# Canonical Representation of a Field Theory of Currents

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Canonically equivalent families of spin-zero meson nonlinear Lagrangian field theories in which the underlying symmetry is $SU(n)\times SU(n)$ are rewritten in terms of currents. A completely symmetric theory is shown to provide a canonical representation of an $SU(n)$ theory of currents of the same structure as Sugawara's. The model does not provide a representation of Sugawara's $K(n)$ theory of currents; however, a comparison is made between the two models.

## I. INTRODUCTION

THE possibility that strong-interaction physics can be formulated in terms of current densities without introducing explicit particle fields has been explored by various authors, and the numerous merits of such a theory have been discussed.[1–4] The feasibility of such a theory was demonstrated by the fact that certain familiar field-theoretic models (in particular, the quark model and the neutral scalar-meson model) can be rewritten in terms of current theories.

Recently Sugawara[5] has constructed a completely internally consistent field theory in which the only dynamical variables that appear are currents obeying the algebra-of-fields[6] commutation relations. Even if we do not take the Sugawara model in its exact (over-symmetric) form very seriously, it has many attractive features, and is at worst a further demonstration of the feasibility of a currents-as-coordinates theory.

With its obvious associations with the algebra of fields, the Sugawara model was shown[7] to be a particular limit of the Yang-Mills[8] theory. As a further development to the understanding of the model, Bardakci and Halpern,[9] and Sugawara and Yoshimura[10] have obtained a canonical representation, closely related to the $\sigma$ model,[11] of Sugawara's theory in which the algebra of the charges associated with the currents is that of $SU(2)$. In this paper we elucidate this approach and extend it to the case of general $SU(n)$.

More precisely, we consider families of canonically equivalent Lagrangian field theories in which the only fields appearing are a set of $N=n^2-1$ spin-zero mesons transforming nonlinearly under $K(n)=SU(n)\times SU(n)$.[12] The Lagrangian is constructed so that it is $K(n)$-symmetric—the mesons are then massless. On introducing canonical quantization for the fields, the currents defined by Noether's theorem provide a representation of an $SU(n)$ theory of currents having the same algebraic structure as Sugawara's symmetric $SU(n)$ theory of currents. The model does not, however, provide a canonical representation of Sugawara's $K(n)$ theory of currents.

In Sec. V, for a more complete translation of the spin-zero-meson model in terms of currents, we introduce a $K(n)$-symmetry-breaking term into the Lagrangian and write the stress-tensor–current commutation relations.

For completeness and for later reference we will specify Sugawara's symmetric $SU(n)$ and $K(n)$ theories of currents.[5]

In the symmetric $SU(n)$ theory of currents, we have currents $C^{\mu}{}_i$, $i=1,\cdots,N$, obeying the algebra-of-fields equal-time commutation relations[13]

$$[C^0{}_i(x),C^0{}_j(y)]_{x_0=y_0}=ic_{ijk}C^0{}_k(x)\delta(\mathbf{x}-\mathbf{y}),\qquad(1)$$

$$[C^0{}_i(x),C^{\alpha}{}_j(y)]_{x_0=y_0}=ic_{ijk}C^{\alpha}{}_k(x)\delta(\mathbf{x}-\mathbf{y})+ia\delta_{ij}\frac{\partial}{\partial x_\alpha}\delta(\mathbf{x}-\mathbf{y}),\qquad(2)$$

$$[C^{\alpha}{}_i(x),C^{\beta}{}_j(y)]_{x_0=y_0}=0,\qquad(3)$$

where $a=$ const, and $c_{ijk}$ are the structure constants of $SU(n)$. The stress tensor is given by

$$\theta^{\mu\nu}=-\tfrac{1}{2}a^{-1}(\{C^{\mu}{}_i,C^{\nu}{}_i\}-g^{\mu\nu}C^{\rho}{}_iC_{\rho i})\qquad(4)$$

and the currents obey the equations of motion

$$\partial^{\mu}C^{\nu}{}_i-\partial^{\nu}C^{\mu}{}_i=\tfrac{1}{2}a^{-1}c_{ijk}\{C^{\mu}{}_j,C^{\nu}{}_k\}.\qquad(5)$$

We shall denote this symmetric $SU(n)$ Sugawara theory by $S_n(C^{\mu};a)$.

Sugawara's symmetric $K(n)$ theory of currents is the direct sum of two symmetric $SU(n)$ theories of currents $S_n(J^{\mu}{}_+;\tfrac{1}{2}c)\oplus S_n(J^{\mu}{}_-;\tfrac{1}{2}c)$, where the currents $J^{\mu}{}_{+i}$ and $J^{\nu}{}_{-j}$ commute at equal times. The currents are defined through

$$J^{\mu}{}_{\pm i}=\tfrac{1}{2}(V^{\mu}{}_i\pm A^{\mu}{}_i),\qquad(6)$$

where $V^{\mu}{}_i$ and $A^{\mu}{}_i$ are interpreted as the usual (physi-

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[1] R. F. Dashen and D. H. Sharp, Phys. Rev. **165**, 1857 (1968).
[2] D. H. Sharp, Phys. Rev. **165**, 1867 (1968).
[3] C. G. Callan, R. F. Dashen, and D. H. Sharp, Phys. Rev. **165**, 1883 (1968).
[4] M. B. Halpern, Phys. Rev. **164**, 1878 (1967).
[5] H. Sugawara, Phys. Rev. **170**, 1659 (1968).
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[8] C. N. Yang and R. L. Mills, Phys. Rev. **96**, 191 (1954).
[9] K. Bardakci and M. B. Halpern, Phys. Rev. **172**, 1542 (1968).
[10] H. Sugawara and M. Yoshimura, Phys. Rev. **173**, 1419 (1968).
[11] M. Gell-Mann and M. Lévy, Nuovo Cimento **16**, 705 (1960).
[12] A. J. Macfarlane and P. H. Weisz, Nuovo Cimento **55A**, 853 (1968); and (to be published).

[13] Throughout this paper Latin letters are used for $SU(n)$ indices and Greek letters for Lorentz indices ($\mu,\nu,\rho=0,1,2,3$; $\alpha,\beta,\gamma=1,2,3$).

cal) vector and axial-vector currents. The stress tensor in this theory is then

$$\theta^{\mu\nu}=\theta^{\mu\nu}{}_{+}+\theta^{\mu\nu}{}_{-}$$
$$=-\tfrac{1}{2}c^{-1}[\{V^{\mu}{}_{i},V^{\nu}{}_{i}\}+\{A^{\mu}{}_{i},A^{\nu}{}_{i}\}-g^{\mu\nu}(V^{\rho}{}_{i}V_{\rho i}+A^{\rho}{}_{i}A_{\rho i})]. \quad (7)$$

## II. MESON FIELD TRANSFORMATION PROPERTIES AND COVARIANT DERIVATIVES

Suppose $T_i$ and $X_i$ $(1\leqslant i\leqslant N=n^2-1)$ are the Hermitian charges associated with a set of currents $t_i$ and $x_i$ and that they obey the $K(n)$ commutation relations

$$[T_i,T_j]=[X_i,X_j]=ic_{ijk}T_k, \quad (8)$$

$$[T_i,X_j]=ic_{ijk}X_k. \quad (9)$$

Then the charges $Q_{\pm i}=\frac{1}{2}(T_i\pm X_i)$ generate the commuting $SU(n)$ factors of $K(n)$. These charges are assumed to be functionals only of a set of $N$ spin-zero meson fields and their space-time derivatives. Assume the transformation properties of these fields $M_i(x)$ to be given as

$$[T_i,M_j]=ic_{ijk}M_k, \quad (10)$$

i.e., $M_i$ transforms as a vector under the $SU(n)$ subgroup generated by the charges $T_i$, and

$$[X_i,M_j]=iF_{ij}(M), \quad (11)$$

where $F_{ij}(M)$ is independent of the field derivatives and satisfies[14]: (i) analyticity at the origin $M_i=0$, since inverse powers of the fields are not defined, and (ii) consistency conditions arising from the Jacobi identities, viz.: (a) $F_{ij}$ transforms as a tensor under the $SU(n)$ subgroup generated by the $T_i$, and (b) $F_{ij}$ must satisfy the equation

$$\frac{\partial F_{jk}}{\partial M_l}F_{il}-\frac{\partial F_{ik}}{\partial M_l}F_{jl}=c_{ijl}c_{lkm}M_m. \quad (12)$$

Two solutions of Eq. (12) are $F_{ij}=\pm c_{ijk}M_k$ [then $M$ would transform linearly under the $K(n)$ group]. It can be shown that all other solutions of Eq. (12) are nonsingular, i.e., $F_{ij}(0)=f\delta_{ij}$, where $f$ is a nonzero constant, and nonlinear in $M$, e.g., $\partial^2F_{ij}(0)/\partial M_l\partial M_m\neq 0$. Furthermore, it can be shown that the nonsingular solutions of Eq. (12) are canonically equivalent to one another.[12] In the following, we shall only consider the fields with nonlinear transformations.

It will be convenient to use the charges $Q_{\pm i}$ and write

$$[Q_{\pm i},M_j]=i\tfrac{1}{2}(c_{ijk}M_k\pm F_{ij})\equiv id^{-1}{}_{\pm ij}(M). \quad (13)$$

[14] For a detailed treatment of the tensor $F_{ij}$ in the case of $K(2)$, see S. Weinberg, Phys. Rev. **166**, 1568 (1968). For a full treatment in the general case $K(n)$ or for the particular case $K(3)$, see Ref. 12.

In terms of the transformation tensors $d^{-1}{}_{\pm ij}$,[15] the Jacobi identities read

$$\frac{\partial d^{-1}{}_{\pm jk}}{\partial M_l}d^{-1}{}_{\pm il}-\frac{\partial d^{-1}{}_{\pm ik}}{\partial M_l}d^{-1}{}_{\pm jl}=c_{ijl}d^{-1}{}_{\pm lk}, \quad (14)$$

$$\frac{\partial d^{-1}{}_{-jk}}{\partial M_l}d^{-1}{}_{+il}-\frac{\partial d^{-1}{}_{+ik}}{\partial M_l}d^{-1}{}_{-jl}=0; \quad (15)$$

or, since the transformation tensors $d^{-1}{}_{\pm ij}$ are invertible (as implied by the notation), Eq. (14) can be written

$$\frac{\partial d_{\pm ik}}{\partial M_j}-\frac{\partial d_{\pm jk}}{\partial M_i}=c_{lmk}d_{\pm il}d_{\pm jm}. \quad (16)$$

The important result

$$d_{+ik}d_{+jk}=d_{-ik}d_{-jk} \quad (17)$$

can be proved directly from Eqs. (14)–(16) after some algebraic manipulations.[12]

If we now make the assumption (which is really an assumption on the symmetry breaking, if any)

$$[\partial^0Q_{\pm i}(t),M_j(x)]_{x_0=t}=0, \quad (18)$$

then the transformation properties of the field derivatives are determined by the transformation properties of the fields. The meson-field covariant derivatives are defined by $d^{\mu}{}_{\pm i}=\partial^{\mu}M_jd_{\pm ji}$. They are linear in the meson-field first derivatives, and they transform according to the representations $(N,0)$ and $(0,N)$, respectively, of $K(n)$,

$$[Q_{\pm i},d^{\mu}{}_{\pm j}]=ic_{ijk}d^{\mu}{}_{\pm k}, \quad (19)$$

$$[Q_{\pm i},d^{\mu}{}_{\mp j}]=0. \quad (20)$$

This can be seen as a direct consequence of the Jacobi-identity conditions (14) and (15). In terms of the covariant derivatives, Eqs. (16) and (17) can now be written, respectively,

$$\partial^{\nu}d^{\mu}{}_{\pm i}-\partial^{\mu}d^{\nu}{}_{\pm i}=\tfrac{1}{2}c_{ijk}\{d^{\mu}{}_{\pm j},d^{\nu}{}_{\pm k}\}, \quad (21)$$

$$d^{\mu}{}_{+i}d^{\mu}{}_{+i}=d^{\mu}{}_{-i}d^{\mu}{}_{-i}, \quad \text{for all } \mu. \quad (22)$$

## III. SYMMETRIC LAGRANGIAN

Up to this point all the results are algebraic consequences of the equal-time commutation relations Eqs. (8)–(11) of the fields $M_i$ and the charges $Q_{\pm i}(M)$ whose existence as functionals of the fields is assumed, and of the assumption (18). The existence of the functions $d_{\pm ij}(M)$ satisfying Eqs. (14) and (15) is known.[12,14] In the following, we proceed to realize the charges as functionals of $M$. To do this we will proceed inductively, for the reason of plausibility, and postulate a Lagrangian and canonical commutation relations for the fields,

[15] The notation here coincides with that used by L. S. Brown, Phys. Rev. **163**, 1802 (1967).

thereby obtaining the required currents (and hence charges) by Noether's theorem. We stress, however, that an alternative procedure at this stage would be simply to postulate the appropriate commutators [Eq. (31)] in an apparently *ad hoc* manner and verify that the proposed functionals [Eq. (32)] satisfy all the requirements to represent the currents, and then show equivalence with the Lagrangian formalism. Equations of motion for the fields are not utilized until Sec. IV.

Consider the construction of a $K(n)$-invariant meson Lagrangian $\mathfrak{L}_s$ which is at most quadratic in the meson first derivatives and contains no higher derivatives. First, we note that no nontrivial $K(n)$-invariant function of the $M_i$ alone can be constructed[12]; hence there is no mass term. Accordingly, consider a term of the form[16]

$$\mathfrak{L}_s = L_{ij}(M)\partial^\mu M_i \partial_\mu M_j \equiv \hat{L}_{\pm ij}(M) d^\mu{}_{\pm i}(M) d_{\mu\pm j}. \quad (23)$$

It can be shown that for $K(n)$ invariance $\hat{L}_{\pm ij}$ must be of the simple form

$$\hat{L}_{\pm ij} = -\tfrac{1}{8} f^2 \delta_{ij}, \quad (24)$$

where the constant has been chosen so that the Lagrangian contains the conventional kinetic term $\mathfrak{L}_s = -\frac{1}{2}\partial^\mu M_i \partial_\mu M_i + \cdots$. The required currents $C^\mu{}_{\pm i} = \frac{1}{2}(t^\mu{}_i \pm x^\mu{}_i)$ now turn out, according to Noether's theorem,

$$C^\mu{}_{\pm i} = -i \frac{\partial \mathfrak{L}_s}{\partial(\partial_\mu M_j)} [Q_{+i}, M_j] = -\tfrac{1}{4} f^2 d^\mu{}_{\pm i}, \quad (25)$$

to be proportional to the meson covariant derivatives. Hence in terms of the currents the Lagrangian takes the familiar current × current form

$$\begin{aligned}\mathfrak{L}_s &= -2f^{-2} C^\mu{}_{+i} C_{\mu+i} = -2f^{-2} C^\mu{}_{-i} C_{\mu-i} \\ &= -\tfrac{1}{2} f^{-2} (t^\mu{}_i t_{\mu i} + x^\mu{}_i x_{\mu i}). \end{aligned} \quad (26)$$

Note that for the last equality in Eq. (26) we have used

$$\{t^\mu{}_i, x_{\mu i}\} = 0, \quad (27)$$

which, in turn, is a direct consequence of Eq. (22). The usual definition of the stress-energy-momentum tensor, when written in terms of currents, yields

$$\theta^{\mu\nu}{}_s = -2f^{-2}(\{C^\mu{}_{+i}, C^\nu{}_{+i}\} - g^{\mu\nu} C^\rho{}_{+i} C_{\rho+i}) \quad (28a)$$

$$= -2f^{-2}(\{C^\mu{}_{-i}, C^\nu{}_{-i}\} - g^{\mu\nu} C^\rho{}_{-i} C_{\rho-i}) \quad (28b)$$

$$= -\tfrac{1}{2} f^{-2}[\{t^\mu{}_i, t^\nu{}_i\} + \{x^\mu{}_i, x^\nu{}_i\} - g^{\mu\nu}(t^\rho{}_i t_{\rho i} + x^\rho{}_i x_{\rho i})]. \quad (29)$$

Equation (29) has the same form as the stress tensor as introduced by Sugawara in his $K(n)$ theory of currents [Eq. (7) with $c = f^2$, and where $t$ and $x$ appear instead of $V$ and $A$]. Equations (28a) and (28b) have the same structure as the stress tensor in Sugawara's $SU(n)$ theory of currents [Eq. (4) with $a = \frac{1}{4} f^2$, and where $C_\pm$ appear instead of $C$]. Again, in terms of currents, Eq. (21) takes the form

$$\partial^\mu C^\nu{}_{\pm i} - \partial^\nu C^\mu{}_{\pm i} = 2f^{-2} c_{ijk} \{C^\mu{}_{\pm j}, C^\nu{}_{\pm k}\}. \quad (30)$$

This is precisely the form of the $SU(n)$ current equations of motion which arises in Sugawara's theory[5] [Eq. (5) with $a = \frac{1}{4} f^2$] on imposing Heisenberg's equations of motion and Schwinger's condition.[17]

We now introduce canonical commutation relations for the fields and the canonically conjugate fields

$$\pi_i = \frac{\partial \mathfrak{L}_s}{\partial(\partial_0 M_i)} = 2L_{ij}(M)\partial^0 M_j,$$

i.e.,

$$[M_i(x), M_j(y)]_{x_0=y_0} = [\pi_i(x), \pi_j(y)]_{x_0=y_0} = 0, \quad (31a)$$

$$[\pi_i(x), M_j(y)]_{x_0=y_0} = i\delta_{ij}\delta(\mathbf{x}-\mathbf{y}). \quad (31b)$$

The currents are now given by

$$C^\alpha{}_{\pm i} = -\tfrac{1}{4} f^2 d_{\pm ji} \partial^\alpha M_j, \quad (32a)$$

$$C^0{}_{\pm i} = d^{-1}{}_{\pm ij} \pi_j, \quad (32b)$$

and their equal-time commutation relations are given by[13]

$$[C^0{}_{\pm i}(x), C^0{}_{\pm j}(y)]_{x_0=y_0} = i c_{ijk} C^0{}_{\pm k}(x) \delta(\mathbf{x}-\mathbf{y}), \quad (33a)$$

$$[C^0{}_{\pm i}(x), C^\alpha{}_{\pm j}(y)]_{x_0=y_0} = i c_{ijk} C^\alpha{}_{\pm k}(x)\delta(\mathbf{x}-\mathbf{y}) + i\tfrac{1}{4} f^2 \delta_{ij} \frac{\partial}{\partial x_\alpha} \delta(\mathbf{x}-\mathbf{y}), \quad (33b)$$

$$\begin{aligned}[C^0{}_{\pm i}(x), C^\alpha{}_{\mp j}(y)]_{x_0=y_0} &= i\tfrac{1}{4} f^2 d^{-1}{}_{\pm ik}(y) d_{\mp kj}(y) \frac{\partial}{\partial x_\alpha} \delta(\mathbf{x}-\mathbf{y}) \\ &= [-i\tfrac{1}{4} f^2 \delta_{ij} + \text{nonlinear terms in } M(y)] \times \frac{\partial}{\partial x_\alpha} \delta(\mathbf{x}-\mathbf{y}), \end{aligned} \quad (34)$$

while all the other commutators vanish. In deriving these commutation relations, use is made of Eqs. (14) and (15). We see that the currents $C^\mu{}_{\pm i}$ separately obey the $SU(n)$ algebra-of-fields commutation relations. Hence the currents $C^\mu{}_\pm$ separately provide us with a canonical representation of Sugawara's $SU(n)$ theory of currents, $S_n(C_\pm; \frac{1}{4} f^2)$.

At this stage, we remark that we have two interpretations of the structure we have obtained. In the first interpretation, we do not specify any space-reflection properties of the fields. In this case what we have, as stated above, is a canonical representation of Sugawara's $SU(n)$ theory of (vector) currents only in terms of spin-zero fields supplied either by the currents $C_+$ or by the

[16] In the following, we neglect all problems associated with the ordering of the operators.

[17] J. Schwinger, Phys. Rev. **130**, 406 (1963).

$C_-$, the representation containing a higher-symmetry $K(n)$ [not identified with chiral $K(n)$] than the symmetry $SU(n)$ of the theory of currents.

In the second interpretation, we specify the parity properties of the fields and currents. Indeed, we specify the $t^\mu{}_i$ and $x^\mu{}_i$ as vector and axial-vector currents, respectively, and suppose the fields $M_i$ to be pseudoscalar.[18] Again the currents $C_\pm$ supply a canonical representation of an $SU(n)$ theory of currents of the same structure as Sugawara's. However, the currents $t$ and $x$ do not supply a representation of Sugawara's chiral $K(n)$ theory of currents, since we note, e.g., that the time-space commutation relations of currents associated with opposite factors of $K(n)$, i.e., Eq. (34), are nonvanishing, containing operator-type Schwinger terms (which cannot be written in terms of currents alone), as opposed to the corresponding commutators in Sugawara's $K(n)$ theory which vanish.

One way of comparing this model to Sugawara's $K(n)$ theory is to put the stress tensors of the two models, i.e., Eqs. (29) and (7), into correspondence by normalizing the stress tensor (7) by putting $c=f^2$. Then the commutators of the $J_\pm$ currents in Sugawara's $K(n)$ are

$$[J^{0}{}_{\pm i}(x),J^0{}_{\pm j}(y)]_{x0=y0}=ic_{ijk}J^0{}_{\pm k}(x)\delta(\mathbf{x}-\mathbf{y})\,,$$

$$[J^0{}_{\pm i}(x),J^\alpha{}_{\pm j}(y)]_{x0=y0}=ic_{ijk}J^\alpha{}_{\pm k}(x)\delta(\mathbf{x}-\mathbf{y}) +i\tfrac{1}{2}f^2\delta_{ij}\frac{\partial}{\partial x_\alpha}\delta(\mathbf{x}-\mathbf{y})\,,$$

while all the other commutators vanish. These commutators can then be contrasted (or compared) with the commutators [Eqs. (33)–(34)] of the meson model.[19] The presence of the different Schwinger terms is also reflected in the fact that the equations of motion of the currents in Sugawara's $K(n)$ theory, when $c=f^2$, are

$$\partial^\mu J^\nu{}_{\pm i}-\partial^\nu J^\mu{}_{\pm i}=f^{-2}c_{ijk}\{J^\mu{}_{\pm j},J^\nu{}_{\pm k}\}\,,$$

differing by a factor of $\frac{1}{2}$ from the corresponding equations of motion (30) in the meson model.

## IV. SYMMETRY BREAKING AND STRESS-TENSOR–CURRENT COMMUTATORS

For a more complete treatment of the spin-zero-meson model, and with the hope that a clue may be obtained as how to introduce symmetry breaking in the framework of a Sugawara-type theory, we introduce a $K(n)$-symmetry-breaking Lagrangian term $\mathfrak{L}_B$.[20] This function will be assumed to be independent of the meson-field derivatives. Such an additional term in the Lagrangian does not affect the structure of the canonically conjugate fields nor the currents, and hence does not affect the current equal-time commutation relations. We note that such a term is also consistent with assumption (20). A further consequence of this assumption on $\mathfrak{L}_B$ is

$$[C^\mu{}_{\pm i}(x),\mathfrak{L}_B(y)]_{x0=y0}=ig^{0\mu}\frac{\partial\mathfrak{L}_B(\mu)}{\partial M_j}d^{-1}{}_{\pm ij}(x)\delta(\mathbf{x}-\mathbf{y}) \quad (35)$$

$$=ig^{0\mu}\partial^\nu J_{\nu\pm i}(x)\delta(\mathbf{x}-\mathbf{y})\,, \quad (36)$$

the last (partial conservation) equation being a statement of Lagrange's equations of motion for the fields $M_i(x)$.

The total stress-energy-momentum tensor is now given by

$$\theta^{\mu\nu}=\theta^{\mu\nu}{}_s-g^{\mu\nu}\mathfrak{L}_B\,, \quad (37)$$

and has the following equal-time commutation relations with the currents, where $C^\mu$ denotes any current $C^\mu{}_{\pm i}$:

$$[\theta^{00}(x),C^0(y)]_{x0=y0} =-iC^\alpha(x)\frac{\partial}{\partial x_\alpha}\delta(\mathbf{x}-\mathbf{y})+i\partial_\nu C^\nu(x)\delta(\mathbf{x}-\mathbf{y})\,, \quad (38)$$

$$[\theta^{0\alpha}(x),C^0(y)]_{x0=y0}=-iC^0(x)\frac{\partial}{\partial x_\alpha}\delta(\mathbf{x}-\mathbf{y})\,, \quad (39)$$

$$[\theta^{0\mu}(x),C^\alpha(y)]_{x0=y0} =-iC^\mu(y)\frac{\partial}{\partial x_\alpha}\delta(\mathbf{x}-\mathbf{y})+i\partial^\mu C^\alpha(x)\delta(\mathbf{x}-\mathbf{y})\,, \quad (40)$$

$$[\theta^{\alpha\beta}(x),C^0(y)]_{x0=y0} =i[C^\alpha(x)g^{\gamma\beta}+C^\beta(x)g^{\gamma\alpha}-C^\gamma(x)g^{\alpha\beta}] \times\frac{\partial}{\partial x_\gamma}(\mathbf{x}-\mathbf{y})+ig^{\alpha\beta}\partial_\mu C^\mu(x)\delta(\mathbf{x}-\mathbf{y})\,, \quad (41)$$

$$[\theta^{\alpha\beta}(x),C^\gamma(y)]_{x0=y0}=-g^{\alpha\beta}[\theta^{00}(x),C^\gamma(y)]_{x0=y0}\,. \quad (42)$$

These are the same stress-tensor–current commutation relations as in Sugawara's symmetric-theory-of-currents model, on putting $\partial_\mu C^\mu=0$. We note that the right-hand side of the commutators can be written in terms of the current only. We mention here that the commutation relation (38) arises in quite general models[21] from Schwinger's action principle. Further, the commutator (39) arises in a model-independent way from the canonical commutation relation.[22] The commutator (40) with $\mu=0$ then follows from the commutators (38) and (39) by Lorentz invariance.

Note also the integrated commutators

$$[\theta^{\mu\nu}(x),Q_{\pm i}(t)]_{x_0=t}=ig^{\mu\nu}\partial_\rho C^\rho{}_{\pm i}(x)\,. \quad (43)$$

[18] In this case we must have $F_{ij}(M)=F_{ij}(-M)$; thus the condition that $F_{ij}$ is nonsingular is imposed.

[19] Note the property that the tensor $R_{ij}=d^{-1}{}_{+ik}d_{-kj}=d^{-1}{}_{-jk}d_{+ki}$ appearing in the commutator (34) is orthogonal, i.e., $R_{ik}R_{jk}=\delta_{ij}$.

[20] $\mathfrak{L}_B$ contains the mass term $\mathfrak{L}_B=\frac{1}{2}m^2M_iM_i+\cdots$.

[21] D. J. Gross and R. Jackiw, Phys. Rev. **163**, 1688 (1967).

[22] R. Jackiw, Phys. Rev. **175**, 2058 (1968). We point out that Jackiw's proof extends trivially to the case in which the currents arise from nonlinear transformations.

The commutators with the stress tensor and $\mathcal{L}_B$ are

$$[\theta^{\mu 0}(x),\mathcal{L}_B(y)]_{x_0=y_0} = -2if^{-2}\{C^{\mu}{}_{\pm i}(x),\partial_\nu C^{\nu}{}_{\pm i}(x)\}\delta(\mathbf{x}-\mathbf{y}) = i\partial^\mu \mathcal{L}_B(x)\delta(\mathbf{x}-\mathbf{y}), \tag{44}$$

$$[\theta^{\alpha\beta}(x),\mathcal{L}_B(y)]_{x_0=y_0} = -g^{\alpha\beta}[\theta^{00}(x),\mathcal{L}_B(y)]_{x_0=y_0}. \tag{45}$$

## V. POINCARÉ ALGEBRA

First, it is easy to check that Schwinger's condition[17]

$$[\theta^{00}(x),\theta^{00}(y)]_{x_0=y_0} = -i[\theta^{0\alpha}(x)+\theta^{0\alpha}(y)]\frac{\partial}{\partial x_\alpha}\delta(\mathbf{x}-\mathbf{y}) \tag{46}$$

and the conservation of energy momentum

$$\partial_\mu \theta^{\mu\nu} = 0 \tag{47}$$

hold in the model (with symmetry breaking).

The existence of Poincaré generators follows in the usual way. Define a momentum tensor $P^\mu$ by

$$P^\mu = -\int \theta^{\mu 0}(x) d^3x; \tag{48}$$

then, integrating the commutation relations (38)–(40) and Eq. (44), the Heisenberg equations of motion are obtained:

$$[P^\mu, C^\nu{}_{\pm i}(x)] = -i\partial^\mu C^\nu{}_{\pm i}(x), \tag{49}$$

$$[P^\mu, \mathcal{L}_B(x)] = -i\partial^\mu \mathcal{L}_B(x). \tag{50}$$

Also, defining the Lorentz generators $M^{\mu\nu}$ by

$$M^{\mu\nu} = \int [x^\nu \theta^{0\mu}(x) - x^\mu \theta^{0\nu}(x)] d^3x, \tag{51}$$

these can be checked to have the correct commutation relations with the currents and with $\mathcal{L}_B$,

$$[M^{\mu\nu}, C^\rho{}_{\pm i}(x)] = -i(x^\mu\partial^\nu - x^\nu\partial^\mu)C^\rho{}_{\pm i}(x) - i[g^{\mu\rho}C^\nu{}_{\pm i}(x) - g^{\nu\rho}C^\mu{}_{\pm i}(x)], \tag{52}$$

$$[M^{\mu\nu}, \mathcal{L}_B(x)] = -i(x^\mu\partial^\nu - x^\nu\partial^\mu)\mathcal{L}_B(x). \tag{53}$$

It can also be checked that $P^\mu$ and $M^{\mu\nu}$ together generate the Poincaré algebra. For this purpose, we require the stress-tensor equal-time commutation relations. In this model they turn out to be of the simplest possible structure,

$$[\theta^{0\alpha}(x),\theta^{0\beta}(y)]_{x_0=y_0} = i[\theta^{0\alpha}(y)g^{\gamma\beta} + \theta^{0\beta}(x)g^{\gamma\alpha}] \times \frac{\partial}{\partial x_\gamma}\delta(\mathbf{x}-\mathbf{y}), \tag{54}$$

which arises in a model-independent way from the canonical commutation relations, and

$$[\theta^{00}(x),\theta^{0\alpha}(y)]_{x_0=y_0} = i[\theta^{00}(y)g^{\alpha\beta} - \theta^{\alpha\beta}(x)] \times \frac{\partial}{\partial x_\beta}\delta(\mathbf{x}-\mathbf{y}). \tag{55}$$

Finally, we note the absence of Schwinger terms which are proportional to the third derivatives of the spatial $\delta$ function arising from naïve computation of the stress-tensor commutation relations in this model (and in Sugawara's model). The presence of such Schwinger terms would be required in a realistic model.[23,24]

*Note added in manuscript.* On completion of this manuscript, a paper by Sakita,[25] in which similar ideas are presented, came to my attention.

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