Divergent Electromagnetic Masses and Deep-Inelastic Electron-Hadron Scattering*

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We show that the electromagnetic self-energies of the hadrons and the mass splittings are divergent unless there is very special behavior of the electron-hadron scattering amplitude in the deep-inelastic region. We assume only the validity of the Cottingham formula, a nontrivial Bjorken limit of the scattering-structure functions, and analyticity of a partial-wave amplitude in angular momentum J.

THE purpose of this paper is to show that the electromagnetic self-mass of any hadron and the mass splittings are at least logarithmically divergent, and possibly quadratically divergent, unless the electron-hadron scattering amplitude has very special behavior in the deep-inelastic region. This conclusion is shown to follow from the following essential assumptions: (i) the validity of the Cottingham formula[1] relating the electromagnetic self-mass to integrals over the forward virtual Compton amplitudes $T_{1,2}(q^2,\nu)$, with these amplitudes having the limiting behavior $\nu^2 T_1(q^2,\nu)\to 0$, $T_2(q^2,\nu)\to 0$ as $\nu\to\infty$; (ii) a nontrivial Bjorken limit[2] for the inelastic electron scattering amplitudes $W_{1,2}(q^2,\nu)$, as indicated by experiments on the proton; (iii) analyticity of a partial-wave amplitude $A^{(+)}(q^2,J)$ in the J plane for $\mathrm{Re}J>-\frac{1}{2}$. The simplest way to avoid this conclusion is to suppose the presence of a nonanalytic piece δ_{J0} in the partial-wave amplitude.

We will assume that the electromagnetic self-mass of a hadron, given by

$$\delta M=\frac{i}{(2\pi)^2}\int_{-\infty}^{+\infty}\frac{d^4q}{q^2}g^{\mu\nu}T_{\mu\nu}(p,q)\,,$$

is finite. Here $T_{\mu\nu}(p,q)$ is the forward virtual Compton amplitude of photons of mass q^2 scattering on a target hadron with mass normalized according to $p^2=1$:

$$T_{\mu\nu}(p,q)=T_1(q^2,\nu)\left(-g_{\mu\nu}+\frac{q_\mu q_\nu}{q^2}\right)+T_2(q^2,\nu)\left(p_\mu-\frac{p\cdot q}{q^2}q_\mu\right)\left(p_\nu-\frac{p\cdot q}{q^2}q_\nu\right),\quad(1)$$

where $\nu=p\cdot q$. If δM is finite, one may perform a Wick rotation to obtain the Cottingham formula

$$\delta M=-\tfrac{1}{4}\pi\int_0^{-\infty}\frac{dq^2}{q^2}\int_{-|q|}^{+|q|}d\nu(-q^2-\nu^2)^{1/2}T_\mu{}^\mu(q^2,i\nu)\,,\qquad |q|=\surd(-q^2)\,.\quad(2)$$

Following standard Regge lore, we assume that $T_1(q^2,\nu)$ obeys a once-subtracted dispersion relation in ν, and $T_2(q^2,\nu)$ obeys an unsubtracted dispersion relation. Introducing $\omega=-q^2/\nu$, these dispersion relations read

$$T_1(q^2,\omega)=T_1(q^2,\infty)-\int_0^4\frac{d\omega'^2W_1(q^2,\omega')}{\omega'^2-\omega^2}\,,$$
$$T_2(q^2,\omega)=-\omega^2\int_0^4\frac{d\omega'^2W_2(q^2,\omega')}{\omega'^2(\omega'^2-\omega^2)}\,,\quad(3)$$

where $\pi W_{1,2}(q^2,\omega)=\mathrm{Im}T_{1,2}(q^2,\omega)$ are the usual structure functions for the electron scattering process and $T_1(q^2,\infty)$ is the subtraction constant. Writing $\nu=y|q|$, we have

$$\delta M=\tfrac{1}{4}\pi\int_0^{-\infty}\frac{dq^2}{q^2}\int_{-1}^{+1}dy(1-y^2)^{1/2}q^2T_\mu{}^\mu(q^2,i\nu)\,,\quad(4)$$

with

$$T_\mu{}^\mu(q^2,i\nu)=-3T_1(q^2,\infty)+3\int_0^4\frac{d\omega'^2W_1(q^2,\omega')}{\omega'^2-q^2/y^2}-(1-y^2)\frac{q^2}{y^2}\int_0^4\frac{d\omega'^2W_2(q^2\omega)}{\omega'^2(\omega'^2-q^2/y^2)}\,.$$

We now assume a nontrivial Bjorken limit for the structure functions. With ω fixed,

$$\lim_{-q^2\to\infty}W_1(q^2,\omega)=F_1(\omega)\,,$$
$$\lim_{-q^2\to\infty}q^2W_2(q^2,\omega)=-\omega F_2(\omega)\,,\quad(5)$$

so that[2]

$$\lim_{-q^2\to\infty}q^2T_\mu{}^\mu(q^2,i\nu)=\lim_{-q^2\to\infty}-3q^2T_1(q^2,\infty)-3y^2\int_0^4d\omega^2F_1(\omega)-(1-y^2)\int_0^4\frac{d\omega^2}{\omega}F_2(\omega)\,.$$

Taking the limit as $-q^2\to\infty$ and performing the y integration in (4), the condition for the absence of a logarithmic divergence is easily seen to be

$$\lim_{-q^2\to\infty}2q^2T_1(q^2,\infty)+\int_0^2d\omega[F_2(\omega)+\omega F_1(\omega)]=0\,.\quad(6)$$

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† A. P. Sloan Foundation Fellow, 1968–1969.
[1] W. N. Cottingham, Ann. Phys. (N. Y.) **25**, 424 (1963).
[2] J. D. Bjorken, Phys. Rev. **179**, 1547 (1969). We also assume that the integrals over $W_{1,2}(q^2,\omega')$ converge uniformly as $-q^2\to\infty$.

The combination of deep-inelastic structure functions can be determined from the longitudinal and transverse cross sections

$$F_2(\omega)+\omega F_1(\omega) = \lim_{-q^2\to\infty} \tfrac{1}{4}\pi^2\alpha q^2[2\sigma_T(q^2,\omega)+\sigma_L(q^2,\omega)]<0. \quad (7)$$

Before we can assert whether or not condition (6) is satisfied, we must obtain information about the subtraction term $T_1(q^2,\infty)$ as $-q^2\to\infty$. To this end we will introduce the assumption of analyticity in the J plane as specified below.

It can be shown[3] that $T_1(q^2,\nu)=F_{00}{}^t(q^2,\nu)$, where $F_{00}{}^t$ is a crossed-channel helicity-nonflip amplitude. We define

$$F^{(+)}(q^2,\nu)=T_1(q^2,\infty)+\frac{2\nu}{\pi}\int_{\nu_0}^{\infty}\frac{d\nu'\,\mathrm{Im}T_1(q^2,\nu')}{\nu('\nu'-\nu)}, \quad (8)$$

with $\nu_0=-q^2/2$, so that

$$T_1(q^2,\nu)=\tfrac{1}{2}[F^{(+)}(q^2,\nu)+F^{(+)}(q^2,-\nu)].$$

We may decompose $F^{(+)}(q^2,\nu)$ into partial waves,

$$F^{(+)}(q^2\ \nu)=\sum_{J=0}^{\infty}(2J+1)A^{(+)}(q^2,J)P_J(z), \quad (9)$$

where $z=\nu/\sqrt{q^2}$. Using Eqs. (8) and (9), we have the Froissart-Gribov definition of the partial-wave amplitude

$$A^{(+)}(q^2\ J)=\frac{2}{\pi}\int_{z_0}^{\infty}dz\,Q_J(z)\,\mathrm{Im}T_1(q^2,z), \quad (10)$$

with $\mathrm{Re}J>1$. For the moment we restrict ourselves to $\mathrm{Re}J>1$ because of the presence of the subtraction term $T_1(q^2,\infty)$. Introducing $\omega=-i|q|/z$, we rewrite (10) as

$$A^{(+)}(q^2,J)=\frac{-2i|q|}{\pi}\int_0^2\frac{d\omega}{\omega^2}Q_J\left(\frac{-i|q|}{\omega}\right)\mathrm{Im}T_1(q^2,\omega), \quad (11)$$

for $\mathrm{Re}J>1$. We write

$$\pi^{-1}\,\mathrm{Im}T_1(q^2,\omega)=F_1(\omega)+R_1(q^2,\omega), \quad (12)$$

with $R_1(q^2,\omega)\to 0$ as $-q^2\to\infty$. The next-to-leading term must be kept in what follows. In principle, it can be determined experimentally by

$$[(q^2/4\pi^2\alpha)\sigma_T(q^2,\omega)-\omega F_1(\omega)]\underset{-q^2\to\infty}{\sim}\omega R(q^2,\omega).$$

Using

$$Q_J(-i|q|/\omega)=h(J)(i\omega/|q|)^{J+1}\{1+\tfrac{1}{2}[(\omega^2/q^2)(J+2)\times(J+1)(J+3)]+O(q^{-4})\}, \quad |q|\to\infty$$

with

$$h(J)=[\pi^{1/2}\Gamma(J+1)]/[2^{J+1}\Gamma(J+\tfrac{3}{2})],$$

we obtain from Eqs. (11) and (12) the limit as $-q^2\to\infty$ of $A^{(+)}(q^2,J)$ for $\mathrm{Re}J>1$:

$$A^{(+)}(q^2,J)=[2h(J)/|q|^J][f(J)+g(J)/q^2+r(q^2,J)+O(q^{-4})], \quad (13)$$

as $|q|\to\infty$, with

$$f(J)=\int_0^2\frac{d\omega}{\omega}(i\omega)^J F_1(\omega),$$

$$r(q^2,J)=\int_0^2\frac{d\omega}{\omega}(i\omega)^J R(q^2,\omega), \quad (14)$$

$$g(J)=-f(J+2)\tfrac{1}{2}[(J+2)(J+1)]/(J+3).$$

We now assume $A^{(+)}(q^2,J)$ is meromorphic for $\mathrm{Re}J>-\frac{1}{2}$ so that it may be analytically continued to $J=0$. Therefore, $f(J)$ and $r(q^2,J)$ are meromorphic in J; however, the analytic continuation of these amplitudes may not be defined by (14) for $J\leqslant 1$.[4]

From (9) we find for the subtraction term

$$\begin{aligned}T_1(q^2,\infty)&=F^{(+)}(q^2,0)=\sum_{J=0}^{\infty}(2J+1)A^{(+)}(q^2,J)P_J(0)\\&=A^{(+)}(q^2,0)P_0(0)+5A^{(+)}(q^2,2)P_2(0)+B(q^2), \quad (15)\end{aligned}$$

with $B(q^2)=\sum_{J=4}^{\infty}(2J+1)A^{(+)}(q^2,J)P_J(0)$. It can be shown that $q^2B(q^2)\to 0$ as $-q^2\to\infty$.[5]

Since the integral in (6) is bounded,[4] a necessary condition for a finite δM is $T_1(q^2,\infty)\to 0$ as $-q^2\to\infty$. From (13) and (15) this implies that $A^{(+)}(q^2,0)\to 0$ as $-q^2\to\infty$, and this is true only if $f(0)=0$. If $f(0)\neq 0$, then δM is quadratically divergent. In specific models one may examine whether or not this condition is fulfilled. For example, if we consider Ref. 4 and keep only the Pomeranchon contribution so that $F_1{}^P(\omega)=-C/\omega$, then we find from (14) that $f^P(J)=-e^{i\pi J/2}\times C2^{J-1}/J-1$ for $\mathrm{Re}J>1$. We may now analytically continue this function to $J=0$, $f^P(0)=\frac{1}{2}C\neq 0$. So if the Pomeranchon alone contributes, the mass is quadratically divergent. It is, of course, unrealistic to retain just the Pomeranchon; but the condition $f(0)=0$ requires a remarkable cancellation among all other trajectories,

[3] D. J. Gross and H. Pagels, Phys. Rev. **172**, 1381 (1968).

[4] H. D. I. Abarbanel, M. Goldberger, and S. Treiman, Phys. Rev. Letters **22**, 500 (1969). The behavior of the Regge residues as $-q^2\to\infty$ in this reference is the special case of (13) when evaluated at the Regge pole $J=\alpha_i$. One expects the behavior $F_1(\omega)\sim\omega^{-\alpha_p}$, $F_2(\omega)\sim\omega^{-\alpha_p+1}$ as $\omega\to 0$, with $\alpha_p=1$ the leading trajectory, so that $f(J)$ is not defined by (14) for $J\leqslant 1$.

[5] We will indicate the proof. Writing the sum on J as a contour integral and opening the contour to a line, we have $-2iB(q^2)=\int_{J_0-i\infty}^{J_0+i\infty}dJ\,(2J+1)A^{(+)}(q^2,J)P_J(0)/\sin\pi J$ with $2<J_0<4$. Using Eq. (13) and introducing $iy=J-J_0$, we have, as $-q^2\to\infty$, $B(q^2)=|q|^{-J_0}S(\ln|q|)$, where $S(\ln|q|)=\int_{-\infty}^{+\infty}dy\,e^{-iy\ln|q|}R(y)\times W(y)$. $R(y)$ is defined by $R(y)=(2J+1)h(J)P_J(0)e^{i\pi J/2}/\sin\pi J$, and $W(y)=\int_0^2 d\omega\,F_1(\omega)\omega^{J_0-1-iy}$. Since $\omega^{J_0}F_1(\omega)\to 0$ as $\omega\to 0$ (as is consistent with the proton experiments and Ref. 4), we have $|W(y)|\to|y|^{-1}$ as $|y|\to\infty$, and since $|R(y)|\leqslant$const as $|y|\to\infty$, $\int_{-\infty}^{+\infty}dy\,|R(y)W(y)|^2$ converges. The Riemann-Lebesgue lemma then implies that $S(\ln|q|)\to 0$ as $|q|\to\infty$, so that $q^2B(q^2)\to 0$ as $-q^2\to\infty$.

cuts, and background that one might retain.[6] Another alternative is to drop the analyticity assumption and introduce a Kronecker delta so that[7]

$$f(J) \to f(J) - f(0)\delta_{J0}\,;$$

then $f(0)=0$ is automatic.

Let us assume $f(0)=0$ and examine the question of lower-order divergences. From Eqs. (13)–(15) it follows that

$$\lim_{-q^2\to\infty} q^2 T_1(q^2,\infty) \to 2[h(0)g(0)P_0(0) - 5h(2)f(2)P_2(0) + h(0)q^2 r(q^2,0)P_0(0)] \underset{-q^2\to\infty}{\sim} 2q^2 r(q^2,0)\,,$$

so that the behavior of the subtraction term as $-q^2 \to \infty$ depends only on the nonleading terms. We may distinguish three cases as $-q^2 \to \infty$.

If $q^2 r(q^2,0) \to \infty$, then δM is worse than logarithmically divergent. If $q^2 r(q^2,0) \to 0$, then condition (6) is simply

$$\int_0^2 d\omega[\omega F_1(\omega) + F_2(\omega)] = 0\,. \qquad (16)$$

But for a self-energy the integrand is negative definite, so that $F_1(\omega) = F_2(\omega) = 0$, in contradiction to the assumption of a nontrivial Bjorken limit (and the experiments on the proton). In this case, therefore, we conclude that the self-energies of the hadrons are logarithmically divergent. For mass differences, we lose this conclusion, since $F_{1,2}(\omega)$ are related to differences in cross sections. However, (16) is still a nontrivial condition for a finite δM and can be tested in theoretical models which embrace our general assumptions. For example, in the model of Ref. 4 and that of Drell, Levy, and Yan,[8] Eq. (16) is not, in general, satisfied, and δM is logarithmically divergent.

Finally there is the possibility that $q^2 r(q^2,0) \to C \neq 0$ so (6) is

$$2C + \int_0^\infty d\omega[F_2(\omega) + \omega F_1(\omega)] = 0\,. \qquad (17)$$

If $C<0$, we again conclude that the self-mass of a hadron is logarithmically divergent, but for $C>0$ there may be a cancellation. The question of the sign of C is related to the behavior

$$\lim_{-q^2\to\infty} q^2[(q^2/4\pi^2\alpha)\sigma_T(q^2,\omega) - \omega F_1(\omega)] \to q^2\omega R(q^2,\omega)\,,$$

which is difficult to determine experimentally. Again condition (17) can be examined using specific theoretical models, and in general one would not expect it to be satisfied. An exception is the model of Harari,[9] who has suggested that only the Pomeranchon couples as $-q^2 \to \infty$, the residue functions of other trajectories falling off rapidly. Then (17) is trivially satisfied for mass differences which would then be finite.

The purpose of this paper has been to show that, under the assumptions of J-plane analyticity and the Bjorken limit, one expects self-energies and mass splittings to be at least logarithmically divergent unless very special cancellations occur. Whether such cancellations do occur can be answered in principle by difficult experiments. It is also interesting to examine theoretical models.

If the Bjorken-limit assumption is false, then our conclusions do not follow. This assumption, essentially, amounts to the observations that $q^2\sigma_{T,L}(q^2,\omega)$ is dimensionless and that if (as $-q^2 \to \infty$) there is no elementary length or fundamental mass to scale q^2, then this quantity approaches a (presumably nontrivial) function of the dimensionless variable ω.[10] Should there be a scaling parameter at high $-q^2$, then the Bjorken-limit assumption is probably wrong.

Finally, the Cottingham formula[11] may be the wrong expression for δM, as would be expected if there were a nonelectromagnetic contribution to the mass splittings on if the perturbative approach in $\alpha = 1/137$ failed altogether.

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[6] Such a cancellation is required separately for states of definite isospin in the crossed channel. If this is the case, then $f(J) \sim J$, as $J \to 0$.

[7] This could arise from the exchange of an elementary vacuum tadpole. On this assumption, it is always possible to insert elementary particles giving rise to Kronecker deltas to cancel divergences.

[8] S. D. Drell, D. J. Levy, and T. Yan, Phys. Rev. Letters **22**, 744 (1969).

[9] H. Harari, Phys. Rev. Letters **22**, 1078 (1969).

[10] H. Cheng and T. T. Wu, Phys. Rev. **182**, 1852 (1969).

[11] It should also be mentioned that if $T_{1,2}(q^2,\nu)$ both obey unsubtracted dispersion relations and the Bjorken limit is nontrivial, then δM is quadratically divergent like $\int^{-\infty} dq^2 \int_0^2 d\omega \times F_1(\omega)/\omega$. This would be a good argument that $T_1(q^2,\nu)$ requires at least one subtraction, as is assumed in this paper.