infty}d\mathcal{E}_2\,\delta(\mathcal{E}_1-\mathcal{E}_3-\omega)\,\delta(\mathcal{E}_2-\mathcal{E}_4+\omega) . \quad (15)$$

The δ functions restrict the range of integration over q to values greater than $|\omega|/v_F$, since the energy transfer in a collision in which the momentum transfer is q cannot exceed v_Fq, $(q\ll p_F)$. The integrations over the angular variables may be simplified in the way described in Ref. (8) and the integration over q may be converted into an integral over $s=(\omega/v_Fq)$. The contribution to $I|\Phi\rangle$ from small momentum transfer processes contains a part which contributes to $I^{(0)}|\Phi\rangle$: This part depends on the cutoff q_c. There is also a term linear in T which we write as $(T/T_F)\times I^{(1)}|\Phi\rangle$, where $T_F=p_F{}^2/2m^*k_B$. The integrations over the variables s and χ which occur can be separated from the integrations over $\mathcal{E}_2$ and ω, just as they were in the variational calculation.[8] $I^{(1)}$ may then be written in the form

$$I^{(1)}=\Xi_{\rm I}{}^K I_{\rm I}{}^{(1)}+\Xi_{\rm II}{}^K I_{\rm II}{}^{(1)} , \quad (16)$$

where the Ξ's contain only the s and χ integrations. The Ξ's are essentially averages over s and χ of the frequency-dependent part of the quasiparticle transition rate and are defined in Appendix A. In ΞI, the appropriate average contains the familar factor $(1-\cos\theta)$, and gives by far the more important contribution to the transport coefficient. The components $I_{\rm I}{}^{(1)}|\Phi\rangle$ and $I_{\rm II}{}^{(1)}|\Phi\rangle$ are given by

$$(I_j{}^{(1)}|\Phi\rangle)_1=-\frac{\pi}{\hbar\nu^2(0)\langle W\rangle}\vec{v}_i\cdot\hat{u}\int_{-\infty}^{\infty}dx_2$$

$$\times\int_{-\infty}^{\infty}d\bar{\omega}\,|\bar{\omega}|n_1n_2(1-n_3)(1-n_4)Q_j{}^K . \quad (17)$$

Here, $\bar{\omega}=\omega/k_BT, x_3=x_1+\bar{\omega}, x_4=x_2-\bar{\omega}$,

$$Q_{\mathrm{I}}^{K}=c(x_4)-c(x_2)\ ,$$

and $$Q_{\mathrm{II}}^{K}=c(x_1)+c(x_2)-c(x_3)-c(x_4), \tag{18}$$

where $c(x)$ is defined in Eq. (8).

Making appropriate changes in the spin and angular dependence of Φ, one may decompose the vector $I^{(1)}|\Phi\rangle$ in the case of spin diffusion and viscosity in a similar fashion. For viscosity, one needs only the term involving

$$Q_{\mathrm{II}}^{\eta}=c(x_1)-c(x_3)\ , \tag{19}$$

and in the case of spin diffusion, one needs

$$Q_{\mathrm{I}}^{D}=c(x_2)+c(x_4),$$

$$Q_{\mathrm{II}}^{D}=c(x_1)-c(x_2)+c(x_3)-c(x_4), \tag{20}$$

and $$Q_{\mathrm{III}}^{D}=c(x_1)-c(x_3)\ .$$

The corresponding Ξ's for viscosity and spin diffusion are defined in Appendix A. The physical significance of the various terms will be discussed in Sec. III.

III. SOLUTION OF TRANSPORT EQUATION

In Sec. II, it was shown that the transport equation (1) can be written in the form

$$|X\rangle=I^{(0)}|\Phi\rangle+(T/T_F)I^{(1)}|\Phi\rangle\ , \tag{21}$$

if higher-order terms are neglected. Equation (21) may be solved by perturbation theory; assuming a solution of the form $|\Phi\rangle=|\Phi^{(0)}\rangle+(T/T_F)|\Phi^{(1)}\rangle+\cdots$, one finds

$$|X\rangle=I^{(0)}|\Phi^{(0)}\rangle, \tag{22}$$

$$0=I^{(0)}|\Phi^{(1)}\rangle+I^{(1)}|\Phi^{(0)}\rangle\ . \tag{23}$$

KT is given by Eq. (5), and has the form $\langle X|\Phi^{(0)}\rangle+(T/T_F)\langle X|\Phi^{(1)}\rangle$; the second term may be rewritten by using the following relations which follow from Eqs. (22) and (23)

$$\langle X|\Phi^{(1)}\rangle=\langle\Phi^{(0)}|I^{(0)}|\Phi^{(1)}\rangle$$

$$=-\langle\Phi^{(0)}|I^{(1)}|\Phi^{(0)}\rangle\ . \tag{24}$$

Thus, the leading finite-temperature contributions to the thermal conductivity are given by

$$\frac{1}{KT}-\left(\frac{1}{KT}\right)_{T=0}=\frac{T}{T_F}\frac{\langle\Phi^{(0)}|I^{(1)}|\Phi^{(0)}\rangle}{\langle X|\Phi^{(0)}\rangle^2}\ . \tag{25}$$

Similar expressions may be obtained for the finite-temperature contributions to $(DT)^2$ and $(\eta T)^{-2}$. To the order to which we are working, the result (25) may be written

$$1/KT=\langle\Phi^{(0)}|I|\Phi^{(0)}\rangle/\langle X|\Phi^{(0)}\rangle^2\ , \tag{26}$$

which is the same as the variational expression for $(KT)^{-1}$ evaluated using as a trial function the exact solution to the transport equation in the extreme low-temperature limit. Equation (25) shows that the first-order changes in the transport coefficients may be calculated using the zeroth-order solution; this result is analogous to the well-known result in quantum mechanics that first-order shifts in energy levels may be evaluated using the wave functions for the unperturbed system.

The equation $|X\rangle=I^{(0)}|\Phi^{(0)}\rangle$ was first discussed by Abrikosov and Khalatnikov[1] and it has been solved analytically by Brooker and Sykes,[9] and by Jensen, Smith, and Wilkins[10]; these solutions are discussed in Appendix B. By making use of the fact that Φ has the form given in Eq. (8), the finite-temperature contributions to the transport coefficients, given in Eq. (25), may be rewritten in the following way:

$$\frac{1}{KT}-\left(\frac{1}{KT}\right)_{T=0}$$

$$=-810\frac{\zeta(5)}{\pi}\frac{m^{*3}}{p_F^{\,7}}\hbar^2k_B\left(\sum_{i=I}^{\mathrm{II}}\gamma_i^{K}\Xi_i^{K}\right)T \tag{27}$$

for the thermal conductivity,

$$\frac{1}{DT^2}-\left(\frac{1}{DT^2}\right)_{T=0}$$

$$=-18\pi\zeta(3)\frac{m^{*4}}{p_F^{\,6}}\frac{k_B^{\,3}}{1+F_0^a}\left(\sum_{i=I}^{\mathrm{III}}\gamma_i^{D}\Xi_i^{D}\right)T \tag{28}$$

for the spin-diffusion coefficient and

$$\frac{1}{\eta T^2}-\left(\frac{1}{\eta T^2}\right)_{T=0}$$

$$=-90\pi^3\zeta(3)\frac{m^{*2}}{p_F^{\,7}}\hbar^2k_B^{\,2}\gamma_{\mathrm{II}}^{\eta}\Xi_{\mathrm{II}}^{\eta}T \tag{29}$$

for the coefficient of viscosity. Here, $\zeta(n)$ is the

Riemann ζ function of order n. The coefficients γ_i are the values of $\langle\Phi^{(0)}|I_i^{(1)}|\Phi^{(0)}\rangle/\langle X|\Phi^{(0)}\rangle^2$ normalized to give the results of the variational calculation[7,8] when the γ_I's are put equal to unity and the remaining γ's are put equal to zero. Explicit expressions for the γ's are given in Appendix C, where we also give the results of calculations of γ using analytical approximations and also numerical expressions for $c^{(0)}$.

The most important term in the finite-temperature contribution to the thermal conductivity is the term involving $\gamma_I{}^K\Xi_I{}^K$. As mentioned previously, $\Xi_I{}^K$ is an average of the frequency-dependent part of the quasiparticle transition rate weighted by a factor $1-\cos\theta$. This weight factor is easily understood physically since head-on collisions ($\theta=\pi$) are the most effective at reducing the energy current, whereas, collinear collisions ($\theta=0$) leave the energy current unchanged. In the case of spin diffusion, the dominant finite-temperature contribution to $(DT)^{-2}$ comes from small momentum transfer collisions accompained by a spin flip. This is given by the $\gamma_I{}^D\Xi_I{}^D$ term which is also weighted by a factor $1-\cos\theta$ that measures how effectively a collision reduces the spin current. The term involving $\gamma_{III}{}^D\Xi_{III}{}^D$ comes[19] from scattering processes involving quasiparticles of the same spin; such processes leave the spin current unchanged, but they do alter the form of the quasiparticle distribution function and thereby affect the value of the spin-diffusion coefficient. In the case of viscosity, the finite-temperature contributions to $(\eta T)^{-2}$ are rather small since small momentum transfer processes contribute little to the change in momentum flux; this accounts for the absence of a term of the form $\gamma_I{}^\eta\Xi_I{}^\eta$.

IV. QUASIPARTICLE SCATTERING AMPLITUDE

To determine the solution $c^{(0)}$ for calculating the γ's and the transport coefficients in the extreme low-temperature limit, one needs an expression for the transition probabilities $W_{\uparrow\uparrow}(\theta,\phi)$; these may be expressed in terms of the corresponding scattering amplitudes, which we denote by $A_{\uparrow\uparrow}(\theta,\phi)/\nu(0)$ and $A_{\uparrow\downarrow}(\theta,\phi)/\nu(0)$:

$$W_{\uparrow\uparrow}(\theta,\phi)=\frac{2\pi}{\hbar}\left|\frac{A_{\uparrow\uparrow}(\theta,\phi)}{\nu(0)}\right|^2, \tag{30}$$

$$W_{\uparrow\downarrow}(\theta,\phi)=\frac{2\pi}{\hbar}\left|\frac{A_{\uparrow\downarrow}(\theta,\phi)}{\nu(0)}\right|^2. \tag{31}$$

The amplitudes $A_{\uparrow\uparrow}$ and $A_{\uparrow\downarrow}$ may be decomposed in terms of the singlet and triplet amplitudes, A_s and A_t:

$$A_{\uparrow\uparrow}(\theta,\phi)=A_t(\theta,\phi), \tag{32}$$

$$A_{\uparrow\downarrow}(\theta,\phi)=\tfrac{1}{2}[A_s(\theta,\phi)+A_t(\theta,\phi)]. \tag{33}$$

The angle through which the relative momentum of the two quasiparticles is scattered is given by ϕ. Therefore, it follows from the Pauli principle and invariance under the parity transformation that A_S can contain only even partial waves and A_t can contain only odd partial waves; that is,

$$A_s(\theta,\phi)=\sum_{l\ \text{even}} C_l(\theta)P_l(\cos\phi), \tag{34}$$

$$A_t(\theta,\phi)=\sum_{l\ \text{odd}} C_l(\theta)P_l(\cos\phi). \tag{35}$$

The coefficients $C_l(\theta)$ are defined by these equations.

The scattering amplitudes for $\phi=0$ are given in terms of the Landau parameters by means of the equations[15]

$$A_s(\theta,0)=\textstyle\sum_l (A_l{}^s-3A_l{}^a)P_l(\cos\theta), \tag{36}$$

$$A_t(\theta,0)=\textstyle\sum_l (A_l{}^s+A_l{}^a)P_l(\cos\theta), \tag{37}$$

$$\text{where}\quad A_l{}^i=\frac{F_l{}^i}{1+F_l{}^i/(2l+1)}, \quad (i=a \text{ or } s). \tag{38}$$

Information about the Landau parameters $F_0{}^s$, $F_1{}^s$, and $F_0{}^a$ may be obtained from measurements of thermodynamic properties of the liquid, and these can in turn provide one with information about the scattering amplitude, but only for $\phi=0$. However, the ϕ dependence of the scattering amplitude may be determined completely if one assumes that only s and p waves contribute to the scattering. The scattering amplitudes are then given by

$$A_s(\theta,\phi)\approx A_s(\theta,0), \tag{39}$$

$$A_t(\theta,\phi)\approx A_t(\theta,0)\cos\phi, \tag{40}$$

$$\text{or}\quad A_{\uparrow\uparrow}(\theta,\phi)\approx\textstyle\sum_l (A_l{}^s+A_l{}^a)\cos\phi P_l(\cos\theta), \tag{41}$$

$$A_{\uparrow\downarrow}(\theta,\phi)\approx\textstyle\sum_l \tfrac{1}{2}[(A_l{}^s-3A_l{}^a) + (A_l{}^s+A_l{}^a)\cos\phi]P_l(\cos\theta). \tag{42}$$

Equations (41) and (42) are exact for $\phi = 0$ and π, and Eq. (41) is also exact for $\phi = \frac{1}{2}\pi$. For liquid He^3, there is experimental information only about Landau parameters with $l < 2$, and in applying formulas (41) and (42) for the scattering amplitude, the series will be truncated after the second term. In previous calculations for liquid He^3 based on semiphenomenological considerations, it has been customary to replace the scattering amplitudes by their values for $\phi = 0$[9,13,14]; such an approximation clearly violates the requirement that the triplet amplitude contain only odd partial waves, and, furthermore, leads to results for the transport coefficients which are consistently higher than the experimental values.[9] In Sec. V we shall use the s- and p-wave approximation in calculations for liquid He^3.

V. APPLICATION TO LIQUID He^3

A. Extreme Low-Temperature Limit

The transport coefficients of liquid He^3 in the extreme low-temperature limit, calculated using the s- and p-wave approximation [Eq. (41) and (42)] for the scattering amplitude and exact solutions of the transport equation, are shown in Table I. For comparison, we also give results obtained by replacing the scattering amplitudes by their values for $\phi = 0$ (forward-scattering approximation). In all calculations, only Landau parameters with $l < 2$ were taken into account. We also ignored the fact that the measurements of the transport coefficients at low pressure were made at slightly different pressures. For the Landau parameters, we used the values $A_0^S = 0.91$, $A_1^S = 2.0$, $A_0^a = -2.0$ at low pressure and the values $A_0^S = 0.99$, $A_1^S = 2.5$, $A_0^a = -2.7$ at high pressures (27 atm).[20] For A_1^a, we used values obtained from the forward-scattering sum rule,[13,21] assuming Landau parameters with $l \geq 2$ may be neglected ($A_1^a = -0.91$ for low pressures and $A_1^a = -0.73$ for high pressures), and at low pressure we also used the value of A_1^a ($= -0.55$) obtained from finite-temperature contributions to the thermal conductivity, as described in Sec. V B.

The results using the s- and p-wave approximation for the scattering amplitude are in much better agreement with experiment than are results obtained using the forward-scattering approximation; the spin-diffusion coefficient is particularly sensitive to the approximation one uses for the scattering amplitude, since for spin diffusion, the important processes are ones for which $\phi \approx \pi$, and the amplitude for these processes is poorly approximated by its value for $\phi = 0$. The over-all effect of using the s- and p-wave approximation is to increase the amount of scattering. The scattering of quasiparticles of like spin is actually decreased, but this is more than compensated by the increase in scattering of quasiparticles of opposite spin when $\phi \approx \pi$.

The two values of A_1^a for liquid He^3 at low pressure were both estimated assuming $F_l^{S,a} = 0$ ($l \geq 2$); the difference between these two values gives some indication of how good an approximation it is to neglect higher Landau parameters. One piece of information which bears directly on this assumption is that the observed velocity and attenuation of zero sound in the liquid are not inconsistent with $F_2^S = 0$.[22,23]

To give some idea of the sensitivity of the results to changes in the value of A_1^a, we give in Figs. 1–3 a plot of the transport coefficients at

TABLE I. Comparison of theoretical and experimental values of the transport coefficients of liquid He^3 in the extreme low-temperature limit.

Pressure			KT erg/cm sec	DT^2 10^{-6} cm^2 °K^2/sec	ηT^2 10^{-6}poise °K^2
0.28 atm	Experiment[a]		35	1.4	2
	Theory	s- and p-wave approximation[b]	33	1.6	1.6
		s- and p-wave approximation[c]	39	1.9	2.2
		forward-scattering approximation[b]	47	4.3	2.3
		forward-scattering approximation[c]	54	4.6	2.8
27 atm	Experiment[a]		<12	0.17	...
	Theory	s- and p-wave approximation[c]	8.6	0.16	0.54
		forward-scattering approximation[c]	13	0.49	0.76

[a]Experimental data are taken from Refs. 20 and 26.

[b]Values in this row are calculated using the value of A_1^a obtained from finite-temperature contributions to thermal conductivity.

[c]Values in this row are calculated using the value of A_1^a obtained from the forward-scattering sum rule.

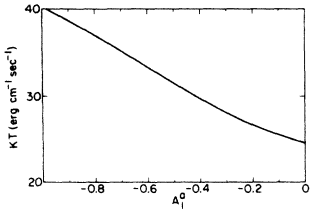

FIG. 1. Plot of the calculated value of KT in the extreme low-temperature limit as a function of A_1^a. The calculations were performed using the s- and p-wave approximation for the scattering amplitude, and the values of A_0^s, A_0^a, and A_1^s for liquid He^3 at low pressure.

low pressure for values of A_1^a between 0 and -1. The results are not overly sensitive to the value of A_1^a, and one can see from the results given in Table I that neither of the two values of A_1^a obtained from experiments on liquid He^3 leads to results for the transport coefficients which differ appreciably from the experimental values.

B. Finite-Temperature Contributions

The most reliable data of finite-temperature contributions to the transport coefficients are those of Abel, Johnson, Wheatley, and Zimmermann[24] on the thermal conductivity of liquid He^3 at low pressure and those of Anderson, Reese, Sarwinski, and Wheatley[25] on the spin-diffusion coefficient of liquid He^3 at low pressure. Plots of the data, as recently reanalyzed by Abel and Wheatley,[26] are given in Ref. 8.

For the case of liquid He^3, the theoretical results depend sensitively on the value of the Landau parameter F_1^a, about which there is no direct experimental information. In comparing theory and experiment, we assumed that all Landau parameters with $l \geq 2$ could be neglected, and then determined the value of F_1^a which led to agreement between theory and experiment. From the thermal conductivity data, we found $F_1^a = -0.46$

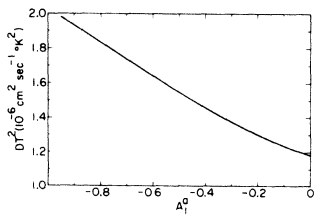

FIG. 2. Plot of the calculated value of DT^2 in the extreme low-temperature limit as a function of A_1^a. The calculations were performed using the s- and p-wave approximation for the scattering amplitude, and the values of A_0^s, A_0^a, and A_1^s for liquid He^3 at low pressure.

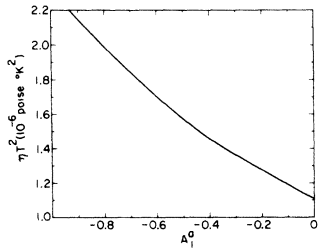

FIG. 3. Plot of the calculated value of ηT^2 in the extreme low-temperature limit as a function of A_1^a. The calculations were performed using the s- and p-wave approximation for the scattering amplitude, and the values of A_0^s, A_0^a, and A_1^s for liquid He^3 at low pressure.

± 0.14, and from the spin-diffusion data, we found $F_1^a = -0.39 \pm 0.14$; the quoted errors include only the contribution from statistical uncertainties in fitting a straight line to the experimental data. These calculations were performed using values of γ obtained from numerical solutions of the transport equation. The values of λ used in calculating γ were obtained using the s- and p-wave approximation for the scattering amplitude discussed in Sec. IV. We consider the value of F_1^a obtained from the thermal conductivity data to be the more reliable of the two estimates since experimentally the linear temperature dependence of $(DT)^{-2}$ is not so well defined as that of $(KT)^{-1}$.

VI. DISCUSSION

One conclusion of the work reported here is that Landau theory is able to give a consistent account of the observed magnitudes of the equilibrium properties and the transport coefficients of liquid He^3 if one takes into account only Landau parameters with $l < 2$, provided one uses an approximation for the quasiparticle scattering amplitude which includes both s- and p-wave contributions. No doubt improved measurements of the properties of liquid He^3 will show that the simple approximation for the scattering amplitude is inadequate; in particular, we note that the viscosity in the extreme low-temperature limit will be rather sensitive to d-wave components in the scattering amplitude.

In this paper, we have not discussed the validity of using the quasiparticle transport equation to calculate finite-temperature contributions to the transport coefficients. It is known that expressions for finite-temperature contributions to the specific heat based on a quasiparticle model can give incorrect results,[27] but one can show that similar difficulties do not occur in the quasiparticle transport equation, at least within the shielded potential approximation.[28]

If one accepts the validity of the quasiparticle transport equation, one may calculate higher-order contributions to the transport coefficients. There are $T^2 \ln T$ contributions to $(KT)^{-1}$, $(DT)^{-2}$, and $(\eta T)^{-2}$ coming from corrections to the quasiparticle density of states, but these are largely canceled by contributions of the same order coming from changes in the renormalization constants, which enter the expression for the effective interaction. The reason for this is that the largest of the logarithmic terms in the quasiparticle self-energy vary as $\omega^3 \ln|\omega|$.[29] This result is very similar to the well-known cancellation of density-of-states corrections and renormalization constants in the electron-phonon problem.[30] There may well be other $T^2 \ln T$ contributions associated with Friedel oscillations in the dressing cloud of a quasiparticle.[31]

The perturbation-theory approach we have used to calculate finite-temperature contributions to the transport coefficients is applicable to a number of other problems; for example, it may be used to calculate the electron-electron scattering contribution to the electrical resistivity of a metal at low temperatures in the presence of electron-impurity scattering. One is then led to the amusing conclusion that the standard "approximate" variational calculation of the T^2 term[4] leads to the correct result, but only when impurity scattering dominates.

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We are indebted to Professor David Pines and Professor Gordon Baym for many helpful converstations during the course of this work. One of the authors (CJP) wishes to acknowledge the hospitality of the Aspen Center for Physics during the summer of 1968; at Aspen, he also enjoyed a number of profitable discussions with Professor E. Feenberg.

APPENDIX A: DEFINITION OF Ξ

Here we give the definitions of the quantities Ξ which appear in Eqs. (27)–(29) for the finite-temperature contributions to the transport coefficients. These are most easily defined in terms of the quantities Ξ^i and $\tilde{\Xi}^i$ ($i = a$ or s), which are given by the equations

$$\Xi^i = \int_0^{2\pi} \frac{d\chi}{2\pi} (1-\cos\chi) \left\{ |\mathfrak{a}^i(0,\chi)|^2 + \int_0^1 ds \left[\frac{|\mathfrak{a}^i(0,\chi)|^2 - |\mathfrak{a}^i(s,\chi)|^2}{s^2} + |\mathfrak{a}^i(s,\chi)|^2 \right] \right\} \tag{A1}$$

and

$$\tilde{\Xi}^i = \int_0^{2\pi} \frac{d\chi}{2\pi} \left\{ |\mathfrak{a}^i(0,\chi)|^2 + \int_0^1 ds \left[\frac{|\mathfrak{a}^i(0,\chi)|^2 - |\mathfrak{a}^i(s,\chi)|^2}{s^2} \right] \right\}, \tag{A2}$$

where $\mathfrak{a}^i(s,\chi) \equiv A^i(s,s,\chi;s)$. The evaluation of Ξ^i is described in Ref. 8. $\tilde{\Xi}$ may be evaluated analytically using the techniques described in Ref. 8 and if only Landau parameters with $l < 2$ are taken into account one finds

$$\begin{aligned} \tilde{\Xi}^i = {} & (A_0^i)^2 [\tfrac{1}{4}\pi^2 A_0^i + 1 + A_1^i] - 2A_0^i A_1^i + (A_1^i)[-\tfrac{1}{16}\pi^2 A_1^i + 1] + (2/s_0^i) \\ & \times [F_0^i + A_1^i s_0^{i2}] \left[\frac{d}{ds} (F_0^i + A_1^i s_1^{i2}) \chi(s_0^i) \right]^{-1} + 2(s_1^{i2} - 1) / \frac{d}{ds} [(s_1^{i2} - 1)\chi(s_1^i)] . \end{aligned} \tag{A3}$$

The last two terms comes from the collective modes and must, of course, be omitted if these modes are absent. s_m^i is the velocity, in units of the Fermi velocity, of the spin-symmetric ($i = s$) or spin-antisymmetric ($i = a$) collective mode whose azimuthal dependence is given by the index m. The Ξ are defined by the relations

$$\Xi_1^K = \Xi^s + 3\Xi^a, \quad \Xi_{II}^K = \tilde{\Xi}^s + 3\tilde{\Xi}^a, \quad \Xi_I^D = 2\Xi^a,$$

$$\Xi_{II}^D = 2\tilde{\Xi}^a, \quad \Xi_{III}^D = \tilde{\Xi}^s + \tilde{\Xi}^a, \quad \Xi_{II}^\eta = \Xi_{II}^K = \tilde{\Xi}^s + 3\tilde{\Xi}^a. \tag{A4}$$

Ξ_1^K and Ξ_{II}^K are closely related to the quantities $[w(1-\cos\theta)]_{av}$ and $[w]_{av}$, respectively, introduced by Emery and Cheng.[11] Similarly, Ξ^a and $\tilde{\Xi}^a$ are closely related to their $[\bar{\omega}_D(1-\cos\theta)]_{av}$ and $[\bar{w}_D]_{av}$. The difference in form between our results and theirs is due to the fact that they work in terms of the variables θ and ϕ whereas we work in terms of s and χ.

APPENDIX B: SOLUTION OF TRANSPORT EQUATION IN EXTREME LOW-TEMPERATURE LIMIT

The solution of the equation $|X\rangle = I^{(0)}|\Phi^{(0)}\rangle$ may be written[9,10]

$$c^{(0)}(x) = 2x/(\pi^2+x^2) + \lambda_K \sum_{n=0}^{\infty} \{(x, f_{2n+1})/[(n+1)(2n+3) - \lambda_K]\} f_{2n+1}(x) \tag{B1}$$

for thermal conductivity and

$$c^{(0)}(x) = 2/(\pi^2+x^2) + \lambda \sum_{n=0}^{\infty} \{(1, f_{2n})/[(n+1)(2n+1) - \lambda]\} f_{2n}(x) \tag{B2}$$

for spin diffusion ($\lambda = \lambda_D$) and viscosity ($\lambda = \lambda_\eta$). Here, $f_n(x)$ is an eigenfunction of the equation

$$\tfrac{1}{2}(\pi^2+x^2) f_n(x) - \lambda_n \int dt \, \frac{t}{e^t - 1} \, \frac{(1+e^{-x})}{(1+e^{-(t+x)})} f_n(t+x) = 0 \tag{B3}$$

and $$(A, B) = \int_{-\infty}^{\infty} dx \, (-dn/dx) A(x) B(x) \, , \tag{B4}$$

where $n(x) = (e^x + 1)^{-1}$ is the Fermi function. The eigenvalues are $\lambda_n = \frac{1}{2}(n+1)(n+2)$ and the eigenfunctions are proportional to the residues of the functions $P_n^1(\coth q)\, e^{iq(x/\pi)}$ at $q = 0$. The first four eigenfunctions are

$$f_0(x) = \left(\frac{3}{2}\right)^{1/2} \frac{1}{\pi}, \quad f_1(x) = \left(\frac{5}{2}\right)^{1/2} \frac{x}{\pi^2}, \quad f_2(x) = \left(\frac{7}{170}\right)^{1/2} [3 - 5(x/\pi)^2], \quad f_3(x) = \frac{x}{8\sqrt{2}\pi^2}\left[17 - 7\left(\frac{x}{\pi}\right)^2\right]. \tag{B5}$$

If only the first terms in the sums in (B1) and (B2) are retained, one obtains an approximate solution of the transport equation first derived by Emery and Cheng[11] by a different method.

APPENDIX C: EVALUATION OF γ

The γ are defined by the relations

$$\gamma_j^K = \frac{\pi^4}{9} \frac{1}{5!\zeta(5)} \frac{1}{(x, c^{(0)})^2} \int_{-\infty}^{\infty} dx \, c^{(0)}(x) \times \int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{\infty} d\bar{\omega} \, |\bar{\omega}| n(x) n(x_2) \times [1 - n(x+\bar{\omega})][1 - n(x_2 - \bar{\omega})] Q_j^K, \tag{C1}$$

and

$$\gamma_j^{D,\eta} = \frac{1}{24\zeta(3)} \frac{1}{(1, c^{(0)})^2} \int_{-\infty}^{\infty} dx \, c^{(0)}(x) \times \int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{\infty} d\bar{\omega} \, |\bar{\omega}| n(x) n(x_2) \times [1 - n(x+\bar{\omega})][1 - n(x_2 - \bar{\omega})] Q_j^{D,\eta}. \tag{C2}$$

Here the Q's must be evaluated with c put equal to $c^{(0)}$. If the γ's are evaluated using Emery and Cheng's[11] approximate expressions for $c^{(0)}$,

$$c^{(0)} = \frac{2x}{\pi^2+x^2} + \frac{\lambda_K}{1 - \frac{1}{3}\lambda_K} \left(\frac{5}{2\pi^2}\right) x \tag{C3}$$

for thermal conductivity, and

$$c^{(0)} = 2/(\pi^2+x^2) + (3/2\pi^2)[\lambda/(1-\lambda)] \tag{C4}$$

for spin diffusion and viscosity, one finds the following results:

$$\gamma_{\mathrm{I}}^K = 1 - 0.328\left(\frac{\tau_{QP}}{\tau_0}\right) + 0.037\left(\frac{\tau_{QP}}{\tau_0}\right)^2, \qquad \gamma_{\mathrm{II}}^K = 0.043\left(\frac{\tau_{QP}}{\tau_0}\right)^2, \tag{C5}$$

where, using Emery and Cheng's notation, $\tau_0/\tau_{QP} = \frac{1}{4}(12 - \pi^2) + 5\lambda_K/12(3 - \lambda_K)$ for thermal conductivity.

$$\gamma_{\mathrm{I}}^D = 1 - \frac{\pi^2}{6} \frac{\zeta(3) - 1}{\zeta(3)} \left(\frac{\tau_{QP}}{\tau_0}\right) + 0.0297\left(\frac{\tau_{QP}}{\tau_0}\right)^2, \qquad \gamma_{\mathrm{II}}^D = 0.0161\left(\frac{\tau_{QP}}{\tau_0}\right)^2, \tag{C6}$$

$$\gamma_{III}^{\ D} = 0.0416\left(\frac{\tau_{QP}}{\tau_0}\right)^2,$$

where $\tau_0/\tau_{QP} = \frac{1}{12}\pi^2 + 3\lambda_D/4(1-\lambda_D)$ for spin diffusion, and

$$\gamma_{II}^{\ \eta} = 0.042\left(\frac{\tau_{QP}}{\tau_0}\right)^2, \qquad \text{(C7)}$$

where $\tau_0/\tau_{QP} = \pi^2/12 + 3\lambda_\eta/4(1-\lambda_\eta)$ for viscosity. These results agree with those of Emery and Cheng.[11,32]

The values of finite-temperature contributions calculated using the above expressions for γ differ from results obtained using more precise numerical values of $|\Phi^{(0)}\rangle$ by amounts of the order of a percent or so for values of λ appropriate for liquid He^3. These numerical solutions were obtained by iterating the integral equation for $c^{(0)}$ starting[33] from the solution for $\lambda = 0$; for the values of λ of interest this procedure converges rapidly. To compare the results of the two calculations, we give the values of γ for the case of liquid He^3 at low pressure, using the approximation for the scattering amplitude described in Sec. IV and with $A_1^a = -0.55$. The values of λ are $\lambda_K = 1.02$ and $\lambda_D = -0.125$ and the corresponding values of γ obtained using the numerical solutions of the transport equation are $\gamma_I^{\ K} = 0.606\,(0.628)$, $\gamma_{II}^{\ K} = 0.091\,(0.078)$, $\gamma_I^{\ D} = 0.684\,(0.672)$, $\gamma_{II}^{\ D} = 0.028\,(0.031)$, and $\gamma_{III}^{\ D} = 0.075\,(0.081)$; the values given in parentheses are those calculated using the approximate expressions given above.

We also note that for the particular cases $\lambda_K \to 3$, $\lambda_D \to 1$, and $\lambda_\eta \to 1$ the variational results given in Ref. 8 are exact.

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Attenuation of Zero Sound in a Normal Fermi Liquid*

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The attenuation of zero sound in the collisionless regime ($\omega\tau \gg 1$) is calculated using perturbation theory to treat the effects of collisions. Detailed calculations are performed taking into account Landau parameters with $l \leq 2$; the results are applied to the experimental data for liquid He^3.

1. INTRODUCTION

In the second of his classic papers on normal Fermi liquids, Landau predicted that a new type of sound, which he called zero sound, could propagate in such systems.[1] One of the necessary conditions for the existence of zero sound is that a typical quasiparticle collision frequency be small compared with the frequency of the wave; in other words, zero sound exists in the collisionless region ($\omega\tau \gg 1$). Ordinary sound (first sound), on the other hand, exists in the hydrodynamic regime ($\omega\tau \ll 1$), where the liquid is in local thermodynamic equilibrium.

The velocity and attenuation coefficient of zero sound in a normal Fermi liquid may be determined from the eigenvalues of the quasiparticle transport equation.[1,2] The collision integral in the transport equation is rather difficult to handle, and in previous calculations[3,4] it has been common to replace it by an approximate expression which involves a single relaxation time and which conserves quasiparticle number and total quasiparticle momentum. The first such calculation was that of Khalatnikov and Abrikosov,[3] who took into account only the Landau parameters[5] F_0^S and F_1^S. The calculations were extended by Brooker,[4] who included the Landau parameter F_2^S as well. A microscopic calculation of the attenuation of zero sound has been performed by Eliashberg[6]; in this calculation a number of approximations were made and the results were very similar to those obtained using the quasiparticle transport equation and the relaxation-time approximation.

The work described here was stimulated by the fact that the calculations of Khalatnikov and Abrikosov[3] apparently do not give a consistent account of the observed attenuation of zero sound and first sound in liquid He^3 – the relaxation time required to account for the zero-sound data is somewhat shorter than that required to account for the first-sound data.[7] Here, we derive expressions for the attenuation of zero sound without making the single relaxation-time approximation and show that one can give a consistent account of the data. In the collisionless regime, the properties of the sound wave are little affected by collisions; one may, therefore, use perturbation theory to calculate the effect of collisions on the zero-sound wave. This situation should be contrasted with that in the hydrodynamic regime where it is important to take into account multiple scattering effects. Detailed calculations are performed taking into account Landau parameters with $l \leq 2$ and the results are compared with the experimental data for liquid He^3. By comparing the observed attenuation of zero sound with that of first sound a rough estimate for the Landau parameter F_2^S is obtained, but the uncertainty in its value is rather large as a result of uncertainties in the experimental data. The value of F_2^S is consistent with, but somewhat more uncertain than, the value obtained from measurements of the velocity of zero sound.[4]

In Sec. 2, we describe the perturbation-theory calculation of the attenuation coefficient, and give limiting forms of the result when the zero-sound velocity is very much greater than the Fermi velocity. The method used is modeled closely on the standard calculation of the attenuation of sound in the hydrodynamic regime,[8] and is close in spirit to Gavoret's[9] calculation of the acoustic impedance of liquid He^3. The calculation is applied to liquid He^3 in Sec. 3. In Sec. 4, the in-