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Sugawara Model, Stress Tensor, and Spectral Sum Rules*

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A prescription is presented for modifying the stress tensor of the Sugawara model in order that it become consistent with general postulates of field theory. The prescription involves redefining the stress tensor as the limit of a spatially nonlocal operator. Within the context of the modified theory, sum rules are derived by considering the vacuum expectation value of equal-time stress-tensor commutators. Variations of our limiting procedure are also considered. It is shown that convergent Weinberg second sum rule in the framework of the Sugawara model leads to a null theory. The existence of a nontrivial Sugawara model leads to a specific behavior of the cross section for positron-electron annihilation into hadrons at high energies.

I. INTRODUCTION

THE success of the current algebra[1] in correlating experimental facts has led to a considerable interest in model theories in which currents appear as coordinates.[2–4] An interesting example of such a model which is relativistic is due to Sugawara.[3] It consists in postulating equal-time commutators for two $SU(3)$ octets of vector and axial-vector currents; the energy-momentum stress tensor $\theta_{\mu\nu}$ is then determined, essentially uniquely, to be a sum of bilinear local products of currents under the requirement that it be a unitary singlet, satisfy the Dirac-Schwinger covariant condition,[5] and be a finite polynomial in currents. The stress tensor in turn gives the equations of motion for the currents. An equivalent Lagrangian formulation of the Sugawara model has been found[6–8] and the model has been extended to include various forms of symmetry breaking.[6,8,9] Furthermore, with a view toward testing the theory, sum rules have been derived which might be compared with experiment.[10] But the recent demonstration[11] that the Sugawara model leads to a particle spectrum consisting of parity doublets of degenerate masses makes the physical relevance of the Sugawara model questionable, although the degeneracy in mass is not valid when symmetry is broken in a specified way.

We shall be concerned with the Sugawara theory not as a physical theory but as a model. Difficulties in computing any physical quantity in the framework of the Sugawara model are already known. Our purpose is to study the Sugawara model to get an understanding of some of the formal problems associated with a model with currents as coordinates and their possible solutions.

With Sugawara's expression for $\theta_{\mu\nu}$ and the current commutators, one can compute the equal-time commutators involving components of the stress tensor. All of these commutators contain terms involving at most first derivatives of spatial δ functions. But, in general, the spectral representation of the vacuum expectation value (VEV) of an equal-time stress-tensor commutator (ETSC) leads to not only first derivatives but third and fifth derivatives[12] of δ functions. The absence of the leading Schwinger terms, the third derivatives of δ functions, means the spectral function is identically zero and the theory does not exist. The genesis of the problem is that $\theta_{\mu\nu}$ is a sum of local products of two operators which is ill defined. One is familiar with such difficulties in the definition of current as a local product of two field operators.

In this paper we present a prescription for formulating the Sugawara model so that the ETSC's contain the necessary Schwinger terms. Basically, our prescription is to redefine $\theta_{\mu\nu}$ as a suitable limit of a sum of nonlocal products of operators. The analogous case of currents is well known[13] and has been treated elegantly and in detail by Brandt.[14]

In Sec. II we give our prescription[15] after briefly stating the basic equations of the unmodified Sugawara model and discuss how the equations are modified by our prescription. In Sec. III we evaluate an ETSC using the nonlocal expression for the stress tensor and give arguments for using a particular type of limiting pro-

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[1] S. L. Adler and R. F. Dashen, *Current Algebra and Application to Particle Physics* (W. A. Benjamin, Inc., New York, 1968).
[2] R. F. Dashen and D. H. Sharp, Phys. Rev. **165**, 1857 (1968); D. H. Sharp, *ibid.* **165**, 1867 (1968); C. G. Callan, R. F. Dashen, and D. H. Sharp, *ibid.* **165**, 1883 (1968); W. J. Pardee, L. Schlessinger, and J. Wright, *ibid.* **175**, 2140 (1968); D. J. Gross, *ibid.* **177**, 1843 (1969).
[3] H. Sugawara, Phys. Rev. **170**, 1659 (1968).
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[5] P. A. M. Dirac, Rev. Mod. Phys. **34**, 592 (1962); J. Schwinger, Phys. Rev. **130**, 406 (1963); **130**, 800 (1963).
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[7] K. Bardakci and M. B. Halpern, Phys. Rev. **172**, 1542 (1968); B. Sakita, Ref. 4.
[8] H. Sugawara and M. Yoshimura, Phys. Rev. **173**, 1419 (1968).
[9] H. Sugawara, Phys. Rev. **21**, 772 (1968); D. J. Gross and M. B. Halpern, *ibid.* **179**, 1436 (1969).
[10] D. J. Gross, Phys. Rev. Letters **21**, 308 (1968); D. J. Gross and C. G. Callan, *ibid.* **21**, 311 (1968); H. Grotch, Phys. Rev. (to be published).
[11] R. F. Dashen and Y. Frishman, Phys. Rev. Letters **22**, 572 (1969).
[12] D. G. Boulware and S. Deser, J. Math. Phys. **8**, 1468 (1967); K. T. Mahanthappa, Phys. Rev. **181**, 2087 (1969).
[13] J. Schwinger, Phys. Rev. Letters **3**, 296 (1959).
[14] R. A. Brandt, Phys. Rev. **166**, 1795 (1968).
[15] A prescription similar to ours has been used in connection with the stress tensor of free fermion field theories by S. Coleman, D. J. Gross, and R. Jackiw, Phys. Rev. **180**, 1359 (1969). See also K. T. Mahanthappa, Ref. 12.

cedure. The derivation of spectral sum rules as a consequence of the limiting procedure is given in Sec. IV. We conclude, in Sec. V, with comments on variations from our limiting procedure and Weinberg's second sum rule[16] and with an expression for the high-energy behavior of the cross section for $e^+e^- \to$ hadrons.

II. MODIFIED MODEL

The symmetric Sugawara model is defined by first postulating the equal-time commutators of vector $(V_\mu{}^a)$ and axial-vector $(A_\mu{}^a)$ currents $(x_0=x_0')$:

$$\begin{aligned}
[V_0{}^a(x),V_0{}^b(x')]&=[A_0{}^a(x),A_0{}^b(x')]\\
&=if_{abc}V_0{}^c(x)\delta(\mathbf{x}-\mathbf{x}'),\\
[V_0{}^a(x),V_k{}^b(x')]&=[A_0{}^a(x),A_k{}^b(x')]\\
&=if_{abc}V_k{}^c(x)\delta(\mathbf{x}-\mathbf{x}')\\
&\quad+iC\delta_{ab}\partial_k{}^{(x)}\delta(\mathbf{x}-\mathbf{x}'),\\
[V_0{}^a(x),A_k{}^b(x')]&=[A_0{}^a(x),V_k{}^b(x')]\\
&=if_{abc}A_k{}^c(x)\delta(\mathbf{x}-\mathbf{x}'),\\
[V_k{}^a(x),V_l{}^b(x')]&=[A_k{}^a(x),A_l{}^b(x')]\\
&=[A_k{}^a(x),V_l{}^b(x')]=0,
\end{aligned}\tag{2.1}$$

where $a,b,c=1, \cdots, 8$, f_{abc} are $SU(3)$ structure constants, and C is a finite c number. The stress tensor is obtained by requiring it to be a finite polynomial and obey the Dirac Schwinger covariant condition.[5] It is given by

$$\theta_{\mu\nu}(x)=(1/2C)\{[V_\mu{}^a(x),V_\nu{}^a(x)]_+ -g_{\mu\nu}V_\lambda{}^a(x)V^{a\lambda}+(V\to A)\}. \tag{2.2}$$

The Heisenberg equations of motion are

$$\partial^\mu V_\mu{}^a(x)=0=\partial^\mu A_\mu{}^a(x), \tag{2.3a}$$

$$\partial_\mu V_\nu{}^a(x)-\partial_\nu V_\mu{}^a(x)=(1/2C)f_{abc}\{[V_\mu{}^b(x),V_\nu{}^c(x)]_+ +[A_\mu{}^b(x),A_\nu{}^c(x)]_+\}, \tag{2.3b}$$

$$\partial_\mu A_\nu{}^a(x)-\partial_\nu A_\mu{}^a(x)=(1/2C)f_{abc}\{[V_\mu{}^b(x),A_\nu{}^c(x)]_+ +[A_\mu{}^b(x),V_\nu{}^c(x)]_+\}. \tag{2.3c}$$

For the reasons mentioned in the Introduction, the stress tensor has to be defined as a limit of a spatially nonlocal operator. In analogy with the case of currents,[13,14] we redefine the stress tensor as

$$\theta_{\mu\nu}(x)=\lim_{\xi\to0}[\theta_{\mu\nu}(x;\xi)-\langle0|\theta_{\mu\nu}(x;\xi)|0\rangle], \tag{2.4a}$$

where[17] $\xi=(0,\boldsymbol{\xi})$ and

$$\begin{aligned}\theta_{\mu\nu}(x;\xi)=[f(\xi)/4C]\{&[V_\mu{}^a(x),V_\nu{}^a(x+\xi)]_+\\ &+[V_\mu{}^a(x+\xi),V_\nu{}^a(x)]_+\\ &-g_{\mu\nu}[V_\lambda{}^a(x),V^{a\lambda}(x+\xi)]_+ +(V\to A)\};\end{aligned}\tag{2.4b}$$

$f(\xi)$ is a function of ξ and the vanishing of $f(0)$ is chosen to cancel singularities of local products like $V_\mu{}^a(x)V^{a\mu}(x)$.[18] We shall suppress the appearance of the function $f(\xi)$ until we come to Sec. III. Commutators involving $\theta_{\mu\nu}(x)$ are to be calculated using $\theta_{\mu\nu}(x;\xi)$ and then the limit $\xi\to0$ is to be taken.

The equations of motion for currents are assumed to be given by

$$\partial_\mu J(x)=i\lim_{\xi\to0}\int d\mathbf{x}'[\theta_{0\mu}(x_0,\mathbf{x}';\xi),J(x)], \tag{2.5}$$

where J represents either vector or axial-vector current. Equation (2.5) gives consistent results if

$$[J_{b\mu}(x+\xi),J_{c\nu}(x)]_+=[J_{b\mu}(x),J_{c\nu}(x-\xi)]_+ \quad \text{as} \quad \xi\to0.$$

With this condition as a part of our prescription, the equations of motion for the currents become

$$\partial^\mu V_\mu{}^a(x)=0=\partial^\mu A_\mu{}^a(x), \tag{2.6a}$$

$$\begin{aligned}\partial_\mu V_\nu{}^a(x)-\partial_\nu V_\mu{}^a(x)&\\ =\lim_{\xi\to0}(1/8C)f_{abc}\{&[V_\mu{}^b(x+\xi),V_\nu{}^c(x)]_+\\ &+[V_\mu{}^b(x-\xi),V_\nu{}^c(x)]_+ +[V_\mu{}^b(x),V_\nu{}^c(x-\xi)]_+\\ &+[V_\mu{}^b(x),V_\nu{}^c(x+\xi)]_+ +(V\to A)\},\end{aligned}\tag{2.6b}$$

$$\begin{aligned}\partial_\mu A_\nu{}^a(x)-\partial_\nu A_\mu{}^a(x)&\\ =\lim_{\xi\to0}(1/8C)f_{abc}\{&[A_\mu{}^b(x+\xi),V_\nu{}^c(x)]_+\\ &+[A_\mu{}^b(x-\xi),V_\nu{}^c(x)]_+ +[A_\mu{}^b(x),V_\nu{}^c(x-\xi)]_+\\ &+[A_\mu{}^b(x),V_\mu{}^c(x+\xi)]_+ +(V\to A)\}.\end{aligned}\tag{2.6c}$$

Using the equations of motion, one can determine the form of various equal-time commutators of the stress tensor with the currents $(x_0=x_0')$:

$$\begin{aligned}[\theta_{00}(x),J_0{}^a(x')]&=\lim_{\xi\to0}[\theta_{00}(x;\xi),J_0{}^a(x')]\\ &=iJ_k{}^a(x)\partial_k{}^{(x)}(\mathbf{x}-\mathbf{x}'),\end{aligned}\tag{2.7a}$$

$$\begin{aligned}[\theta_{00}(x),J_k{}^a(x')]&=\lim_{\xi\to0}[\theta_{00}(x;\xi),J_k{}^a(x')]\\ &=iJ_0{}^a(x')\partial_k{}^{(x)}\delta(\mathbf{x}-\mathbf{x}')\\ &\quad-i\partial_0J_k{}^a(x)\delta(\mathbf{x}-\mathbf{x}'),\end{aligned}\tag{2.7b}$$

$$\begin{aligned}[\theta_{0k}(x),J_0{}^a(x')]&=\lim_{\xi\to0}[\theta_{0k}(x;\xi),J_0{}^a(x')]\\ &=iJ_0{}^a(x)\partial_k{}^{(x)}\delta(\mathbf{x}-\mathbf{x}'),\end{aligned}\tag{2.7c}$$

$$\begin{aligned}[\theta_{0k}(x),J_m{}^a(x')]&=iJ_k{}^a(x')\partial_m{}^{(x)}\delta(\mathbf{x}-\mathbf{x}')\\ &\quad-i\partial_k{}^{(x)}J_m{}^a(x)\delta(\mathbf{x}-\mathbf{x}').\end{aligned}\tag{2.7d}$$

There appear additional terms of the form

$$\lim_{\xi\to0}f_{abc}[J^b(x+\xi),J^c(x)]_+(\delta(\mathbf{x}+\boldsymbol{\xi}-\mathbf{x}')-\delta(\mathbf{x}-\mathbf{x}')) \tag{2.8}$$

in the commutators (2.7). If we assume that

$$\lim_{\xi\to0}[J(x),J(x+\xi)]_+-\langle0|[J(x),J(x+\xi)]_+|0\rangle$$

[16] S. Weinberg, Phys. Rev. Letters **18**, 507 (1967).

[17] We use the notations and conventions by Bjorken and Drell [J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill Book Co., New York, 1965)].

[18] See, for example, R. A. Brandt, Ann. Phys. (N. Y.) **44**, 221 (1967); W. Zimmermann, Commun. Math. Phys. **6**, 161 (1967).

exists, we have

$$\lim_{\xi\to 0} \xi^n[J(x),J(x+\xi)]_+ = \lim_{\xi\to 0} \xi^n\langle 0|[J(x),J(x+\xi)]_+|0\rangle \quad (2.9)$$

with $n>0$, and hence we can replace (2.8) by terms like that on the right-hand side of (2.9) involving vacuum expectation values. But the internal symmetric nature of these terms lead to vanishing of their vacuum expectation values (VEV). The fact that commutators (2.7) are the same as those derived in a model-independent way[19] lends credence to the validity of Eq. (2.9). Commutators (2.7) are also the same as those found in the unmodified Sugawara model and hence we can continue from here on in the same fashion as before[3] to show that P_μ and $M_{\mu\nu}$ constitute the Poincaré algebra and that the currents have the correct transformation properties under the Lorentz group.

III. STRESS-TENSOR COMMUTATORS

Now the stress-tensor commutators are given by

$$[\theta_{\mu\nu}(x),\theta_{\lambda\sigma}(x')] = \lim_{\xi,\xi'\to 0} [\theta_{\mu\nu}(x;\,\xi),\theta_{\lambda\sigma}(x';\,\xi)]. \quad (3.1)$$

In order to compute ETSC's we can make use of commutators (2.7) and hence write ($x_0=x_0'$)

$$[\theta_{\mu\nu}(x),\theta_{\lambda\sigma}(x')] = \lim_{\xi\to 0}[\theta_{\mu\nu}(x;\,\xi),\theta_{\lambda\sigma}(x')]. \quad (3.2)$$

Consider ($x_0=x_0'$)

$$[\theta_{00}(x),\theta_{0m}(x')] = \lim_{\xi\to 0} [\theta_{00}(x;\,\xi),\theta_{0m}(x')], \quad (3.3a)$$

where

$$\begin{aligned}
[\theta_{00}(x;\,\xi),\theta_{0m}(x')] = (i/4C)\{&[V_0^a(x),V_0^a(x')]_+\partial_m^{(x)}\delta(\mathbf{x}+\boldsymbol{\xi}-\mathbf{x}') \\
&+[V_0^a(x+\xi),V_0^a(x')]_+\partial_m^{(x)}\delta(\mathbf{x}-\mathbf{x}') \\
&+[V_k^a(x),V_m^a(x')]_+\partial_k^{(x)}\delta(\mathbf{x}+\boldsymbol{\xi}-\mathbf{x}') \\
&+[V_k^a(x+\xi),V_m^a(x')]_+\partial_k^{(x)}\delta(\mathbf{x}-\mathbf{x}') \\
&+[V_k^a(x),\partial_m V_k^a(x+\xi)-\partial_k V_m^a(x+\xi)]_+\delta(\mathbf{x}+\boldsymbol{\xi}-\mathbf{x}') \\
&+[V_k^a(x+\xi),\partial_m V_k^a(x)-\partial_k V_m^a(x)]_+\delta(\mathbf{x}-\mathbf{x}') \\
&+(V\to A)\}. \quad (3.3b)
\end{aligned}$$

After some manipulations, one obtains ($x_0=x_0'$)

$$[\theta_{00}(x),\theta_{0m}(x')] = i(\theta_{00}(x')\delta_{km}+\theta_{km}(x))\partial_k^{(x)}\delta(\mathbf{x}-\mathbf{x}') + \lim_{\xi\to 0}\tau(x,x';\,\xi), \quad (3.4a)$$

with

$$\begin{aligned}
\tau(x,x';\,\xi) = (i/4C)f(\xi)\{&([V_0^a(x),V_0^a(x+\xi)]_+\delta_{mk} \\
&+[V_k^a(x),V_m^a(x+\xi)]_+)\partial_k^{(x)}(\delta(\mathbf{x}+\boldsymbol{\xi}-\mathbf{x}')+\delta(\mathbf{x}-\mathbf{x}')) \\
&+([V_0^a(x),\partial_m V_0^a(x+\xi)]_+ + [V_k^a(x),\partial_m V_k^a(x+\xi)]_+) \\
&\times(\delta(\mathbf{x}+\boldsymbol{\xi}-\mathbf{x}')-\delta(\mathbf{x}-\mathbf{x}'))+(V\to A)\} \\
&+i\langle 0|\theta_{00}(x';\,\xi)\delta_{km}+\theta_{km}(x;\,\xi)|0\rangle\partial_k^{(x)}(\mathbf{x}-\mathbf{x}'), \quad (3.4b)
\end{aligned}$$

where we have explicitly exhibited the presence of the function $f(\xi)$. Using Eq. (2.9), we can show that $\tau(x,x';\,\xi)=\langle 0|\tau(x,x';\,\xi)|0\rangle$.

At the outset there is nothing in the model to dictate the form of $f(\xi)$. But Bardakci *et al.* (BFH)[6] have shown that the massive Yang-Mills theory[20] in a certain formal limit reduces to the Sugawara model. The massive Yang-Mills Lagrangian is given by

$$L(x) = -\tfrac{1}{4}F_{\mu\nu}{}^a(x)F^{a\mu\nu}(x)+\tfrac{1}{2}m_0^2\varphi_\mu{}^a(x)\varphi^{a\mu}(x), \quad (3.5a)$$

where

$$F_{\mu\nu}{}^a = \partial_\mu\varphi_\nu{}^a-\partial_\nu\varphi_\mu{}^a-\tfrac{1}{2}g_0 f_{abc}(\varphi_\mu{}^b\varphi_\nu{}^b+\varphi_\nu{}^c\varphi_\mu{}^b). \quad (3.5b)$$

Let

$$\lambda = m_0^2/g_0,\quad C=m_0^2/g_0^2. \quad (3.5c)$$

The hadronic currents are assumed to be given by

$$J_\mu{}^a=\lambda\varphi_\mu{}^a.$$

In the BFH limit

$$m_0\to 0,\quad g_0\to 0,\quad C=\text{const}$$

we get the Sugawara model. We shall look at this limit in a slightly different way to arrive at the nature of $f(\xi)$. We are dealing with local product of operators. In analogy with soluble models and perturbation theory and also what we have done in Sec. II, we assume that they are appropriate limits of nonlocal products. Then the mass term in (3.5a) becomes

$$\lim_{\xi\to 0}\tfrac{1}{2}[m_0(\xi)]^2\varphi_\mu{}^a(x+\xi)\varphi^{a\mu}(x), \quad (3.6)$$

where $m_0(0)$ is assumed to cancel the singularities of the local product $\varphi_\mu{}^a(x)\varphi^{a\mu}(x)$. Appropriate modifications have to be made in equations of motion and commutators involving local bilinear product of fields.[21] In order to get the BFH limit, we write

$$m_0(\xi)=m_0h(\xi),\quad g_0(\xi)=g_0[h(\xi)]^2,$$
$$C(\xi)=[m_0(\xi)]^2/[g_0(\xi)]^2=C/[h(\xi)]^2$$

[with $h(\xi)\to 0$ as $\xi\to 0$], and let $m_0\to 0$, $g_0\to 0$, and $C=$const. Under this procedure,[22] the existence of (3.6) leads to the existence of

$$\lim_{\xi\to 0}\frac{h(\xi)}{2C}J_\mu{}^a(x+\xi)J^{\mu a}(x). \quad (3.7)$$

Note that the BFH limit and $\xi\to 0$ limit are to be taken in that order. The dimensionality of the product $\varphi_\mu(x+\xi)\varphi^\mu(x)$ is l^{-2} and hence we take its leading singularity on the light cone to be like ξ^{-2}. This leads us to have $h(\xi)\sim\xi^2$ and hence $f(\xi)\sim\xi^2$. The leading singularities of the product of operators being given by

[19] R. Jackiw, Phys. Rev. **175**, 2058 (1968); D. J. Gross and R. Jackiw, *ibid.* **163**, 1688 (1967).

[20] C. N. Yang and R. L. Mills, Phys. Rev. **96**, 191 (1954).

[21] A similar procedure has been used by J. D. Bjorken and R. A. Brandt [Phys. Rev. **177**, 2331 (1969)] to get "minimal algebra." But we differ from them in that we keep the limiting procedure to get local products fron nonlocal ones and the BFH limit separate.

[22] This procedure is by no means unique.

the dimensional consideration is not true in general. But it is true in perturbation theory and in models and is suggested by axiomatic results[18]; and it follows if there is exact scale invariance.[23] In Sec. IV we study the consequence[24] of $f(\xi)\sim\xi^2$ and mention the consequences of having $f(\xi)\sim\xi^n$ $(n\geqslant 0)$ in Sec. V.

IV. SPECTRAL SUM RULES

Now we take the VEV of Eq. (3.4) to derive a set of spectral sum rules. For this purpose we write the Källen-Lehmann spectral representation for conserved vector currents as

$$\langle 0|[V_\mu{}^a(x),V_\nu{}^b(0)]_+|0\rangle = -\delta_{ab}\int_0^\infty ds\rho^V(s)\left(g_{\mu\nu}+\frac{\partial_\mu\partial_\nu}{s}\right)\Delta^{(1)}(x;s)\,, \quad (4.1)$$

where $\Delta^{(1)}(x;s)$ is the even, invariant solution to a Klein-Gordon equation of mass $s^{1/2}$. There exists a corresponding expression for axial-vector currents. We first evaluate $\langle 0|\tau(x,x';\xi)|0\rangle$ by expanding $\delta(\mathbf{x}+\boldsymbol{\xi}-\mathbf{x}')$ about $\xi=0$ and obtain an expression involving ξ_m, $\xi_m\xi_n$, $\xi_m\xi_n\xi_p$, etc. Because the limit $\xi\to 0$ should be taken in a rotationally invariant way to preserve spatial symmetry, we should have

$$\begin{aligned}\xi_m\xi_n &\to a(\boldsymbol{\xi}\cdot\boldsymbol{\xi})\delta_{mn}\,,\\ \xi_m\xi_n\xi_l\xi_p &\to b(\boldsymbol{\xi}\cdot\boldsymbol{\xi})^2(\delta_{mn}\delta_{lp}+\delta_{ml}\delta_{np}+\delta_{mp}\delta_{nl})\,, \quad (4.2)\\ \xi_m\xi_n\xi_l\xi_p\xi_q\xi_r &\to d(\boldsymbol{\xi}\cdot\boldsymbol{\xi})^3(\delta_{mn}\delta_{lp}\delta_{qr}+\text{permutations})\,,\end{aligned}$$

and

$$\xi_m\xi_n\cdots\to 0$$

for any product of an odd number of ξ's, where a, b, and d are pure numbers and are undetermined. They need not necessarily take the values obtained by averaging with a spatially symmetric function with unit integral (that is, $a=\frac{1}{3}$, $b=1/15$, $d=1/168$). Using (4.2), we arrive at

$$\begin{aligned}\langle 0|\tau(x,x';\xi)|0\rangle &= \frac{2i}{\pi^2C}\frac{f(\xi)}{\boldsymbol{\xi}\cdot\boldsymbol{\xi}}(1-3a)\int ds\,\rho^V(s)\partial_m{}^{(x)}\\ &\times\delta(\mathbf{x}-\mathbf{x}')+\frac{2i}{\pi^2C}\frac{f(\xi)}{\boldsymbol{\xi}\cdot\boldsymbol{\xi}}\Big[\tfrac{1}{4}\boldsymbol{\xi}\cdot\boldsymbol{\xi}(a-6b)\int ds\,\rho^V(s)\\ &-10b\int ds\,s^{-1}\rho^V(s)\Big]\Delta\partial_m{}^{(x)}\delta(\mathbf{x}-\mathbf{x}')-\frac{28i}{\pi^2C}f(\xi)d\\ &\times\int ds\,s^{-1}\rho^V(s)\Delta^2\partial_m{}^{(x)}\delta(\mathbf{x}-\mathbf{x}')+(V\to A)\,. \quad (4.3)\end{aligned}$$

We have not included terms which obviously vanish as $\xi\to 0$.

Expression (4.3) may be simplified by using Weinberg's sum rules.[16] Taking the VEV of commutators (2.1), we get the first sum rule:

$$\int ds\,s^{-1}\rho^V(s)=\int ds\,s^{-1}\rho^A(s)=C\,. \quad (4.4)$$

The second sum rule is obtained by considering the VEV of the commutator $(x_0=x_0')$

$$\begin{aligned}&[V_i{}^a(x),\partial_jV_0{}^b(x')-\partial_0V_j{}^b(x')]\\ &=[A_i{}^a(x),\partial_jA_0{}^b(x')-\partial_0A_j{}^b(x')]\\ &=\lim_{\xi\to 0}\frac{if(\xi)}{8C}f_{bcd}f_{ade}([V_j{}^c(x+\xi),V_i{}^e(x)]_+\\ &+[V_j{}^c(x),V_i{}^e(x+\xi)]_+)\\ &\times(\delta(\mathbf{x}-\mathbf{x}')+\delta(\mathbf{x}+\boldsymbol{\xi}-\mathbf{x}'))+(V\to A)\,. \quad (4.5)\end{aligned}$$

We have made use of the equations of motion to arrive at Eq. (4.5). The VEV of Eq. (4.5) gives

$$\begin{aligned}&\delta(\mathbf{x}-\mathbf{x}')i\delta_{ab}g_{ij}\int\rho^V(s)ds=\delta(\mathbf{x}-\mathbf{x}')i\delta_{ab}g_{ij}\\ &\times\int\rho^A(s)ds=\lim_{\xi\to 0}\frac{3i}{8\pi^2}\delta_{ab}\frac{f(\xi)}{C}\\ &\times\int ds[\rho^V(s)+\rho^A(s)]\Big[\frac{2}{s(\boldsymbol{\xi}\cdot\boldsymbol{\xi})^2}(s\boldsymbol{\xi}\cdot\boldsymbol{\xi}+2+a)\\ &\times\delta(\mathbf{x}-\mathbf{x}')g_{ij}-\frac{1}{s(\boldsymbol{\xi}\cdot\boldsymbol{\xi})}[\tfrac{1}{2}as(\boldsymbol{\xi}\cdot\boldsymbol{\xi})+4b]\Delta\delta(\mathbf{x}-\mathbf{x}')g_{ij}\\ &+\frac{8b}{s(\boldsymbol{\xi}\cdot\boldsymbol{\xi})}\partial_i{}^{(x)}\partial_j{}^{(x)}\delta(\mathbf{x}-\mathbf{x}')\Big]\,. \quad (4.6)\end{aligned}$$

Taking $f(\xi)=\pi^2NC\boldsymbol{\xi}\cdot\boldsymbol{\xi}$, where N is a dimensionless constant (π^2 is included merely for convenience), in the limit $\xi\to 0$ we obtain from Eq. (4.6) the relation

$$\int ds\,\rho^V(s)=\int ds\,\rho^A(s)=\tfrac{3}{2}NC(2+a)/(1-\tfrac{3}{4}N)\frac{1}{\boldsymbol{\xi}\cdot\boldsymbol{\xi}}\,. \quad (4.7)$$

Combining Eq. (4.4) and Eq. (4.7) with Eq. (4.3) gives, with $f(\xi)=\pi^2NC\boldsymbol{\xi}\cdot\boldsymbol{\xi}$,

$$\begin{aligned}\langle 0|\tau(x,x';\xi)|0\rangle &= \frac{6iN^2C}{\boldsymbol{\xi}\cdot\boldsymbol{\xi}}(1-3a)(2+a)\partial_m{}^{(x)}\delta(\mathbf{x}-\mathbf{x}')\\ &+4iNC[\tfrac{3}{8}N(a-6b)(2+a)/(1-\tfrac{3}{4}N)-10b]\\ &\times\Delta\partial_m{}^{(x)}\delta(\mathbf{x}-\mathbf{x}')-56iN\boldsymbol{\xi}\cdot\boldsymbol{\xi}cd\Delta^2\partial_m{}^{(x)}\delta(\mathbf{x}-\mathbf{x}')\,. \quad (4.8)\end{aligned}$$

Now we have from Eq. (3.4) (for $x_0=x_0'$)

$$\langle 0|[\theta_{00}(x),\theta_{0m}(x')]|0\rangle=\lim_{\xi\to 0}\langle 0|\tau(x,x';\xi)|0\rangle\,. \quad (4.9)$$

[23] K. G. Wilson, Phys. Rev. **179**, 1499 (1969).

[24] Coleman *et al.* (Ref. 15) conclude that $f(\xi)\sim\xi^2$ is necessary for massive free-fermion theories.

By making use of the Källen-Lehmann representation for stress-tensor commutators, we[12] can show that (for $x_0=x_0'$)

$$\langle 0|[\theta_{00}(x),\theta_{0m}(x')]|0\rangle = -i\int_0^\infty ds[\tfrac{4}{3}\pi_2(s)+\pi_0(s)]\Delta\partial_m{}^{(x)}\delta(\mathbf{x}-\mathbf{x}') - i[\tfrac{4}{3}\pi_2(\infty)+\pi_0(\infty)]\Delta^2\partial_m{}^{(x)}\delta(\mathbf{x}-\mathbf{x}'), \quad (4.10)$$

where π_2 and π_0 are spin-2 and spin-0 spectral functions, respectively, and $\pi_i(\infty)=\lim_{s\to\infty}\pi_i(s)$. Comparison of both sides of Eq. (4.9) using Eq. (4.8) and Eq. (4.10) gives

$$a=\tfrac{1}{3}, \quad (4.11a)$$

$$\int_0^\infty ds[\tfrac{4}{3}\pi_2(s)+\pi_0(s)] = 4NC[10b+\tfrac{3}{8}N(2+a)/(1-\tfrac{3}{4}N)(6b-a)], \quad (4.11b)$$

$$\tfrac{4}{3}\pi_2(\infty)+\pi_0(\infty)=0. \quad (4.11c)$$

Equation (4.11a) gives the same value for a as that obtained by averaging with a spatially symmetric function with unit integral. If we assume that b takes its spatially averaged values 1/15, then Eq. (4.11b) becomes

$$\int_0^\infty ds[\tfrac{4}{3}\pi_2(s)+\pi_0(s)]=\frac{2}{15}NC(8+N)/(4-3N). \quad (4.12)$$

Since the spectral functions π_i are positive definite, Eq. (4.12) implies $0<N<\frac{4}{3}$. Note that Eqs. (4.11c) and (4.12) are consistent with one another. One can use the pole-dominance approximation in Eq. (4.12) to determine the value of N. But at present the relevant coupling constants are not known.

V. COMMENTS

It is evident from the above considerations that, because the Sugawara model has all the pathologies of a local quantum field theory, one should not draw conclusions from naive manipulations of local products of currents. In conclusion we want to make the following remarks: (a) We have used the BFH limit of the massive Yang-Mills theory and dimensionality arguments to conclude $f(\xi)\sim\xi^2$. But if one chooses to have $f(\xi)\sim\xi^n$, the consequences are interesting. With $n=0$ and $n=1$ we can arrive at different spectral sum rules. If $n>2$, we get $\lim_{\xi\to 0}\langle 0|\tau(x,x';\xi)|0\rangle=0$ leading to vanishing π_i $(i=0,2)$, which means the theory does not exist. The existence of Weinberg's second sum rule [Eq. (4.6)] requires $n=4$ and thus leads to a null theory. (b) With our limiting procedure, we can show that a nontrivial Sugawara model $(0\leq n\leq 2)$ leads to a divergent Weinberg's second sum rule [see Eq. (4.6)][25]

$$\int ds\,\rho^V(s)=\int ds\,\rho^A(s)=\infty\,, \quad (5.1)$$

which, when combined with (4.7) and the expression for the total cross section $\sigma_{\text{tot}}(s)$ for $e^+e^-\to$ hadrons,[26]

$$\sigma_{\text{tot}}(s)=\frac{16\pi^3\alpha^2}{s^2}\rho_V{}^{\text{em}}(s)\,, \quad (5.2)$$

leads to[27]

$$\sigma_{\text{tot}}(s)\underset{s\to\infty}{\sim} s^{-2-\epsilon}\quad (0<\epsilon\leq 1). \quad (5.3)$$

It will be interesting to see whether experimental cross section for $e^+e^-\to$ hadrons will show behavior (5.3).

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[25] This result has been obtained by D. Corrigan and J. Kuriyan, University of California, Los Angeles Report (unpublished). Their arrival at it involves manipulations of local products of currents which are highly questionable. The basis of their conclusion is the elimination of an inconsistency which, in fact, disappears in our limiting procedure.

[26] J. D. Bjorken, Phys. Rev. 148, 1467 (1966); J. Dooher, Phys. Rev. Letters 19, 600 (1967).

[27] We have ignored log factors.