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Energy-Momentum Tensor of Electromagnetic Field in Moving Dispersive Media and Instability: Relativistic Formulation

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Based on the Lagrangian principle, the energy-momentum tensor of the electromagnetic field is obtained for general linear anisotropic dispersive media whose components are moving with different uniform velocities. The field is assumed to change in space and time with a *center* wave-number 4-vector ω_ν. The energy-momentum tensor obtained can be classified into three parts of the pure electromagnetic field, the medium and the interaction with the external system, and the explicit expression for each part is given in terms of the two 4-forces acting on unit electric and magnetic charges. When $\omega_\nu^2 > 0$, the energy density is not positive definite, depending upon the frame of reference, while the "rest" energy of the field is nonvanishing (in contrast with the case of nondispersive media) and has a positive sign at least in isotropic media with no external source; thus the energy density is positive definite in the rest frame of the wave itself. The direction of the power flow 4-vector agrees with that of the wave packet. The instability of the wave is discussed from a general point of view based on the energy-momentum tensor and the covariance of equations, and the wave is found to become unstable when the group velocity reaches the velocity of light in vacuum, over a finite range of the wave number; the two beam instability in a plasma belongs to this type of instability. The wave becomes unstable also when, in any frame of reference, the conductivities of one or more of the medium components become negative *and* the field energy is positive, so that the total energy increases in timelike direction.

1. INTRODUCTION

The electrodynamics in a moving medium has been fully treated at least in the case of a nondispersive isotropic medium of a single moving component,[1,2] and the energy-momentum tensor introduced by Minkowski has been shown to give consistent results in various respects, although the asymmetry of this tensor has given rise to many discussions.[3] The energy density in a dispersive medium has also been found in the rest frame of reference of the medium.[4] On the other hand, there has been great interest in the electrodynamics of a dispersive medium of multicomponents mainly in connection with plasma physics, and the problems of the instability also has come out. A detailed historical review was recently given by Penfield and Haus.[5]

In the case of plasma physics, the main interest has been in actual computations and numerical results, and not much attention seems to have been paid to the over-all dynamics of the system; provided that the force equations are *given* on the basis of suitable assumptions, the orthodox way is to find the Lagrangian density function giving the prescribed force equations and then to construct the energy-momentum tensor according to the usual method, taking into account relativity, gauge invariance and other invariances, symmetries, like in field theory. The latter restrictions usually remove the arbitrariness of the energy-momentum tensor, and it is also known that the arbitrariness of Lagrangian density function itself can be reduced to canonical transformation in dynamics.[6]

The energy-momentum tensor thus obtained is important not only because the force equations are reproduced through the Hamilton's canonical equations, but also because it has a close connection with the direction of wave propagation and the instability of the wave; within such a development, relativity plays an essential role. It is believed that it is always possible to construct the energy-momentum tensor by the Lagrangian principle when the medium is nondissipative, but otherwise this is not possible in the dissipative case; it is usually defined to be that obtained from the nondissipative energy-momentum tensor by an adiabatic change of the medium into a dissipative one.

In this paper, the Maxwell's equations are briefly examined in the case in which the medium of one single component is time dispersive, anisotropic, and inhomogeneous in space, but is not in motion (Sec. 2). Then the equations are expressed in covariant form, in the general case in which the media consist of several components moving with different uniform velocities and are dispersive and inhomogeneous in space and time. In particular, the case is treated in detail in which each component of the media is only time dispersive in its own rest frame of reference (as in a plasma). The external current is also introduced and is assumed to be caused by small displacements of continuous media. The "packet" field is introduced, whose envelope changes very slowly compared to the phase term, and its covariant equations are obtained (Sec. 3). The energy-momentum tensor of the packet field is obtained based on the Lagrangian variational principle in the case of the nondissipative media (Sec. 4), and then it is extended to the dissipative case (Sec. 6). The classification and the representation of the energy-momentum tensor are also treated together with the nonvanishing rest energy therefrom (Sec. 5). Finally, the instability of the field in the dispersive media is discussed based on the covariance of the equations (Sec. 7). The entire analysis is based on a linear (small amplitude) excitation of the media.

2. MAXWELL'S EQUATIONS IN DISPERSIVE MEDIUM

Using the usual notation, Maxwell's equations in heaviside units are given by[7]

$$\mathrm{rot}\tilde{\vec{B}} = c^{-1}\,\partial\tilde{\vec{E}}/\partial t + \tilde{\vec{j}} + \tilde{\vec{j}}^{e}, \quad \mathrm{div}\tilde{\vec{E}} = \tilde{\rho} + \tilde{\rho}^{e}, \tag{2.1}$$

$$\mathrm{rot}\tilde{\vec{E}} = -c^{-1}\,\partial\tilde{\vec{B}}/\partial t, \qquad \mathrm{div}\tilde{\vec{B}} = 0. \tag{2.2}$$

Here, $\tilde{\vec{j}}^{e}$ and $\tilde{\rho}^{e}$ are the externally applied current and charge densities, while $\tilde{\vec{j}}$ and $\tilde{\rho}$ are the same densities induced by the electromagnetic field. Hence, they independently satisfy the conservation law:

$$\mathrm{div}\tilde{\vec{j}}^{e} + c^{-1}\partial\tilde{\rho}^{e}/\partial t = 0,$$

$$\mathrm{div}\tilde{\vec{j}} + c^{-1}\partial\tilde{\rho}/\partial t = 0. \tag{2.3}$$

Generally, $\tilde{\vec{j}}$ can be expressed in terms of two vector fields $\tilde{\vec{P}}$ and $\tilde{\vec{M}}$:

$$\tilde{\vec{j}} = c^{-1}\partial\tilde{\vec{P}}/\partial t + \mathrm{rot}\tilde{\vec{M}}. \tag{2.4}$$

Here, in order to satisfy (2.3),

$$\tilde{\rho} = -\,\mathrm{div}\tilde{\vec{P}}. \tag{2.5}$$

Thus substituting (2.4) and (2.5) in (2.1) and putting

$$\tilde{\vec{H}} = \tilde{\vec{B}} - \tilde{\vec{M}}, \quad \tilde{\vec{D}} = \tilde{\vec{E}} + \tilde{\vec{P}}, \tag{2.6}$$

we have

$$\mathrm{rot}\tilde{\vec{H}} = c^{-1}\,\partial\tilde{\vec{D}}/\partial t + \tilde{\vec{j}}^{e}, \quad \mathrm{div}\tilde{\vec{D}} = \tilde{\rho}^{e}. \tag{2.7}$$

When the medium is linear and not dispersive and is not composed of several components of different velocities, we usually have, in the rest frame of the medium, the relations

$$\tilde{B}_i = \tilde{\mu}_{ij}\tilde{H}_j, \quad \tilde{D}_i = \tilde{\epsilon}_{ij}\tilde{E}_j, \tag{2.8}$$

where $\tilde{B}_i$ $(i = 1, 2, 3)$ is the component of the space vector, and a repeated index is to be summed. Thus it follows from (2.6) that

$$\tilde{M}_i = (1 - \tilde{\mu}^{-1})_{ij}\tilde{B}_j, \quad \tilde{P}_i = (\tilde{\epsilon} - 1)_{ij}\tilde{E}_j. \tag{2.9}$$

On the other hand, when the medium is linear but dispersive, the relations (2.9) are replaced by

$$\tilde{M}_i(\vec{r}, t) = \int_{-\infty}^{\infty} dt'\, \tilde{\nu}_{ij}(\vec{r}, t - t')\tilde{B}_j(\vec{r}, t'),$$

$$\tilde{P}_i(\vec{r}, t) = \int_{-\infty}^{\infty} dt'\, \tilde{\kappa}_{ij}(\vec{r}, t - t')\tilde{E}_j(\vec{r}, t'), \tag{2.10}$$

$$\text{where} \quad \tilde{\nu}_{ij}(\vec{r}, t) = \tilde{\kappa}_{ij}(\vec{r}, t) = 0, \quad \text{for } t < 0. \tag{2.11}$$

In view of the condition (2.11), the Fourier transforms $\nu_{ij}(\vec{r}, \omega)$ and $\kappa_{ij}(\vec{r}, \omega)$ defined by

$$\nu_{ij}(\vec{r}, \omega) = \int_0^{\infty} dt\, \tilde{\nu}_{ij}(\vec{r}, t)e^{-i\omega t},$$

$$\kappa_{ij}(\vec{r}, \omega) = \int_0^{\infty} dt\, \tilde{\kappa}_{ij}(\vec{r}, t)e^{-i\omega t}, \tag{2.12}$$

are analytic in the lower half of the ω complex plane, and also

$$\nu_{ij}(\vec{r}, \omega)^* = \nu_{ij}(\vec{r}, -\omega),$$
$$\kappa_{ij}(\vec{r}, \omega)^* = \kappa_{ij}(\vec{r}, -\omega), \tag{2.13}$$

where * designates the complex conjugate of the referred quantity.

Omitting the space coordinates $\vec{r}$, $\kappa_{ij}(\vec{r}, \omega)$ may be expressed in the form

$$\kappa_{ij}(\omega) = (\omega)_-^{-1}[\zeta_{ij}(\omega) - i\sigma_{ij}(\omega)]. \tag{2.14}$$

Here, when ω is real, ζ_{ij} and σ_{ij} are Hermitian matrices with respect to the Latin subscripts and hence

$$\zeta_{ij}^*(\omega) = \zeta_{ji}(\omega), \quad \sigma_{ij}^*(\omega) = \sigma_{ji}(\omega), \tag{2.15a}$$

and are also analytic on the real axis as well as in the lower-half plane of ω;

$$(\omega)_\pm^{-1} = (\omega \pm i\epsilon)^{-1} = \mp \pi i \delta(\omega) + P(\omega)^{-1}, \quad \epsilon > 0, \tag{2.15b}$$

where ϵ is an infinitesimal positive number and P denotes the Cauchy principal value. Hence, (2.14) can be given also in another form

$$\kappa_{ij}(\omega) = \pi[\sigma_{ij}(0) + i\zeta_{ij}(0)]\,\delta(\omega) + P(\omega)^{-1}[\zeta_{ij}(\omega) - i\delta_{ij}(\omega)], \tag{2.16}$$

and thus $\kappa_{ij}(\omega)$ becomes Hermitian only when

$$\sigma_{ij}(\omega \neq 0) = \zeta_{ij}(0) = 0.$$

The relations (2.13) and (2.15a) are combined to give the relations

$$\zeta_{ij}(-\omega) = -\zeta_{ji}(\omega), \quad \sigma_{ij}(-\omega) = \sigma_{ji}(\omega). \tag{2.17}$$

Further, $\zeta_{ij}(\omega)$ is not independent of $\sigma_{ij}(\omega)$ but can be expressed in terms of it using the analytic property.

The significance of the first term on the right-hand side of (2.16) becomes clear by considering the special case of constant ζ_{ij} and σ_{ij}: In that case, we have from (2.12) and (2.14)

$$\begin{aligned}\tilde{\kappa}_{ij}(\vec{r}, t) &= (2\pi)^{-1}\int_{-\infty}^{\infty} d\omega\, \kappa_{ij}(\omega)\, e^{i\omega t} \\ &= \sigma_{ij}(0) + i\zeta_{ij}(0), \quad t > 0 \\ &= 0, \quad t < 0.\end{aligned} \tag{2.18}$$

Here, in view of (2.15a) and (2.17), the coefficients $i\zeta_{ij}(0)$ form a real antisymmetric matrix, while the $\sigma_{ij}(0)$ form a real symmetric matrix. Thus we find that (2.18) gives the anisotropic conductivity, as may be seen from (2.4) with (2.10).

We can apply the same considerations to $\nu_{ij}(\omega)$, but no singularity is expected at $\omega = 0$ because of its contribution to the current by (2.4). We shall simply assume the form

$$\nu_{ij}(\omega) = \nu_{ij}{}^S(\omega) - i\nu_{ij}{}^A(\omega), \tag{2.19}$$

in which case

$$\nu_{ij}{}^S(\omega)^* = \nu_{ji}{}^S(\omega) = \nu_{ij}{}^S(-\omega),$$
$$\nu_{ij}{}^A(\omega)^* = \nu_{ji}{}^A(\omega) = -\nu_{ij}{}^A(-\omega). \tag{2.20}$$

Introducing the Fourier transforms of the field variables defined by

$$B_i(\vec{r}, \omega) = \int_{-\infty}^{\infty} dt\, \tilde{B}_i(\vec{r}, t)\, e^{-i\omega t} \tag{2.21}$$

and similar expressions for the other physical quantities, we obtain from (2.10) and (2.12)

$$M_i(\vec{r}, \omega) = \nu_{ij}(\vec{r}, \omega) B_j(\vec{r}, \omega),$$
$$P_i(\vec{r}, \omega) = \kappa_{ij}(\vec{r}, \omega) E_j(\vec{r}, \omega). \tag{2.22}$$

Comparing with (2.9), we find that, if ϵ and μ are the Fourier transforms of $\tilde{\epsilon}$ and $\tilde{\mu}$ similar to (2.12),

$$(\epsilon - 1)_{ij} = \kappa_{ij}(\vec{r}, \omega), \quad (1 - \mu^{-1})_{ij} = \nu_{ij}(\vec{r}, \omega), \tag{2.23}$$

and, by the Fourier transformation of (2.2) and (2.7),

$$\mathrm{rot}\vec{H} = c^{-1}\partial\vec{D}/\partial t + \vec{j}^{\,e}, \quad \mathrm{div}\vec{D} = \rho^e,$$
$$\mathrm{rot}\vec{E} = -c^{-1}\partial\vec{B}/\partial t, \quad \mathrm{div}\vec{B} = 0. \tag{2.24}$$

All physical quantities involved are functions of $\vec{r}$ and ω, and are denoted by the lightface letters.

3. COVARIANT FORM OF THE EQUATIONS

In this section, we employ the following notation: Greek subscripts assume values ranging from 1 to 4, while Latin subscripts assume values ranging from 1 to 3, and a repeated index is to be summed. The coordinate vector of a four-dimensional point x is denoted by $x_\mu = (\vec{r}, ict)$ with $x_4 = ix_0 = ict$, and the four-vector derivative by $\partial_\mu = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3, \partial/\partial x_4)$. The four-dimensional element of volume is defined as $(dx) = dx_0 dx_1 dx_2 dx_3$.

A. Covariant Form of Maxwell's Equations in Moving Dispersive Media

The electromagnetic field is characterized by the antisymmetric tensors $\tilde{F}_{\nu\mu} = -\tilde{F}_{\mu\nu}$ and $\tilde{H}_{\nu\mu} = -\tilde{H}_{\mu\nu}$ defined by

$$\tilde{B}_k=(\tilde{F}_{23},\ \tilde{F}_{31},\ \tilde{F}_{12}),$$
$$i\tilde{E}_k=(\tilde{F}_{41},\ \tilde{F}_{42},\ \tilde{F}_{43}),$$
$$\tilde{H}_k=(\tilde{H}_{23},\ \tilde{H}_{31},\ \tilde{H}_{12}),$$
$$i\tilde{D}_k=(\tilde{H}_{41},\ \tilde{H}_{42},\ \tilde{H}_{43}), \tag{3.1}$$

and also by the four-vectors $\tilde{j}_\mu{}^e=(\tilde{\vec{j}}^e,\tilde{\rho}^e)$ and $\tilde{j}_\mu=(\tilde{\vec{j}},\tilde{\rho})$; Eqs. (2.1)–(2.3) are then expressed, respectively, by

$$\partial_\nu\tilde{F}_{\nu\mu}=-\tilde{j}_\mu-\tilde{j}_\mu{}^e, \tag{3.2}$$

$$\partial_\alpha\tilde{F}_{\beta\gamma}+\partial_\beta\tilde{F}_{\gamma\alpha}+\partial_\gamma\tilde{F}_{\alpha\beta}=0, \tag{3.3}$$

$$\partial_\mu\tilde{j}_\mu=\partial_\mu\tilde{j}_\mu{}^e=0. \tag{3.4}$$

If we further introduce the antisymmetrical polarization tensor $\tilde{\Pi}_{\nu\mu}=-\tilde{\Pi}_{\mu\nu}$ defined by

$$-\tilde{M}_k=(\tilde{\Pi}_{23},\ \tilde{\Pi}_{31},\ \tilde{\Pi}_{12}),$$
$$i\tilde{P}_k=(\tilde{\Pi}_{41},\ \tilde{\Pi}_{42},\ \tilde{\Pi}_{43}). \tag{3.5}$$

Equations (2.4) and (2.5) are expressed by

$$\tilde{j}_\mu=\partial_\nu\tilde{\Pi}_{\nu\mu},\quad \tilde{\Pi}_{\nu\mu}=-\tilde{\Pi}_{\mu\nu}, \tag{3.6}$$

and the conservation law (3.4) of $\tilde{j}_\mu$ is guaranteed. Equation (3.2) is thus expressed as

$$\partial_\nu\tilde{H}_{\nu\mu}=-\tilde{j}_\mu{}^e, \tag{3.7}$$

$$\text{with}\quad \tilde{H}_{\nu\mu}=\tilde{F}_{\nu\mu}+\tilde{\Pi}_{\nu\mu}. \tag{3.8}$$

When the external current $\tilde{j}_\mu{}^e$ is caused by small displacements of continuous media, it is always possible to express it by the equation

$$\tilde{j}_\mu{}^e=\partial_\nu\tilde{S}_{\nu\mu},\quad \tilde{S}_{\nu\mu}=-\tilde{S}_{\mu\nu} \tag{3.9}$$

of the same form as (3.6). Using (3.9), Eq. (3.7) is further reduced to

$$\partial_\nu(\tilde{H}_{\nu\mu}+\tilde{S}_{\nu\mu})=0, \tag{3.10}$$

which means the conservation of the total "displacements" including those of the external system.

Generally, the polarization $\tilde{\Pi}_{\nu\mu}$ is considered to be proportional to $\tilde{F}_{\alpha\beta}$ to first order in field strength, and thus, when the medium is dispersive, it takes the form, as in (2.10),

$$\tilde{\Pi}_{\nu\mu}(x)=\int(dx')\tilde{\kappa}_{\nu\mu\alpha\beta}(x,x')\tilde{F}_{\alpha\beta}(x'). \tag{3.11}$$

Here, from the symmetry of $\tilde{\Pi}_{\nu\mu}$ and $\tilde{F}_{\alpha\beta}$,

$$\tilde{\kappa}_{\nu\mu\alpha\beta}=-\tilde{\kappa}_{\mu\nu\alpha\beta}=-\tilde{\kappa}_{\nu\mu\beta\alpha}, \tag{3.12}$$

and, from the relativistic causality,

$$\begin{aligned}\tilde{\kappa}_{\nu\mu\alpha\beta}(x,x')&=0,\quad \text{if } x_0-x_0'<0,\\ &=0,\quad \text{if } (x_\nu-x_\nu')^2>0,\end{aligned} \tag{3.13}$$

i.e., it also vanishes if the points x and x' are spacelike to each other. It follows from this fact that the Fourier transform defined by

$$\kappa_{\nu\mu\alpha\beta}(x,\omega)=\int(dx')\exp[i\omega_\lambda(x_\lambda-x_\lambda')]\,\tilde{\kappa}_{\nu\mu\alpha\beta}(x,x') \tag{3.14}$$

is analytic in the lower half of the complex plane of ω_0 or of any timelike component of the 4-vector ω_λ.[8] When the medium is homogeneous, $\tilde{\kappa}_{\nu\mu\alpha\beta}(x,x')$ becomes a function only of $x-x'$ and hence $\kappa_{\nu\mu\alpha\beta}(x,\omega)$ is independent of the coordinates x.

Introducing the new quantity

$$m_{\nu\mu\alpha\beta}(x,\omega)=\tfrac{1}{2}(\delta_{\nu\alpha}\delta_{\mu\beta}-\delta_{\mu\alpha}\delta_{\nu\beta})+\kappa_{\nu\mu\alpha\beta}(x,\omega), \tag{3.15a}$$

with the same symmetry as in (3.12), i.e.,

$$m_{\nu\mu\alpha\beta}=-m_{\mu\nu\alpha\beta}=-m_{\nu\mu\beta\alpha}, \tag{3.15b}$$

we obtain, from (3.11) and (3.8),

$$\begin{aligned}\tilde{\Pi}_{\nu\mu}(x)&=(2\pi)^{-4}\int(d\omega)\kappa_{\nu\mu\alpha\beta}(x,\omega)\\ &\quad\times F_{\alpha\beta}(\omega)\exp(-i\omega_\lambda x_\lambda),\\ \tilde{H}_{\nu\mu}(x)&=(2\pi)^{-4}\int(d\omega)m_{\nu\mu\alpha\beta}(x,\omega)\\ &\quad\times F_{\alpha\beta}(\omega)\exp(-i\omega_\lambda x_\lambda),\end{aligned} \tag{3.16}$$

$$(d\omega)=d\omega_0d\omega_1d\omega_2d\omega_3,$$

in terms of the notation

$$Q(\omega')=\int(dx)\exp(+i\omega_\lambda'x_\lambda)\tilde{Q}(x). \tag{3.17}$$

The tensor $\kappa_{\nu\mu\alpha\beta}(x,\omega)$ can be expressed in terms of ν_{ij} and κ_{ij} introduced in (2.12); when the medium is composed of a single component moving with the velocity $\vec{v}$, it is found that[9]

$$\begin{aligned}\kappa_{\nu\mu\alpha\beta}&=\tfrac{1}{2}[(n_\mu\kappa_{\nu\beta}-n_\nu\kappa_{\mu\beta})n_\alpha\\ &\quad+(n_\nu\kappa_{\mu\alpha}-n_\mu\kappa_{\nu\alpha})n_\beta\end{aligned}$$

$$+\epsilon_{\nu\mu\lambda\pi} n_\pi n_\lambda \xi n_\eta \epsilon_{\xi\eta\alpha\beta}]. \quad (3.18a)$$

Here,

$$n_\nu = (\gamma\vec{v}/c, i\gamma), \quad n_\nu^2 = -1,$$

$$\gamma = [1 - (\vec{v}/c)^2]^{-1/2}, \quad (3.18b)$$

and $\kappa_{\nu\beta} = \kappa_{\nu\beta}(x, \omega)$ is a function only of $\omega = -n_\lambda\omega_\lambda$ and the spacelike coordinates, say $\vec{x}_\nu = x_\nu + n_\nu(n_\lambda x_\lambda)$ with $n_\lambda \vec{x}_\lambda = 0$; $\epsilon_{\nu\mu\lambda\pi}$ is the Levi-Civitá symbol[10] which is antisymmetrical with respect to all subscripts and is +1 or −1 or 0 according as (ν, μ, λ, π) is an even or an odd permutation of (1, 2, 3, 4) or otherwise, respectively. Indeed, (3.18a) satisfies the same symmetry as (3.12) and also, in the rest frame of reference of the medium in which $n_\nu = (0, 0, 0, i)$, we readily find the equivalence of $\tilde{\Pi}_{\nu\mu}(x)$ in (3.16) with the Fourier (inverse) transform of (2.22) [Note that $\kappa_{\nu\mu\alpha\beta}(x, \omega)$ is a function only of the space coordinates $\vec{r}$ and the time frequency ω used in Sec. 2 and is independent of the space components of ω_ν]. The covariant expression of $\kappa_{\nu\mu}$ in the case of the anisotropic plasma is given by (5.17).

It is straightforward to extend these considerations to the general case in which the medium is composed of several components of different kinds and/or velocities; $\kappa_{\nu\mu\alpha\beta}(x, \omega)$ in (3.15) and (3.16) is then replaced by the sum of their independent contributions

$$\kappa_{\nu\mu\alpha\beta} = \sum_n \kappa_{\nu\mu\alpha\beta}^{(n)}, \quad (3.19)$$

where the superscript (n) expresses the dependence of each component on the timelike unit vector n_μ characterizing its velocity.

B. 'Packet' Field and Its Covariant Equations

Let $\tilde{Q}(x)$ be any tensor expressed by

$$\tilde{Q}(x) = 2^{-1/2}[\exp(-i\omega_\lambda x_\lambda)Q(x, \omega) + \exp(+i\omega_\lambda x_\lambda)Q(x, -\omega)], \quad (3.20)$$

and let $\tilde{Q}(x)$ be defined to be "real" if $\tilde{Q}^\dagger(x) = \tilde{Q}(x)$, where

$$\tilde{Q}^\dagger(x) = \text{time reversal of } \tilde{Q}^*(x). \quad (3.21)$$

Then, since $x_\nu^\dagger = x_\nu$ is real, the real condition of $\tilde{Q}(x)$ for any real 4-vector $\omega_\lambda^\dagger = \omega_\lambda$ becomes

$$Q^\dagger(x, \omega) = Q(x, -\omega), \quad (3.22)$$

and the periodic mean value of the product $\tilde{Q}(x)\tilde{P}(x)$, say $\langle\tilde{Q}(x)\tilde{P}(x)\rangle$, of any real $\tilde{Q}(x)$ and $\tilde{P}(x)$ of the form of (3.20) over the period characterized by the vector ω_ν can be shown to be

$$\langle\tilde{Q}(x)\tilde{P}(x)\rangle = \tfrac{1}{2}[Q^\dagger(x, \omega)P(x, \omega) + Q(x, \omega)P^\dagger(x, \omega)]. \quad (3.23)$$

When the space-time change of $Q(x, \omega)$ in the direction of ω_ν is very small within the range of the period of ω_ν, $Q(x, \omega)$ describes the envelope of a wave packet in that direction; if ω_ν is timelike, it is a temporal wave packet, while, if ω_ν is spacelike, it is a spatial wave packet. In the following, we shall assume that all the field variables and also the external sources are of the form of (3.20), and look for the covariant equations to be satisfied by the lightface variables like $Q(x, \omega)$.

Assuming the possibility of a Taylor expansion of $Q(x', \omega)$ in a power series of $x' - x$, one expresses the expansion formally by

$$Q(x', \omega) = \exp[(x'_\lambda - x_\lambda)\partial''_\lambda]\, Q(x'', \omega)\big|_{x''=x}. \quad (3.24)$$

Hence, the Fourier transform of $\tilde{Q}(x)$, (3.17), yields, on using (3.20) and (3.24), the term

$$2^{-1/2}\int (dx')\exp[i(\omega'_\lambda - \omega_\lambda - i\partial''_\lambda)x'_\lambda] \times \exp(-x_\lambda\partial''_\lambda)Q(x'', \omega)\big|_{x''=x}$$
$$= 2^{-1/2}(2\pi)^4\,\delta^4(\omega' - \omega - i\partial'')\,Q(x'' - x, \omega)\big|_{x''=x}, \quad (3.25)$$

δ^4 being the δ function in the four-dimensional space. Thus we formally have

$$Q(\omega') = 2^{-1/2}(2\pi)^4[\delta^4(\omega' - \omega - i\partial'')Q(x'' - x, \omega) + \delta^4(\omega' + \omega - i\partial'')Q(x'' - x, -\omega)]\big|_{x''=x}. \quad (3.26)$$

Suppose $\tilde{Q}(x) = \tilde{F}_{\alpha\beta}(x)$ with $Q(\omega') = F_{\alpha\beta}(\omega')$ given by (3.26) and substitute the latter in (3.16) to obtain $\tilde{H}_{\nu\mu}(x)$. Then, we readily find that

$$\tilde{H}_{\nu\mu}(x) = 2^{-1/2}[\exp(-i\omega_\lambda x_\lambda) \times m_{\nu\mu\alpha\beta}(x, \omega + i\partial)F_{\alpha\beta}(x, \omega) + \exp(+i\omega_\lambda x_\lambda) \times m_{\nu\mu\alpha\beta}(x, -\omega + i\partial)F_{\alpha\beta}(x, -\omega)], \quad (3.27)$$

which is again of the form of (3.20). Hence,

$$H_{\nu\mu}(x, \omega) = m_{\nu\mu\alpha\beta}(x, \omega + i\partial)F_{\alpha\beta}(x, \omega). \quad (3.28)$$

Here, $m_{\nu\mu\alpha\beta}(x, \omega+i\partial)$ is to be ordered in such a way that, when expanding it in power series of $(\omega+i\partial)$, the term $(\omega+i\partial)^n$ should be always to the right-hand side of the coefficient (which is generally a function of the coordinates x). However, when the medium components are moving with uniform velocities, this well ordering does not matter, since the differential operators are always commutable with the coordinates involved, as may be seen from (3.18a). Thus, in this case, it holds that

$$\partial_\lambda m_{\nu\mu\alpha\beta}{}^{\lambda}(x, \omega) = 0 \tag{3.29}$$

in terms of the notation

$$m_{\nu\mu\alpha\beta}{}^{\lambda}(x, \omega) \equiv (\partial/\partial\omega_\lambda) m_{\nu\mu\alpha\beta}(x, \omega). \tag{3.30}$$

From the real condition (3.22), we find that both $\tilde{H}_{\nu\mu}$ and $\tilde{F}_{\nu\mu}$ are real if

$$m_{\nu\mu\alpha\beta}(x, \omega)^\dagger = m_{\nu\mu\alpha\beta}(x, -\omega), \tag{3.31}$$

which is guaranteed directly by the definition (3.14).

Now, assuming the form of (3.20) also for $\tilde{S}_{\nu\mu}(x)$, Eqs. (3.9) and (3.7) yield

$$ij_\mu{}^e(x, \omega) = (i\partial_\nu + \omega_\nu) S_{\nu\mu}(x, \omega), \tag{3.32}$$

$$(i\partial_\nu + \omega_\nu) H_{\nu\mu}(x, \omega) = -ij_\mu{}^e(x, \omega), \tag{3.33}$$

and thus

$$(i\partial_\nu + \omega_\nu)[H_{\nu\mu}(x, \omega) + S_{\nu\mu}(x, \omega)] = 0. \tag{3.34}$$

4. LAGRANGIAN PRINCIPLE AND ENERGY-MOMENTUM TENSOR IN NONDISSIPATIVE MEDIA

In this section, we shall find that, in the case of a nondissipative medium, Eq. (3.34) can be derived by the Lagrangian variational principle. Thus the energy-momentum tensor for the "packet" field can be constructed by the orthodox methods used in field theory.

A. Construction of Energy-Momentum Tensor and Boundary Conditions on Discontinuous Surface

Equation (3.3) is automatically satisfied by the introduction of the 4-vector potential $\tilde{\phi}_\mu$:

$$\tilde{F}_{\nu\mu} = \partial_\nu \tilde{\phi}_\mu - \partial_\mu \tilde{\phi}_\nu . \tag{4.1}$$

Assuming the form of (3.20) for $\tilde{\phi}_\mu$ with the additional factor i, we have

$$\begin{aligned} F_{\nu\mu}(x, \omega) &= (i\partial_\nu + \omega_\nu)\phi_\mu(x, \omega) - (i\partial_\mu + \omega_\mu)\phi_\nu(x, \omega), \\ F_{\nu\mu}(x, \omega)^\dagger &= (-i\partial_\nu + \omega_\nu)\phi_\mu(x, \omega)^\dagger - (-i\partial_\mu + \omega_\mu)\phi_\nu(x, \omega)^\dagger, \end{aligned} \tag{4.2}$$

where $\phi_\mu(x, \omega)$ is pure "imaginary" and, instead of (3.22), it has the relation

$$\phi_\mu(x, \omega)^\dagger = -\phi_\mu(x, -\omega). \tag{4.3}$$

$F_{\nu\mu}$ given by (4.2) is invariant for the replacement

$$\phi_\mu \to \phi_\mu + (i\partial_\mu + \omega_\mu)\psi, \quad \phi_\mu{}^\dagger \to \phi_\mu{}^\dagger + (-i\partial_\mu + \omega_\mu)\psi^\dagger, \tag{4.4}$$

where ψ is an arbitrary function. The replacement (4.4) will be called the gauge transformation, and any physically significant quantity should be gauge invariant.

We introduce the gauge-invariant Lagrangian density function $\mathcal{L}$ by

$$\mathcal{L} = -\tfrac{1}{8}(F_{\nu\mu}{}^\dagger H_{\nu\mu} + F_{\nu\mu} H_{\nu\mu}{}^\dagger) - \tfrac{1}{4}(F_{\nu\mu}{}^\dagger S_{\nu\mu} + F_{\nu\mu} S_{\nu\mu}{}^\dagger).$$

Here, $F_{\nu\mu}$ and $F_{\nu\mu}{}^\dagger$ are given by (4.2), and $H_{\nu\mu}$ is by (3.28), while $S_{\nu\mu}$ is to be a physical quantity of the external system giving (3.32).

The medium will be found to be nondissipative if

$$m_{\nu\mu\alpha\beta}(x, \omega)^\dagger = m_{\alpha\beta\nu\mu}(x, \omega), \quad \text{or using (3.31),} \quad m_{\nu\mu\alpha\beta}(x, -\omega) = m_{\alpha\beta\nu\mu}(x, \omega). \tag{4.5}$$

Hence, assuming the condition (4.5) and also the very small change of the envelope of the packet field within the period of ω, we have, to first order of ∂,

$$H_{\nu\mu}(x,\omega)\simeq[m_{\nu\mu\alpha\beta}(x,\omega)+m_{\nu\mu\alpha\beta}{}^{\lambda}(x,\omega)(i\partial_\lambda)]F_{\alpha\beta}(x,\omega),$$

$$H_{\nu\mu}(x,\omega)^{\dagger}\simeq F_{\alpha\beta}(x,\omega)^{\dagger}[m_{\alpha\beta\nu\mu}(x,\omega)+(-i\overleftarrow{\partial}_\lambda)m_{\alpha\beta\nu\mu}{}^{\lambda}(x,\omega)] \tag{4.6}$$

in terms of the notation (3.30).

Now, $\mathcal{L}$ can be regarded as a function of ϕ_ν, $\phi_\nu{}^\dagger$ and their derivatives, and also of $S_{\nu\mu}$ and $S_{\nu\mu}{}^\dagger$. In order to see the contributions of their variations, we first note that, using (4.6) with the relation (4.5),

$$F_{\nu\mu}{}^{\dagger}\delta H_{\nu\mu}=H_{\nu\mu}{}^{\dagger}\delta F_{\nu\mu}+i\partial_\lambda(\Pi_{\nu\mu\lambda}{}^{\dagger}\delta F_{\nu\mu}),\quad F_{\nu\mu}\delta H_{\nu\mu}{}^{\dagger}=H_{\nu\mu}\delta F_{\nu\mu}{}^{\dagger}-i\partial_\lambda(\Pi_{\nu\mu}{}^{\lambda}\delta F_{\nu\mu}{}^{\dagger}). \tag{4.7}$$

Here, with reference to (4.6),

$$H_{\nu\mu}=F_{\nu\mu}+\Pi_{\nu\mu}+i\partial_\lambda\Pi_{\nu\mu}{}^{\lambda},\quad \Pi_{\nu\mu}=\kappa_{\nu\mu\alpha\beta}F_{\alpha\beta},\quad \Pi_{\nu\mu}{}^{\lambda}=m_{\nu\mu\alpha\beta}{}^{\lambda}F_{\alpha\beta},\quad \Pi_{\nu\mu}{}^{\lambda\dagger}=F_{\alpha\beta}{}^{\dagger}m_{\alpha\beta\nu\mu}{}^{\lambda},\ \text{etc.}, \tag{4.8}$$

and the condition (3.29) is also taken into account. Thus

$$\delta\mathcal{L}=-\tfrac{1}{4}[(H_{\nu\mu}{}^{\dagger}+S_{\nu\mu}{}^{\dagger})\delta F_{\nu\mu}+(H_{\nu\mu}+S_{\nu\mu})\delta F_{\nu\mu}{}^{\dagger}]$$

$$-(i/8)\partial_\lambda[\Pi_{\nu\mu}{}^{\lambda\dagger}\delta F_{\nu\mu}-\Pi_{\nu\mu}{}^{\lambda}\delta F_{\nu\mu}{}^{\dagger}]-\tfrac{1}{4}[F_{\nu\mu}\delta S_{\nu\mu}{}^{\dagger}+F_{\nu\mu}{}^{\dagger}\delta S_{\nu\mu}], \tag{4.9}$$

with $\delta F_{\nu\mu}=(i\partial_\nu+\omega_\nu)\delta\phi_\mu-(i\partial_\mu+\omega_\mu)\delta\phi_\nu$. (4.10)

The integration of (4.9) over the four-dimensional space enclosed by two spacelike surfaces, say σ_1 and σ_2, yields, with the aid of partial integration,

$$\int_{\sigma_2}^{\sigma_1}\delta\mathcal{L}(dx)=-\tfrac{1}{2}\int_{\sigma_2}^{\sigma_1}(dx)[\delta\phi_\mu(-i\partial_\nu+\omega_\nu)(H_{\nu\mu}{}^{\dagger}+S_{\nu\mu}{}^{\dagger})+\delta\phi_\mu{}^{\dagger}(i\partial_\nu+\omega_\nu)(H_{\nu\mu}+S_{\nu\mu})+\tfrac{1}{2}(F_{\nu\mu}\delta S_{\nu\mu}{}^{\dagger}+F_{\nu\mu}{}^{\dagger}\delta S_{\nu\mu})]$$

$$+\int_\sigma d\sigma_\lambda\{-(i/2)[(H_{\lambda\mu}{}^{\dagger}+S_{\lambda\mu}{}^{\dagger})\delta\phi_\mu-(H_{\lambda\mu}+S_{\lambda\mu})\delta\phi_\mu{}^{\dagger}]+(i/8)[\Pi_{\nu\mu}{}^{\lambda}\delta F_{\nu\mu}{}^{\dagger}-\Pi_{\nu\mu}{}^{\lambda\dagger}\delta F_{\nu\mu}]\}\Big|_{\sigma=\sigma_2}^{\sigma_1}. \tag{4.11}$$

Here, $d\sigma_\lambda$ is the 4-vector differential surface area

$$d\sigma_\lambda=(dx_2dx_3dx_0,\ dx_3dx_1dx_0,\ dx_1dx_2dx_0,\ dx_1dx_2dx_3/i), \tag{4.12}$$

and is normal to the spacelike surface σ. Thus regarding $\delta\phi_\mu$ and $\delta\phi_\mu{}^\dagger$ as independent variations in (4.11), we find Eq. (3.34) and its conjugate equation according to the Lagrangian variational principle.

It is straightforward to construct the energy-momentum tensor $T_{\lambda\sigma}$ according to the ordinary method in field theory; it is obtained from the integrand of the surface integral in (4.11) by the replacement $\delta\phi_\mu\to-\partial_\sigma\phi_\mu$, $\delta F_{\nu\mu}\to-\partial_\sigma F_{\nu\mu}$ and the corresponding replacement for the conjugate variables with the addition of $\mathcal{L}\delta_{\lambda\sigma}$:

$$T_{\lambda\sigma}=\tfrac{1}{2}[(H_{\lambda\mu}{}^{\dagger}+S_{\lambda\mu}{}^{\dagger})(i\partial_\sigma\phi_\mu)+(H_{\lambda\mu}+S_{\lambda\mu})(-i\partial_\sigma\phi_\mu{}^{\dagger})]+\tfrac{1}{8}[\Pi_{\lambda\mu}{}^{\lambda\dagger}(i\partial_\sigma F_{\nu\mu})+\Pi_{\nu\mu}{}^{\lambda}(-i\partial_\sigma F_{\nu\mu}{}^{\dagger})]+\mathcal{L}\delta_{\lambda\sigma}, \tag{4.13}$$

and it satisfies the conservation equation

$$\partial_\lambda T_{\lambda\sigma}=\partial_\sigma\mathcal{L}\equiv-\tfrac{1}{4}(F_{\nu\mu}{}^{\dagger}\partial_\sigma S_{\nu\mu}+F_{\nu\mu}\partial_\sigma S_{\nu\mu}{}^{\dagger})-\tfrac{1}{8}\{F_{\nu\mu}{}^{\dagger}[\partial_\sigma m_{\nu\mu\alpha\beta}(x,\omega+i\partial)]F_{\alpha\beta}$$

$$+F_{\alpha\beta}{}^{\dagger}[\partial_\sigma m_{\alpha\beta\nu\mu}(x,\omega-i\overleftarrow{\partial})]F_{\nu\mu}\}. \tag{4.14}$$

However, $T_{\lambda\sigma}$ is not gauge-invariant.

In order to obtain the gauge-invariant energy-momentum tensor, we note the invariance of $\mathcal{L}$ under variations of the physical variables caused by a change of the phases of a constant amount:

$$\phi_\mu \to \phi_\mu \exp(i\theta), \quad \phi_\mu{}^\dagger \to \phi_\mu{}^\dagger \exp(-i\theta), \quad S_{\nu\mu} \to S_{\nu\mu} \exp(i\theta), \quad S_{\nu\mu}{}^\dagger \to S_{\nu\mu}{}^\dagger \exp(-i\theta).$$

Hence, for the infinitesimal variations $\delta\phi_\mu = i\phi_\mu \delta\theta$, $\delta F_{\nu\mu} = iF_{\nu\mu}\delta\theta$, $\delta S_{\nu\mu} = iS_{\nu\mu}\delta\theta$, and corresponding variations for the conjugate variables, the right-hand side of (4.11) should vanish. Thus we find the relation

$$\int_\sigma d\sigma_\lambda S_\lambda \Big|_{\sigma=\sigma_2}^{\sigma_1} + (i/4)\int_{\sigma_2}^{\sigma_1}(dx)(F_{\nu\mu}S_{\nu\mu}{}^\dagger - F_{\nu\mu}{}^\dagger S_{\nu\mu}) = 0\,, \tag{4.15}$$

where $$S_\lambda = \tfrac{1}{2}[(H_{\lambda\mu}{}^\dagger + S_{\lambda\mu}{}^\dagger)\phi_\mu + (H_{\lambda\mu} + S_{\lambda\mu})\phi_\mu{}^\dagger] + \tfrac{1}{8}(\Pi_{\nu\mu}{}^\lambda F_{\nu\mu}{}^\dagger + \Pi_{\nu\mu}{}^{\lambda\dagger} F_{\nu\mu}). \tag{4.16}$$

Equation (4.15) is equivalent to

$$\partial_\lambda S_\lambda = (i/4)(F_{\nu\mu}{}^\dagger S_{\nu\mu} - F_{\nu\mu}S_{\nu\mu}{}^\dagger). \tag{4.17}$$

Now, we introduce the tensor $\theta_{\lambda\sigma}$ defined by

$$\theta_{\lambda\sigma} = T_{\lambda\sigma} + S_\lambda \omega_\sigma + (i/2)\partial_\mu[(H_{\lambda\mu} + S_{\lambda\mu})\phi_\sigma{}^\dagger - (H_{\lambda\mu}{}^\dagger + S_{\lambda\mu}{}^\dagger)\phi_\sigma]\,, \tag{4.18}$$

whose explicit expression is

$$\theta_{\lambda\sigma} = \tfrac{1}{2}[(H_{\lambda\mu}{}^\dagger + S_{\lambda\mu}{}^\dagger)F_{\sigma\mu} + (H_{\lambda\mu} + S_{\lambda\mu})F_{\sigma\mu}{}^\dagger] + \mathcal{L}\delta_{\lambda\sigma} + \tfrac{1}{8}[\Pi_{\nu\mu}{}^{\lambda\dagger}(i\partial_\sigma + \omega_\sigma)F_{\nu\mu} + \Pi_{\nu\mu}{}^\lambda(-i\partial_\sigma + \omega_\sigma)F_{\nu\mu}{}^\dagger]\,. \tag{4.19}$$

The tensor $\theta_{\lambda\sigma}$ is evidently gauge-invariant and, on using (4.14) and (4.17), its divergence is found to be

$$\partial_\lambda \theta_{\lambda\sigma} = (i/4)[F_{\nu\mu}{}^\dagger(i\partial_\sigma + \omega_\sigma)S_{\nu\mu} - F_{\nu\mu}(-i\partial_\sigma + \omega_\sigma)S_{\nu\mu}{}^\dagger] - \tfrac{1}{8}F_{\nu\mu}{}^\dagger\{[\partial_\sigma m_{\nu\mu\alpha\beta}(x, \omega + i\vec{\partial})] + [\partial_\sigma m_{\nu\mu\alpha\beta}(x, \omega - i\overleftarrow{\partial})]\}F_{\alpha\beta}. \tag{4.20}$$

It is quite reasonable to identify $\theta_{\lambda\sigma}$ with the energy-momentum tensor of the field, and, in Sec. 4 B, we shall indeed find that the direction of $\theta_{\lambda\sigma}$ for any given σ agrees with that of the group velocity ordinarily defined.

The Lagrangian variational principle has the additional advantage of giving the necessary boundary conditions on any surface of discontinuity of the medium.

When the medium is discontinuous across a surface, say σ', in space and time, the right-hand side of (4.11) contains the additional surface integral over σ' with the same integrand as those over σ_1 and σ_2. Here, $\delta\phi_\mu$, $\delta F_{\nu\mu}$ and their conjugate variations involved should be continuous on σ' in order that the partial integration employed in the derivation of (4.11) is possible. Since the variations are arbitrary, it follows from the Lagrangian principle that this additional surface integral also should vanish and hence, if $n_\lambda(\sigma')$ denotes the unit vector normal to the surface σ' at a particular point on it, it must hold, at any point on σ', that

$$n_\lambda(\sigma')(\Delta H_{\lambda\mu} + \Delta S_{\lambda\mu}) = n_\lambda(\sigma')\,\Delta\Pi_{\nu\mu}{}^\lambda = 0\,, \tag{4.21a}$$

and their conjugate equations. Here, Δ denotes the difference of the referred quantity on each side of σ'; $\Delta S_{\lambda\mu}$ is different from zero when the external system has a surface current on σ', as may be seen from (3.32).

On the other hand, the ϕ_μ's also should be continuous on σ' with possible discontinuities of their first derivatives, in order that the field variable $F_{\nu\mu}$ is finite on σ'. This condition can be expressed in the same form as (4.21a) in terms of the dual tensor:

$$n_\lambda(\sigma')\Delta B_{\lambda\mu} = 0, \quad B_{\nu\mu} = (2i)^{-1}\epsilon_{\nu\mu\alpha\beta}F_{\alpha\beta}\,. \tag{4.21b}$$

B. Agreement of the Direction of Power Flow Vector with That of Wave Path

In this section, we assume that the (nondissipative) medium is homogeneous in space and time and no external source is present, so that there exists a plane-wave solution.

If we regard ω_λ in the preceding sections as the wave-number vector of the plane wave, we have from (4.2) and (3.28)

$$F_{\nu\mu}=\omega_\nu\phi_\mu-\omega_\mu\phi_\nu, \quad H_{\nu\mu}=m_{\nu\mu\alpha\beta}(\omega)F_{\alpha\beta}, \tag{4.22}$$

and also, from (3.34),

$$\omega_\nu H_{\nu\mu}=0, \quad \omega_\nu H_{\nu\mu}{}^\dagger=0, \quad \omega_\nu{}^\dagger=\omega_\nu, \tag{4.23}$$

provided that the vector ω_ν is real. Here, taking into account the symmetry of $m_{\nu\mu\alpha\beta}$ in (3.15b), we have from (4.23)

$$\omega_\nu H_{\nu\mu}=2\omega_\nu m_{\nu\mu\alpha\beta}\omega_\alpha\phi_\beta=0, \quad \text{or} \quad D_{\mu\beta}\phi_\beta=0, \quad \phi_\mu{}^\dagger D_{\mu\beta}=0 \tag{4.24}$$

in terms of the notation

$$D_{\mu\beta}\equiv\omega_\nu m_{\nu\mu\alpha\beta}\omega_\alpha=D_{\beta\mu}{}^\dagger, \tag{4.25}$$

and Eq. (4.24) gives the dispersion equation

$$\mathrm{Det}[D_{\mu\beta}]=0. \tag{4.26}$$

Now, we consider the arbitrary variation $\delta\omega_\lambda$ of ω_λ satisfying the dispersion equation or (4.24) and look for the relation to be satisfied by the elements of $\delta\omega_\lambda$; in terms of the notation

$$D\equiv\phi_\mu{}^\dagger D_{\mu\beta}\phi_\beta, \quad D_{\mu\beta}{}^\lambda\equiv(\partial/\partial\omega_\lambda)D_{\mu\beta}, \tag{4.27}$$

we find, on using (4.24), that

$$\delta D=(\phi_\mu{}^\dagger D_{\mu\beta}{}^\lambda\phi_\beta)\delta\omega_\lambda=0. \tag{4.28}$$

Hence, it follows that the direction of the 4-vector $(\phi_\mu{}^\dagger D_{\mu\beta}{}^\lambda\phi_\beta)$ agrees with that of the wave packet. On the other hand, from (4.25) and (4.27),

$$\omega_\sigma\phi_\mu{}^\dagger D_{\mu\beta}{}^\lambda\phi_\beta=\omega_\sigma(\phi_\mu{}^\dagger m_{\lambda\mu\alpha\beta}\omega_\alpha\phi_\beta+\phi_\mu{}^\dagger\omega_\nu m_{\nu\mu\lambda\beta}\phi_\beta+\phi_\mu{}^\dagger\omega_\nu m_{\nu\mu\alpha\beta}{}^\lambda\omega_\alpha\phi_\beta), \tag{4.29}$$

where, on using (4.22),

$$\omega_\sigma\phi_\mu{}^\dagger m_{\lambda\mu\alpha\beta}\omega_\alpha\phi_\beta=\tfrac{1}{2}\omega_\sigma\phi_\mu{}^\dagger H_{\lambda\mu}=\tfrac{1}{2}F_{\sigma\mu}{}^\dagger H_{\lambda\mu}$$

in view of (4.23). In this way, we find that

$$\omega_\sigma\phi_\mu{}^\dagger D_{\mu\beta}{}^\lambda\phi_\beta=\tfrac{1}{2}(H_{\lambda\beta}{}^\dagger F_{\sigma\beta}+H_{\lambda\mu}F_{\sigma\mu}{}^\dagger)+\tfrac{1}{8}\omega_\sigma(\Pi_{\alpha\beta}{}^{\lambda\dagger}F_{\alpha\beta}+\Pi_{\nu\mu}{}^\lambda F_{\nu\mu}{}^\dagger), \tag{4.30}$$

and also $\mathcal{L}=0$. Thus comparing (4.30) with the expression (4.19) of $\theta_{\lambda\sigma}$, we finally have the relation

$$\omega_\sigma\phi_\mu{}^\dagger D_{\mu\beta}{}^\lambda\phi_\beta=\omega_\sigma\partial D/\partial\omega_\lambda=\theta_{\lambda\beta}. \tag{4.31}$$

We conclude from (4.31) that the direction of the wave path agrees with that of the power flow vector $\theta_{\lambda 0}=-i\theta_{\lambda 4}$, and that the momentum density $\theta_{0\sigma}=-i\theta_{4\sigma}$ is always proportional to ω_σ.

5. REPRESENTATION OF ENERGY–MOMENTUM TENSOR AND THE "REST MASS"

The energy-momentum tensor $\theta_{\lambda\sigma}$ in (4.19) may be divided into three parts, say $\theta_{\lambda\sigma}{}^0$, $\theta_{\lambda\sigma}{}^M$, and $\theta_{\lambda\sigma}{}^I$, attributable to the pure electromagnetic field, the medium and the interaction with the external system, respectively; with reference to (4.8), we put

$$\theta_{\lambda\sigma}=\theta_{\lambda\sigma}{}^0+\theta_{\lambda\sigma}{}^M+\theta_{\lambda\sigma}{}^I, \tag{5.1}$$

where $$\theta_{\lambda\sigma}{}^0=\theta_{\sigma\lambda}{}^0=\tfrac{1}{2}(F_{\lambda\mu}{}^\dagger F_{\sigma\mu}+F_{\lambda\mu}F_{\sigma\mu}{}^\dagger)-\tfrac{1}{4}F_{\nu\mu}{}^\dagger F_{\nu\mu}\delta_{\lambda\sigma}, \tag{5.2}$$

$$\theta_{\lambda\sigma}{}^{M}(\neq\theta_{\sigma\lambda}{}^{M})=\tfrac{1}{2}(\Pi_{\lambda\mu}{}^{\dagger}F_{\sigma\mu}+\Pi_{\lambda\mu}F_{\sigma\mu}{}^{\dagger})-\tfrac{1}{8}(\Pi_{\nu\mu}{}^{\dagger}F_{\nu\mu}+\Pi_{\nu\mu}F_{\nu\mu}{}^{\dagger})\delta_{\lambda\sigma}$$

$$+\tfrac{1}{8}[\Pi_{\nu\mu}{}^{\lambda\dagger}(i\partial_{\sigma}+\omega_{\sigma})F_{\nu\mu}+\Pi_{\nu\mu}{}^{\lambda}(-i\partial_{\sigma}+\omega_{\sigma})F_{\nu\mu}{}^{\dagger}],\quad(5.3)$$

$$\theta_{\lambda\sigma}{}^{I}=\tfrac{1}{2}(S_{\lambda\mu}{}^{\dagger}F_{\sigma\mu}+S_{\lambda\mu}F_{\sigma\mu}{}^{\dagger})-\tfrac{1}{4}(S_{\nu\mu}{}^{\dagger}F_{\nu\mu}+S_{\nu\mu}F_{\nu\mu}{}^{\dagger})\delta_{\lambda\sigma}\,,\quad(5.4)$$

where $\Pi_{\lambda\mu}$ and $\Pi_{\lambda\mu}{}^{\dagger}$ are to be abbreviations for $\kappa_{\lambda\mu\alpha\beta}(x,\omega+i\partial)F_{\alpha\beta}$ and its conjugate tensor.

The contraction of each tensor yields

$$\theta_{\lambda\lambda}{}^{0}=0,\quad\theta_{\lambda\lambda}{}^{I}=-\tfrac{1}{2}\{S_{\nu\mu}{}^{\dagger}F_{\nu\mu}+S_{\nu\mu}F_{\nu\mu}{}^{\dagger}\},\quad\theta_{\lambda\lambda}{}^{M}=\tfrac{1}{8}\{\Pi_{\nu\mu}{}^{\lambda\dagger}(i\partial_{\lambda}+\omega_{\lambda})F_{\nu\mu}+\Pi_{\nu\mu}{}^{\lambda}(-i\partial_{\lambda}+\omega_{\lambda})F_{\nu\mu}{}^{\dagger}\}\,.\quad(5.5)$$

In order to see the contribution to $\theta_{\lambda\sigma}{}^{M}$ from each component of the moving media, we introduce the two new 4-vectors

$$E_{\nu}{}^{(n)}=F_{\nu\mu}n_{\mu},\quad B_{\nu}{}^{(n)}=(2i)^{-1}\epsilon_{\nu\mu\alpha\beta}F_{\alpha\beta}\eta_{\mu}\quad,\quad(5.6)$$

with the relations[10] $\epsilon_{\nu\mu\alpha\beta}{}^{\dagger}=-\epsilon_{\nu\mu\alpha\beta}$ and[11]

$$F_{\nu\mu}=n_{\nu}E_{\mu}{}^{(n)}-n_{\mu}E_{\nu}{}^{(n)}+i\epsilon_{\nu\mu\alpha\beta}n_{\alpha}B_{\beta}{}^{(n)}\,,\quad(5.7)$$

and also $P_{\lambda}{}^{(n)}=\kappa_{\lambda\beta}E_{\beta}{}^{(n)},\quad M_{\lambda}{}^{(n)}=\nu_{\lambda\xi}B_{\xi}{}^{(n)}\,.\quad(5.8)$

Then, in the special case of a single component, $\theta_{\lambda\sigma}{}^{M}$ can be shown to be given by

$$\begin{aligned}\theta_{\lambda\sigma}{}^{M}=\tfrac{1}{2}\{&(\tfrac{1}{2}\delta_{\lambda\sigma}+n_{\lambda}n_{\sigma})[E_{\mu}{}^{(n)\dagger}P_{\mu}{}^{(n)}+E_{\mu}{}^{(n)}P_{\mu}{}^{(n)\dagger}-B_{\mu}{}^{(n)\dagger}M_{\mu}{}^{(n)}-B_{\mu}{}^{(n)}M_{\mu}{}^{(n)\dagger}]\\&+[-E_{\sigma}{}^{(n)\dagger}P_{\lambda}{}^{(n)}-E_{\sigma}{}^{(n)}P_{\lambda}{}^{(n)\dagger}+B_{\lambda}{}^{(n)\dagger}M_{\sigma}{}^{(n)}+B_{\lambda}{}^{(n)}M_{\sigma}{}^{(n)\dagger}]\\&+(M^{(n)}\times E^{(n)\dagger}+M^{(n)\dagger}\times E^{(n)})_{\lambda}n_{\sigma}-n_{\lambda}(B^{(n)\dagger}\times P^{(n)}+B^{(n)}\times P^{(n)\dagger})_{\sigma}\}\\&-\tfrac{1}{4}[E_{\nu}{}^{(n)\dagger}\kappa_{\nu\mu}{}^{\lambda}(i\partial_{\sigma}+\omega_{\sigma})E_{\mu}{}^{(n)}+B_{\nu}{}^{(n)\dagger}\nu_{\nu\mu}{}^{\lambda}(i\partial_{\sigma}+\omega_{\sigma})B_{\mu}{}^{(n)}\\&+E_{\mu}{}^{(n)\dagger}(-i\overleftarrow{\partial}_{\sigma}+\omega_{\sigma})\kappa_{\mu\nu}{}^{\lambda}E_{\nu}{}^{(n)}+B_{\mu}{}^{(n)\dagger}(-i\overleftarrow{\partial}_{\sigma}+\omega_{\sigma})\nu_{\mu\nu}{}^{\lambda}B_{\nu}{}^{(n)}]\,,\end{aligned}\quad(5.9)$$

in terms of the notation

$$(A\times B)_{\sigma}\equiv i\epsilon_{\sigma\alpha\beta\gamma}n_{\gamma}A_{\beta}B_{\alpha},\quad\kappa_{\nu\mu}{}^{\lambda}\equiv\partial\kappa_{\nu\mu}/\partial\omega_{\lambda},\quad\text{etc.}\quad(5.10)$$

$\kappa_{\nu\mu}{}^{\lambda}$ and $\nu_{\nu\mu}{}^{\lambda}$ involved in the last term in (5.9) can be expressed in terms of the more common parameters $\epsilon_{\nu\mu}$ and $\mu_{\nu\mu}$ by (2.23); in the special case in which the medium is time dispersive in its rest frame of reference, these quantities are functions only of the frequency $\omega=-n_{\lambda}\omega_{\lambda}$ and thus

$$\kappa_{\nu\mu}{}^{\lambda}=-n_{\lambda}[\partial\epsilon/\partial\omega]_{\nu\mu},\quad\nu_{\nu\mu}{}^{\lambda}=-n_{\lambda}[\mu^{-1}(\partial\mu/\partial\omega)\mu^{-1}]_{\nu\mu}\quad.\quad(5.11)$$

In the same way,

$$\begin{aligned}\theta_{\lambda\sigma}{}^{0}=&-\tfrac{1}{2}(E_{\lambda}{}^{\dagger}E_{\sigma}+E_{\lambda}E_{\sigma}{}^{\dagger}+B_{\lambda}{}^{\dagger}B_{\sigma}+B_{\lambda}B_{\sigma}{}^{\dagger})\\&+(\tfrac{1}{2}\delta_{\lambda\sigma}+n_{\lambda}n_{\sigma})(E_{\mu}{}^{\dagger}E_{\mu}+B_{\mu}{}^{\dagger}B_{\mu})+\tfrac{1}{2}(E^{\dagger}\times B+E\times B^{\dagger})_{\lambda}n_{\sigma}+\tfrac{1}{2}n_{\lambda}(E^{\dagger}\times B+E\times B^{\dagger})_{\sigma}\,.\end{aligned}\quad(5.12)$$

Here, the superscript (n) is dropped, since the expression (5.12) is true for *any* timelike vector n_{μ} and so it is particularly convenient to choose it in the direction of the time axis of the frame of reference. In that case, E_ν and B_ν become the electric field and the magnetic induction, and $(E^\dagger \times B)_\lambda$ becomes the component of the vector product of $E_\nu{}^\dagger$ and B_ν as ordinarily defined.

Further, in the case of the single component medium as we have assumed in the above, $\theta_{\lambda\sigma}{}^0$ and $\theta_{\lambda\sigma}{}^M$ can be combined into a unified form by the introduction of

$$D_\mu{}^{(n)} = E_\mu{}^{(n)} + P_\mu{}^{(n)} = \epsilon_{\mu\nu} E_\nu{}^{(n)}, \quad H_\mu{}^{(n)} = B_\mu{}^{(n)} - M_\mu{}^{(n)} = (\mu^{-1})_{\mu\nu} B_\nu{}^{(n)}; \tag{5.13}$$

thus on using (5.11), it is found to be

$$\begin{aligned}\theta_{\lambda\sigma}{}^0 + \theta_{\lambda\sigma}{}^M = &-\tfrac{1}{2}(D_\lambda{}^{(n)\dagger}E_\sigma{}^{(n)} + D_\lambda{}^{(n)}E_\sigma{}^{(n)\dagger} + B_\lambda{}^{(n)\dagger}H_\sigma{}^{(n)} + B_\lambda{}^{(n)}H_\sigma{}^{(n)\dagger}) \\ &+\tfrac{1}{2}(\tfrac{1}{2}\delta_{\lambda\sigma} + n_\lambda n_\sigma)(D_\mu{}^{(n)\dagger}E_\mu{}^{(n)} + D_\mu{}^{(n)}E_\mu{}^{(n)\dagger} + B_\mu{}^{(n)\dagger}H_\mu{}^{(n)} + B_\mu{}^{(n)}H_\mu{}^{(n)\dagger}) \\ &+\tfrac{1}{2}(E^{(n)\dagger}\times H^{(n)} + E^{(n)}\times H^{(n)\dagger})_\lambda n_\sigma + \tfrac{1}{2}n_\lambda(D^{(n)\dagger}\times B^{(n)} + D^{(n)}\times B^{(n)\dagger})_\sigma \\ &+\tfrac{1}{4}n_\lambda[E_\nu{}^{(n)\dagger}(\partial\kappa/\partial\omega)_{\nu\mu}(i\partial_\sigma + \omega_\sigma)E_\mu{}^{(n)} + H_\nu{}^{(n)\dagger}(\partial\mu/\partial\omega)_{\nu\mu}(i\partial_\sigma + \omega_\sigma)H_\mu{}^{(n)} \\ &+E_\mu{}^{(n)\dagger}(-i\overleftarrow{\partial}_\sigma + \omega_\sigma)(\partial\kappa/\partial\omega)_{\mu\nu}E_\nu{}^{(n)} + H_\mu{}^{(n)\dagger}(-i\overleftarrow{\partial}_\sigma + \omega_\sigma)(\partial\mu/\partial\omega)_{\mu\nu}H_\nu{}^{(n)}],\end{aligned} \tag{5.14}$$

and its energy density in the rest frame of reference of the medium agrees with that ordinarily accepted.[12]

In the general case where the medium is composed of more than two components moving with different velocities, $\theta_{\lambda\sigma}{}^M$ becomes simply the sum of all their contributions, as is evident from (3.19), and thus

$$\theta_{\lambda\sigma}{}^M = \sum_n \theta_{\lambda\sigma}{}^{(n)}, \tag{5.15}$$

where $\theta_{\lambda\sigma}{}^{(n)}$ represents the contribution of each component moving in the direction of n_μ and hence it is given by (5.9) with its *own* $B_\mu{}^{(n)}$ and $E_\mu{}^{(n)}$.

On the other hand, when the velocities of all the components involved are very close to each other so that $|\vec{v}/c| \ll 1$, where $\vec{v}$ is the velocity difference of any pair of the components, then all the $B_\mu{}^{(n)}$ and the $E_\mu{}^{(n)}$ become the same and, effectively, $\kappa_{\nu\mu}$ and $\nu_{\nu\mu}$ in (5.9) can be replaced by

$$\kappa_{\nu\mu} = \sum_n \kappa_{\nu\mu}{}^{(n)}, \quad \nu_{\nu\mu} = \sum_n \nu_{\nu\mu}{}^{(n)}. \tag{5.16}$$

Here, in the case of the anisotropic plasma, for example,

$$\kappa_{\nu\mu}{}^{(n)} = -\frac{\omega_p{}^2}{\omega(\omega - i\nu)}\left((\delta_{\nu\mu} + n_\nu n_\mu) - \frac{(\omega_H)_{\nu\mu}{}^2 + (\omega - i\nu)i(\omega_H)_{\nu\mu}}{(\omega - i\nu)^2 - \omega_H{}^2}\right),$$

$$\text{where}\quad \omega = -n_\lambda\omega_\lambda, \quad (\omega_H)_{\nu\mu}{}^2 = (\omega_H)_{\nu\lambda}(\omega_H)_{\lambda\mu}, \quad \omega_H{}^2 = -\tfrac{1}{2}(\omega_H)_{\lambda\lambda}{}^2, \tag{5.17}$$

and ω_p is the plasma frequency in relativistic units (the ordinary frequency divided by c) and ν is the proper collision frequency; $(\omega_H)_{\nu\mu} = -(\omega_H)_{\mu\nu}$ is the cyclotron frequency tensor and it is proportional to the static (magnetic) field. The inclusion of the collision frequency ν makes $\kappa_{\nu\mu}{}^{(n)}$ non-Hermitian and the medium becomes dissipative (Sec. 6).

The contraction of $\theta_{\lambda\sigma}$ yields a nonvanishing value in the case of a dispersive medium, and the straightforward calculation of $\theta_{\lambda\lambda}{}^M$ by (5.5) with the use of (5.11) shows that

$$\theta_{\lambda\lambda}{}^M = \tfrac{1}{4}\{E_\nu{}^{(n)\dagger}(\partial\epsilon/\partial\omega)_{\nu\mu}n_\lambda(i\partial_\lambda + \omega_\lambda)E_\mu{}^{(n)} + B_\nu{}^{(n)\dagger}[\mu^{-1}(\partial\mu/\partial\omega)\mu^{-1}]_{\nu\mu}n_\lambda(i\partial_\lambda + \omega_\lambda)B_\mu{}^{(n)}\} + \text{c.c.} \tag{5.18}$$

and, in the special case of the plane wave of the wave-number vector ω_ν,

$$\theta_{\lambda\lambda}{}^M = -\tfrac{1}{2}\{E_\nu{}^{(n)\dagger}(\omega\partial\epsilon/\partial\omega)_{\nu\mu}E_\mu{}^{(n)} + B_\nu{}^{(n)\dagger}[\omega\mu^{-1}(\partial\mu/\partial\omega)\mu^{-1}]_{\nu\mu}B_\mu{}^{(n)}\}, \quad \omega = -n_\lambda\omega_\lambda. \tag{5.19}$$

Here, since $E_\nu{}^{(n)}(\omega)^\dagger = E_\nu{}^{(n)}(-\omega)$ by (3.22) and $\epsilon_{\nu\mu}(-\omega) = \epsilon_{\mu\nu}(\omega)$ by (2.23) and (2.16) with the relation (2.17), it follows that

$$E_{\nu}^{(n)\dagger}(\omega)[\omega\partial\epsilon(\omega)/\partial\omega]_{\nu\mu}E_{\mu}^{(n)}(\omega)=E_{\mu}^{(n)\dagger}(-\omega)[\omega\partial\epsilon(-\omega)/\partial\omega]_{\mu\nu}E_{\nu}^{(n)}(-\omega)\,, \tag{5.20}$$

and hence this quantity is positive definite independent of the sign of ω,[13] if $(\partial\epsilon/\partial\omega)_{\nu\mu}$ has only positive eigenvalues for $\omega > 0$. We can do the same consideration also for the last term in (5.19). Thus we conclude that $\theta_{\lambda\lambda}{}^{M}$ is negative definite independent of the sign of $\omega = -n_\lambda\omega_\lambda$, if $(\partial\epsilon/\partial\omega)_{\nu\mu}$ and $(\partial\mu/\partial\omega)_{\nu\mu}$ have *only* positive eigenvalues for $\omega > 0$. Thus, in this case, it follows that the field has a "rest mass" of positive sign, in contrast to the case of a nondispersive medium.

The positiveness (or negativeness) of $(\partial\epsilon/\partial\omega)$ and $(\partial\mu/\partial\omega)$ for $\omega > 0$ (or $\omega < 0$) has been proven in the case of an isotropic medium based on the theory of causality.[14] However, it is not immediately clear in the general case of an anisotropic medium and, indeed, it is not the case, e.g., for the anisotropic plasma characterized by (5.17) in the range of $|\omega| < |\omega_H|/2$. In spite of this fact, $-\theta_{\lambda\lambda}{}^{M}$ will be expected to be proven to be positive definite, probably based on the causality, although, when ω_λ is spacelike, the energy density is not positive definite but changes its sign depending upon the frame of reference in view of (4.31). ($\partial D/\partial\omega_\lambda$ is a null or timelike vector in order to give the group velocity which does not exceed the velocity of light in vacuum, and thus $\partial D/\partial\omega_0$ keeps a definite sign independent of the frame of reference.) Note that the above conclusion concerning the rest energy is valid for the general media of multicomponents moving with different velocities, although it was illustrated by (5.19) for the medium of a single component.[15]

6. ENERGY–MOMENTUM TENSOR IN DISSIPATIVE MEDIUM AND "NEGATIVE" CONDUCTIVITY

When there is dissipation in the medium, the energy and momentum of the field are not conserved even in a homogeneous medium without sources, and it follows that the energy-momentum tensor can be defined in various ways. In this paper, we shall define it by the same tensor as obtained for the nondissipative medium in the preceding sections with the removal of the restriction (4.5). This means the inclusion of the anti-Hermitian part of κ_{ij} in (2.16), say $-i\kappa_{ij}{}^{(D)}$, with

$$\kappa_{ij}{}^{(D)} = P\omega^{-1}\sigma_{ij}(\omega) - \pi\,\zeta_{ij}(0)\delta(\omega)\,, \tag{6.1}$$

and the corresponding additional term $-i\kappa_{\nu\mu\alpha\beta}{}^{(D)}$ in (3.15a), where $\kappa_{\nu\mu\alpha\beta}{}^{(D)}$ has the same expression as $\kappa_{\nu\mu\alpha\beta}$ with $\kappa_{\nu\mu}$ replaced by $\kappa_{\nu\mu}{}^{(D)}$.

The straightforward calculation shows that $\theta_{\lambda\sigma}$ in (5.1) has the additional term $\theta_{\lambda\sigma}{}^{(D)}$ given by

$$\begin{aligned}\theta_{\lambda\sigma}{}^{(D)} &= (i/2)(C^{\dagger}{}_{\lambda\mu}F_{\sigma\mu} - C_{\lambda\mu}F^{\dagger}{}_{\sigma\mu}) - (i/4)(C^{\dagger}{}_{\nu\mu}F_{\nu\mu} - C_{\nu\mu}F^{\dagger}{}_{\nu\mu})\delta_{\lambda\sigma}\\ &\quad + \tfrac{1}{8}\partial_\rho[F^{\dagger}{}_{\nu\mu}(\kappa_{\nu\mu\alpha\beta}{}^{(D)\rho}\delta_{\lambda\sigma} - \kappa_{\nu\mu\alpha\beta}{}^{(D)\lambda}\delta_{\rho\sigma})F_{\alpha\beta}] + \tfrac{1}{8}F^{\dagger}{}_{\nu\mu}(\partial_\sigma\kappa_{\nu\mu\alpha\beta}{}^{(D)\lambda})F_{\alpha\beta}\,,\end{aligned} \tag{6.2}$$

$$\text{with}\quad C_{\lambda\mu} = \kappa_{\lambda\mu\alpha\beta}{}^{(D)}F_{\alpha\beta}\,,\quad \kappa_{\nu\mu\alpha\beta}{}^{(D)\lambda} \equiv (\partial/\partial\omega_\lambda)\kappa_{\nu\mu\alpha\beta}{}^{(D)}\,. \tag{6.3}$$

The first-two terms on the right-hand side of (6.2) are the same as given by $\theta_{\lambda\sigma}{}^{I}$ of (5.4) with $S_{\lambda\mu}$ replaced by $-iC_{\lambda\mu}$, while the other terms in (6.2) are the same as the contribution when the Lagrangian density has the additional divergence term $(\frac{1}{8})\partial_\lambda[F_{\nu\mu}{}^{\dagger}\kappa_{\nu\mu\alpha\beta}{}^{(D)\lambda}F_{\alpha\beta}]$. Hence, with reference to (3.32), the former can be regarded as the contribution of the conduction current $j_\mu{}^{c} = -(i\partial_\nu + \omega_\nu)C_{\nu\mu}$ or

$$j_\mu{}^{c} = (i\partial_\nu + \omega_\nu)[n_\mu\kappa_{\nu\beta}{}^{(D)} - n_\nu\kappa_{\mu\beta}{}^{(D)}]E_\beta{}^{(n)}\,, \tag{6.4}$$

while the latter is the result of the canonical transformation of the dynamical system.[16]

Now, with reference to (4.20), the divergence of $\theta_{\lambda\sigma}$ becomes, in the case of a homogeneous medium of a single component with no external source,

$$\begin{aligned}\partial_\lambda\theta_{\lambda\sigma} &= \tfrac{1}{4}[F^{\dagger}{}_{\nu\mu}(i\partial_\sigma + \omega_\sigma)C_{\nu\mu} + F_{\nu\mu}(-i\partial_\sigma + \omega_\sigma)C^{\dagger}{}_{\nu\mu}]\\ &= -\tfrac{1}{2}[E_\nu{}^{(n)\dagger}(i\partial_\sigma + \omega_\sigma)\kappa_{\nu\mu}{}^{(D)}E_\mu{}^{(n)} + E_\mu{}^{(n)\dagger}\kappa_{\mu\nu}{}^{(D)}(-i\overleftarrow{\partial}_\sigma + \omega_\sigma)E_\nu{}^{(n)}]\,,\end{aligned} \tag{6.5}$$

where $\kappa_{\nu\mu}{}^{(D)}$ is given by (6.1) with $\omega = -n_\lambda\omega_\lambda$ and the Latin subscripts replaced by the Greek subscripts. Hence, in the special case of the plane wave of the (complex) 4-vector $\omega_\nu - i\omega'_\nu$, (6.5) gives

$$\partial_\lambda\theta_{\lambda 0} = -E_\nu{}^{(n)\dagger}\omega_0[P\omega^{-1}\sigma_{\nu\mu}(\omega) - \pi\zeta_{\nu\mu}(0)\delta(\omega)]E_\mu{}^{(n)}\,,\quad \omega = -n_\lambda\omega_\lambda\,, \tag{6.6}$$

where $E_\mu^{(n)}$ and $E_\nu^{(n)\dagger}$ are functions of the coordinates through the factor $e^{-\omega_\nu' x_\nu}$.

Here, as in (5.20),

$$E_\nu^{(n)\dagger}(\omega)\sigma_{\nu\mu}(\omega)E_\mu^{(n)}(\omega) = E_\mu^{(n)\dagger}(-\omega)\sigma_{\mu\nu}(-\omega)E_\nu^{(n)}(-\omega)\,, \tag{6.7}$$

should be positive definite independent of the sign of ω, if the medium is dissipative. On the other hand, when $\omega_\lambda^2 > 0$ or when the phase velocity is smaller than the velocity of light in vacuum, $\omega = -n_\lambda\omega_\lambda$ can assume both signs and, if $\omega_0 > 0$, it is positive or negative according as to whether the medium velocity given by n_λ does not or does exceed the phase velocity. Accordingly, the right-hand side of (6.6) becomes positive when $\omega_0(n_\lambda\omega_\lambda)^{-1} > 0$, i.e., the conductivity of the moving medium effectively becomes negative when the velocity of the medium exceeds the phase velocity of the wave, and, if $\sigma_{\nu\mu}(0) \neq 0$, it tends to infinity as $n_\lambda\omega_\lambda \to 0$ or the two velocities approach each other. Therefore, when this happens in a part of the medium components *and* the total energy of the field is positive in the frame of reference considered, the wave then has a possibility of becoming unstable in the sense that the energy of the field increases with time (see Sec. 7).

7. INSTABILITY

The wave is usually defined to be unstable, when its amplitude grows in the timelike direction. We shall find that the problem of this instability is closely connected with that of whether or not the group velocity of the wave can exceed the velocity of light in vacuum.

A. Case of Nondissipative Medium

Suppose that the medium is nondissipative and homogeneous in space and time with no external source, and that the wave-number vector ω_μ of a plane wave has the form of

$$\omega_\mu = \omega_\mu^0 + in_\mu\nu\,, \tag{7.1}$$

where ω_μ^0 and ν are real, and n_μ is a timelike unit vector with $n_\mu^2 = -1$. Then, the use of (7.1) in the conservation equation (4.20) shows that

$$\partial_\lambda\theta_{\lambda\sigma} = 2\nu n_\lambda\theta_{\lambda\sigma} = 0\,,$$

and hence also that $\theta_{\lambda\sigma}$ for any given σ is a spacelike vector since n_λ is assumed to be timelike; it means that, in the "rest frame" defined so that n_λ has only a time component, the energy and momentum densities vanish with the possible existence of nonvanishing stress and power flow densities. We note that this is the necessary physical condition for an unstable wave to grow in a timelike direction.

On the other hand, consider the case when ν in (7.1) is sufficiently small, say $\nu = \Delta\nu$, so that the unstable wave is produced from a stable wave of the wave number ω_μ^0, by a change of the wave number $\Delta\omega_\mu^0$. Hence, putting

$$\omega_\mu = \omega_\mu^0 + \Delta\omega_\mu^0 + in_\mu\Delta\nu\,, \tag{7.2}$$

then, if $D(\omega) = 0$ is the dispersion equation with $D(\omega)$ given by (4.27), the expansion of $D(\omega)$ in the vicinity of $\omega_\mu = \omega_\mu^0$ yields

$$\begin{aligned} D(\omega) &= D^\lambda\Delta\omega_\lambda^0 + iD^\lambda n_\lambda\Delta\nu \\ &\quad - \tfrac{1}{2}D^{\lambda\mu}n_\lambda n_\mu\Delta\nu^2 + O[\Delta\nu^3] = 0\,, \\ D^\lambda &\equiv \partial D/\partial\omega_\lambda\Big|_{\omega=\omega^0}\,, \\ D^{\lambda\mu} &\equiv \partial^2 D/\partial\omega_\lambda\partial\omega_\mu\Big|_{\omega=\omega^0}\,, \end{aligned} \tag{7.3}$$

where only the first-order term with respect to $\Delta\omega_\lambda^0$ is kept. Equating the real and imaginary parts of (7.3) to zero, we find that

$$D^\lambda n_\lambda = 0, \quad D^\lambda\Delta\omega_\lambda^0 = \tfrac{1}{2}D^{\lambda\mu}n_\lambda n_\mu\Delta\nu^2\,, \tag{7.4}$$

and hence, the vector D^λ is spacelike or $(D^\lambda)^2 > 0$; since $\theta_{\lambda\sigma} = D^\lambda\omega_\sigma$ according to (4.31), it follows that the energy-momentum tensor of the stable wave at $\omega_\mu = \omega_\mu^0$ has the same spacelike property as that of the unstable wave. Also, $\Delta\nu$ is given by

$$\Delta\nu = \pm[(2/D^{\lambda\mu}n_\lambda n_\mu)D^\alpha\Delta\omega_\alpha^0]^{1/2}, \quad (D^\lambda)^2 > 0\,, \tag{7.5}$$

which is indeed real when $\Delta\omega_\alpha^0$ is such that the term inside square brackets is positive. The plane wave at $\omega_\mu = \omega_\mu^0$ is, of course, stable, but it is critical in the sense that a small change of the wave number can turn it into an unstable wave.

On the other hand, since the phase of the critical wave is real, there exists a wave packet and its direction of motion agrees with that of D^λ (see Sec. 4 B). But since D^λ is spacelike, it follows that the group velocity of the critical wave must exceed the velocity of light in vacuum [and even tends to infinity as the frame of reference approaches the rest frame of $n_\lambda = (0, 0, 0, i)$] and evidently loses its physical meaning; the wave packet is a physical quantity filled with the field.[17] Hence, it is concluded that the 4-vector

D^λ of the critical wave cannot be spacelike; but it also cannot be timelike, because the wave then cannot go over into an unstable wave of the form of (7.1).

From the above consideration, we are forced to conclude that the wave of the form of (7.1) cannot exist; thus we simply postulate that the group velocity of the (stable) wave cannot exceed the velocity of light in vacuum, i.e., $(D^\lambda)^2 \leq 0$. It follows then that, if the velocity reaches the velocity of light at a finite value of the wave number, $(D^\lambda)^2 = 0$ is the maximum value.

On the other hand, the maximum value of $(D^\lambda)^2$ on the restriction $D = 0$ is obtained by solving

$$D^\lambda D^{\lambda\mu} - kD^\mu = 0\,, \tag{7.6}$$

where k is a constant and $D^{\lambda\mu}$ is the same as defined in (7.3). Now suppose that a plane wave satisfies $(D^\lambda)^2 = 0$ and hence also (7.6) at $\omega_\nu = \omega_\nu^0$, and that the dispersion equation is expanded in power series of $\Delta\omega_\nu = \omega_\nu - \omega_\nu^0$:

$$D = D^\lambda \Delta\omega_\lambda + \tfrac{1}{2} D^{\lambda\mu} \Delta\omega_\lambda \Delta\omega_\mu + \tfrac{1}{6} D^{\lambda\mu\nu} \Delta\omega_\lambda \Delta\omega_\mu \Delta\omega_\nu + \cdots = 0\,. \tag{7.7}$$

Then, putting $\Delta\omega_\lambda$ in the form

$$\Delta\omega_\lambda = \Delta\omega_\lambda^{\ 0} + D^\lambda \Delta\nu\,, \quad (D^\lambda)^2 = 0 \tag{7.8}$$

for any given $\Delta\omega_\lambda^{\ 0}$, (7.7) becomes, on using the relation (7.6),

$$D = D^\lambda \Delta\omega_\lambda^{\ 0} + \tfrac{1}{6} D^{\lambda\mu\nu} D^\lambda D^\mu D^\nu \Delta\nu^3 + O[4\Delta\nu^4] = 0 \tag{7.9}$$

to the first order of $\Delta\omega_\lambda^{\ 0}$. In the same way, since $\Delta\omega_\lambda^{\ 0} \sim \Delta\nu^3$,

$$\partial D/\partial\omega_\lambda = D^\lambda + \tfrac{1}{2} D^{\lambda\mu\nu} D^\mu D^\nu \Delta\nu^2 + O[\Delta\nu^3]\,,$$

$$(\partial D/\partial\omega_\lambda)^2 = D^{\lambda\mu\nu} D^\lambda D^\mu D^\nu \Delta\nu^2 + O[\Delta\nu^3]\,. \tag{7.10}$$

Thus since $(\partial D/\partial\omega_\lambda)^2 < 0$ for any real $\Delta\nu$,

$$D^{\lambda\mu\nu} D^\lambda D^\mu D^\nu < 0\,. \tag{7.11}$$

Solving (7.9) with respect to $\Delta\nu$ for a given *real* $\Delta\omega_\lambda^{\ 0}$,

$$\Delta\nu = [-6(D^{\lambda\mu\nu} D^\lambda D^\mu D^\nu)^{-1} D^\alpha \Delta\omega_\alpha^{\ 0}]^{\frac{1}{3}}\,, \tag{7.12}$$

and hence the phase of $\Delta\nu$ is found to be 0, $\pm 2\pi/3$ or π, $\pm\pi/3$ according as $D^\alpha \Delta\omega_\alpha^{\ 0}$ is positive or negative. In either case, $\Delta\nu$ can have a complex value whose imaginary part is negative, and the wave is then unstable since it grows with time in any frame of reference in view of (7.8) $(D^0 > 0)$; also the wave of a constant amplitude propagates with the velocity of light. Hence, it follows that, in a dispersive medium, the wave becomes critical when its group velocity reaches the velocity of light at a finite value of the wave number.

The energy-momentum tensor in the unstable state of the wave can be deduced from the conservation equation; on referrring to (7.8),

$$\partial_\lambda \theta_{\lambda\sigma} = i(\Delta\nu^* - \Delta\nu) D^\lambda \theta_{\lambda\sigma} = 0\,, \tag{7.13a}$$

and hence, since $\Delta\nu^* - \Delta\nu \neq 0$ in the unstable state, it follows that

$$D^\lambda \theta_{\lambda\sigma} = 0, \quad \text{or} \quad \theta_{\lambda\sigma} \propto D^\lambda \quad \text{for given } \sigma, \tag{7.13b}$$

in view of the condition[18] $(D^\lambda)^2 = 0$. Thus we find that, in the unstable state, the energy propagates with the velocity of light in vacuum.

It may be interesting to check the two beam instability in a plasma as a simple example: On referring to (5.7),

$$\Pi_{\nu\mu} = \kappa_{\nu\mu\alpha\beta} F_{\alpha\beta} = \textstyle\sum_n [-n_\mu \kappa_{\nu\beta}^{(n)} + n_\nu \kappa_{\mu\beta}^{(n)}] E_\beta^{(n)}\,. \tag{7.14}$$

Hence, the multiplication of a 4-vector m_μ on (4.23) and the use of $H_{\nu\mu} = F_{\nu\mu} + \Pi_{\nu\mu}$ yield

$$\omega_\nu E_\nu^{(m)} + \textstyle\sum_n [-(m_\mu n_\mu)\omega_\nu \kappa_{\nu\beta}^{(n)} + (n_\nu \omega_\nu) m_\mu \kappa_{\mu\beta}^{(n)}] E_\beta^{(n)} = 0\,. \tag{7.15}$$

Here, we assume that the velocities of the beams involved in the plasma are sufficiently close to each other so that their velocity differences can be neglected in comparison with the velocity of light c. Then, it follows that, if we equate m_μ to one of the n_μ's,

$$m_\mu n_\mu \simeq -1, \quad E_\nu^{(m)} \simeq E_\nu^{(n)}, \quad m_\mu \kappa_{\mu\beta}^{(n)} \simeq 0\,, \tag{7.16}$$

for all the n_μ's. Hence, to this approximation, (7.15) becomes

$$\omega_\nu E_\nu^{(n)} + \textstyle\sum_n \omega_\nu \kappa_{\nu\beta}^{(n)} E_\beta^{(n)} = 0\,, \tag{7.17a}$$

or, in the case of the isotropic medium where $\kappa_{\nu\beta}^{(n)} = (\delta_{\nu\beta} + n_\nu n_\beta)\kappa^{(n)}$,

$$[1 + \textstyle\sum_n \kappa^{(n)}]\,\omega_\nu E_\nu^{(n)} = 0, \quad n_\nu E_\nu^{(n)} = 0\,. \tag{7.17b}$$

Now we suppose that the plasma consists of two beams of the same plasma frequencies but with the different velocities characterized by the two timelike unit vectors n'_λ and n''_λ, and no external field exists. Then, using (5.17), (7.17b) gives the dispersion equation

$$D(\omega) \equiv 1 - \omega_p{}^2[(n'_\lambda\omega_\lambda)^{-2} + (n''_\lambda\omega_\lambda)^{-2}] = 0\,, \quad (7.18)$$

which gives, by the straightforward calculation,

$$D^\lambda \equiv \partial D/\partial\omega_\lambda$$

$$= 2\omega_p{}^2[n'_\lambda(n'_\nu\omega_\nu)^{-3} + n''_\lambda(n''_\nu\omega_\nu)^{-3}]\,,$$

$$(D^\lambda)^2 = 4\omega_p{}^{-2}[-1 + 3x^2 + 2(n'_\lambda n''_\lambda)x^3]\,, \quad (7.19)$$

$$\text{with} \quad x = \omega_p{}^2(n'_\nu\omega_\nu)^{-1}(n''_\mu\omega_\mu)^{-1}\,. \quad (7.20)$$

Here, since $n'_\lambda n''_\lambda = -1$ to the approximation used, the last equation in (7.19) is equivalent to

$$(D^\lambda)^2 = -4\omega_p{}^{-2}(1-x)^2(1+2x)\,. \quad (7.21)$$

Now, in terms of the two variables

$$x_1 = \omega_p(n'_\lambda\omega_\lambda)^{-1}, \quad x_2 = \omega_p(n''_\lambda\omega_\lambda)^{-1}\,, \quad (7.22a)$$

x can be expressed by

$$x = x_1x_2\,, \quad x_1{}^2 + x_2{}^2 = 1\,. \quad (7.22b)$$

Hence, $x = \pm\frac{1}{2}$ are found to be the extrema, and $(D^\lambda)^2 < 0$ for $x = \frac{1}{2}$ and $(D^\lambda)^2 = 0$ for $x = -\frac{1}{2}$ according to (7.21). Thus we find that $(D^\lambda)^2 = 0$ is the maximum value, as expected, and the wave becomes critical at $x = -\frac{1}{2}$ where

$$\omega_p(n'_\lambda\omega_\lambda)^{-1} = -\omega_p(n''_\lambda\omega_\lambda)^{-1} = \pm 2^{-1/2}\,.$$

B. Case of "Dissipative" Medium

As it was noted at the end of Sec. 6, the conductivity of the moving medium effectively becomes negative when the velocity of the medium exceeds the phase velocity of the wave. Therefore, when this happens in a part of the medium components so that, according to (6.6), the right-hand side of

$$\partial_\lambda\theta_{\lambda 0} = -\sum_n E_\nu{}^{(n)\dagger}\omega_0[P\omega^{-1}\sigma_{\nu\mu}{}^{(n)}(\omega) - \pi\zeta_{\nu\mu}{}^{(n)}(0)\delta(\omega)]E_\mu{}^{(n)}, \quad \omega = -n_\lambda\omega_\lambda\,, \quad (7.23)$$

becomes positive, then the total energy of the field (and also the amplitude) increases in the timelike direction if the vector $\theta_{\lambda 0}$ is timelike with a positive value of θ_{00}. Here, although the energy density θ_{00} is not positive definite in the case of $\omega_\lambda{}^2 > 0$ (Sec. 5), it is always positive in the frame of reference in which the wave does not propagate and $\theta_{\lambda\sigma}$ has only the component $\theta_{4\sigma}$ for given σ: On referring to (4.31), the contraction of the energy-momentum tensor in this frame of reference yields $-\theta_{\lambda\lambda} = \theta_{00} > 0$, provided that the rest energy of the field is positive. Thus when the right-hand side of (7.23) is positive in the rest frame of the wave itself, the wave is unstable and its amplitude grows in the timelike direction.

8. CONCLUDING REMARKS

The polarization of linear media is characterized by the tensor $\kappa_{\nu\mu\alpha\beta}$ given by (3.19), where each $\kappa_{\nu\mu\alpha\beta}{}^{(n)}$ is the contribution from the medium component moving with the velocity given by the vector n_μ, and it has the same expression as (3.18a) with (3.18b); $\kappa_{\nu\mu}$ in the latter equation is generally a function of the 4-vector ω_ν and the coordinates x_ν, and thus it can cover the general case where the medium is dispersive and inhomogeneous in space and time. In the case of the anisotropic plasma, the contribution of each component is given by (5.17), which depends on the velocity through $\omega = -n_\lambda\omega_\lambda$. The packet field is defined in covariant form, and its energy-momentum tensor is found to be given by (4.19) or (5.1) with (5.9) and (5.12) in the case of a single component, and with (5.15) instead of (5.9) in the case of multicomponents; it is obtained on the condition (3.29) with (3.30) and hence, when the medium is inhomogeneous, each component of the medium is, in its own rest frame of reference, only time dispersive and depends only on the spatial coordinates. However, these restrictions are not necessary when the media are homogeneous. The rest energy obtained by the contraction of the energy-momentum tensor generally does not vanish and its sign is found to be positive definite in the case of isotropic dispersive media, while the energy density is not positive definite when $\omega_\lambda{}^2 > 0$. The power flow density 4-vector is shown to be in the same direction as that of the wave propagation, and the energy-momentum density vector is proportional to the wave-number 4-vector. When the velocity of a component of the media exceeds the phase velocity of the wave, its conductivity has effectively a negative sign, and this phenomenon is one of the causes of the plasma instability (Sec. 7 B). The instability of the wave in dispersive media is finally discussed based on the covariance of the equations, the energy-momentum tensor and the connected relations. If the group velocity of the wave (in nondissipative media) is postulated to be equal to or less than the velocity of light in vacuum, the wave can be shown to become unstable when the velocity reaches the velocity of light over a finite value of the wave number.

The energy-momentum tensor obtained is asymmetrical, and this result is what may be expected from the nonconservation of the total angular momentum of the system when each component of the media is moving in a prescribed direction in the four-dimensional space: The angular momentum density $M_{\lambda\mu\nu}$ is defined by

$$M_{\lambda\mu\nu} = x_\mu \theta_{\lambda\nu} - x_\nu \theta_{\lambda\mu} ,$$

and hence the total angular momentum, say $J_{\mu\nu}[\sigma]$, over any spacelike surface σ is given by

$$J_{\mu\nu}[\sigma] = \int_\sigma d\sigma_\lambda M_{\lambda\mu\nu} .$$

Thus, in the special case of the nondissipative homogeneous media with no external source, it follows that

$$\partial_\lambda M_{\lambda\mu\nu} = \theta_{\mu\nu} - \theta_{\nu\mu} ,$$

and hence also that the energy-momentum tensor is necessarily asymmetrical when the angular momentum is not conserved. On the other hand, the angular momentum of the field should not be conserved, since the media prescribed by the n_μ's are not symmetrical with respect to all the directions in the four-dimensional space; the media are not in the same state when they are viewed from the different frame of references each of which is obtained by rotation of other.

The nonexistence of an unstable wave which grows in timelike direction as given by (7.1), is in a distinct contrast to previously published results where exists an unstable wave growing with time. This discrepancy arises from the fact that the former is fully a relativistic treatment, while the latter is not and correspondingly the onset of instability had to be marked by an infinite group velocity.[19] The relativistic correction is very small, however, in the usual plasmas except for the vicinity of the critical point (where the wave begins to become unstable); according to the conclusion in Sec. 7, the unstable wave should grow with time by a factor of the form of $\exp[+\nu(t \pm x/c)]$, where ν is a real positive number and x is the spatial coordinate (in one-dimensional space for simplicity). Thus after the time interval t, the observers in the range $|x| \ll ct$ will actually find the amplitude of the wave to be $\exp(\nu t)$ times as large as the initial value, and therefore will find the same result as predicted by the previous nonrelativistic theories.

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[1]C. Møller, The Theory of Relativity (Oxford University Press, New York, 1967), Secs. 72–76.

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[3]M. Abraham, Rend. Pal. (Italy) 28, (1909); Ann. Phys. (Paris) 44, 537 (1914). W. Pauli, Encykl. Math. Wiss. 2, 667 (1920); Jg. Tamm, J. Phys. (USSR) 1, 439 (1939).

[4]See L. D. Landau and E. M. Lifshitz, Ref. 2, Secs. 58–64.

[5]P. Penfield, Jr., and H. A. Haus, Electrodynamics of Moving Media (MIT Press, Cambridge, Mass., 1967); they used the Lagrangian principle merely to show that the force equations therefrom agree with those they obtained by the principle of virtual power.

[6]J. Schwinger, Phys. Rev. 82, 914 (1951); 91, 713 (1953).

[7]A tilde is used to distinguish a quantity from its Fourier transform [see Eq. (2.12)].

[8]M. L. Goldberger, Phys. Rev. 99, 979 (1955); R. Karplus and M. A. Ruderman, *ibid.* 98, 771 (1955).

[9]Equation (3.18a) is equivalent to that already given, e.g., in Ref. 1.

[10]$\epsilon_{\nu\mu\alpha\beta}$ is usually defined as a pseudotensor so that it does not change its sign for the time inversion. However, ever, in this paper, it is to be defined as an ordinary tensor and hence it is transformed as its subscripts indicate.

[11]See, for instance, C. Møller, Ref. 1.

[12]L. D. Landau and E. M. Lifshitz, Ref. 2, Sec. 61. Assuming an additional spatial dispersion, T. Musha also found the corresponding energy-momentum tensor for the medium of a single component in the rest frame of reference, which is equivalent to (5.14) with $\kappa_{\nu\mu}{}^{\lambda}$ given by (5.10) instead of (5.11), J. Phys. Soc. Japan 26, 541 (1969).

[13]$\omega = -n_\lambda \omega_\lambda$ is possible to become negative when $\omega_\lambda{}^2 > 0$, and then the medium velocity exceeds the phase velocity of the wave.

[14]L. D. Landau and E. M. Lifshitz, Ref. 2, Sec. 64.

[15]The discussions on the negative energy can be seen in P. A. Sturrock, J. Appl. Phys. 31, 2052 (1960); G. C. Van Hoven and T. Wesselberg, *ibid.* 34, 1834 (1963); A. Bers and S. Gruber, Appl. Phys. Letters 6, 27 (1965).

[16]J. Schwinger, Ref. 6.

[17]Also no one can expect that it is possible to transmit communication signals faster than the light in vacuum, even with an infinite velocity, through these media.

[18]This conclusion may be derived directly from the expression (4.31) for $\theta_{\lambda\sigma}$, to the zeroth order of $\Delta\nu$; however, note that the expression is generally true only for the real ω_σ.

[19]R. J. Briggs, Electron-Stream Interactions with Plasmas (MIT Press, Cambridge, Mass., 1964).