T Products at High Energy and Commutators*

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(Received 10 April 1969)

We show that in perturbation theory the high-energy behavior of T products is not, in general, governed by canonical equal-time commutators. Criteria for the applicability of canonical commutators in this limit are developed. Interaction-dependent modifications to commutators are calculated.

I. INTRODUCTION

Some time ago, Bjorken[1] and Johnson and Low[2] pointed out that the high-energy behavior of T products is governed by equal-time commutators (ETC): a result which we shall call the BJL theorem. The usefulness of this theorem lies, not in defining such ETC's through the high-energy limit of T products, but rather to derive the high-energy behavior of the amplitudes with the help of *canonical* commutators.

This has been the route that many authors have followed to study weak and electromagnetic (e.m.) cross sections at high energy, e.m. mass shifts, and to determine the nature of the divergences in e.m. and weak higher-order corrections.

Johnson and Low[2,3] observed, however, in the context of their study of triangle graphs, that *canonical* commutators may be in fact incorrect in describing this asymptotic behavior as calculated in perturbation theory: a state of affairs which shall be referred to in the following as the "failure" of the BJL theorem. Furthermore, Vainshtein and Ioffe[4] exhibited the failure of the BJL theorem for the leading asymptotic term in the spin-dependent Compton amplitude (off the photon mass shell) in pseudoscalar meson theory.

Recently, we have shown[5] that the tensor structure of the high-energy asymptotic form of the forward Compton amplitude, as calculated in perturbation theory with scalar, pseudoscalar, or vector mesons, does not coincide with the predictions of the BJL theorem and canonical commutators. This investigation disproved the Callan-Gross electroproduction sum rule[6] in the perturbative framework for quark models. These results were independently arrived at by Adler and Tung,[7] who also did the calculation of Vainshtein and Ioffe in vector-meson theory. They found the BJL theorem to fail here as well, thus they invalidated various perturbative investigations of radiative corrections to β decay.[8]

The purpose of this paper is to examine further the validity of the BJL theorem and/or canonical commutators.

In Sec. II we rederive this theorem in a simple fashion, suitable for purposes of studying its validity. We shall show that, in general, it should be satisfied: (a) when the operator whose momentum is getting large is the *time* component of either a conserved or of a "partially" conserved axial-vector current (PCAC), and (b) when the commutators of this operator with the other operators in the T product are well defined. It will be further demonstrated that this desirable state of affairs is a consequence of the constraints that gauge invariance and/or PCAC impose on the theory. On the other hand, the space component of a current will be seen to violate the BJL theorem, in the general case.

Section III will be devoted to a catalogue of examples of successes and failurs of the BJL theorem. Although contemporary investigations have exhibited failures in rather complicated matrix elements, we shall present other failures drawn from simple, classic results about field theory. Indeed, we shall demonstrate that the anomalous behavior of the Compton amplitude is closely related to these simple failures. From this investigation, a simple diagrammatic criterion for the validity of the BJL theorem will emerge.

In Sec. IV we shall inquire to what extent canonical commutators can be modified so that the BJL theorem is valid *by definition*. We shall see that time components

* This work is supported in part by the Office of Naval Research under Contract No. Nonr-1866(55).

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[1] J. D. Bjorken, Phys. Rev. **148**, 1467 (1966).

[2] K. Johnson and F. E. Low, Progr. Theoret. Phys. (Kyoto) Suppl. **37–38**, 74 (1966).

[3] The modification of commutators from their canonical value in the BJL context has also been studied by B. Hamprecht, Nuovo Cimento **50A**, 449 (1967), and by J. C. Polkinghorne, *ibid.* **52A**, 351 (1967). The latter author gave a graphical analysis of the BJL theorem, similar to the present discussion in Sec. IIID. The possibility of obtaining noncanonical equal-time commutators in perturbation theory has been demonstrated by several authors, who adopt a special definition for the limiting procedure to equal times. However, it is not clear that commutators calculated by this method are physically significant. For examples of such investigations, see R. A. Brandt, Phys. Rev. **166**, 1795 (1968); R. A. Brandt and C. A. Orzalesi, *ibid.* **162**, 1747 (1967); B. Schroer and P. Stichel, Commun. Math. Phys. (Germany) **3**, 258 (1966); and J. Langerholc and B. Schroer, *ibid.* **4**, 123 (1967). A critique of the point of view expressed by these investigators is to be found in J. S. Bell and R. Jackiw, Nuovo Cimento **60**, 47 (1969), and D. G. Boulware and R. Jackiw, Phys. Rev. **186**, 1442 (1969).

[4] A. I. Vainshtein and B. L. Ioffe, Zh. Eksperim. i Teor. Fiz. Pis'ma v Redaktsiyu **6**, 917 (1967) [English transl.: Soviet Phys.—JETP Letters **6**, 341 (1967)].

[5] R. Jackiw and G. Preparata, Phys. Rev. Letters **22**, 975 (1969); **22**, 1162(E) (1969); Phys. Rev. **185**, 1748 (1969).

[6] C. Callan and D. Gross, Phys. Rev. Letters **22**, 156 (1969).

[7] S. L. Adler and Wu Ki Tung, Phys. Rev. Letters **22**, 978 (1969).

[8] For a summary, see G. Preparata and W. Weisberger, Phys. Rev. **175**, 1965 (1968).

of conserved and PCAC currents satisfy canonical commutators. However, in perturbation theory, other commutators cannot be maintained. For example, one finds that space-space current commutators possess interaction-dependent terms. Similarly, canonical field commutators are not maintained.

Concluding remarks comprise Sec. V.

II. BJL THEOREM

Consider a covariant T^* product of two operators A and B between arbitrary states:

$$T_{AB}^*(q)\equiv\int d^4x\, e^{iqx}\langle\alpha|T^*A(x)B(0)|\beta\rangle. \tag{2.1}$$

We assume that T_{AB}^* is a well-defined object, as will be the case, for example, for scattering amplitudes in a renormalizable field theory. In position space, T_{AB}^* will be composed of an ordinary T product, which is undefined at equal times, but otherwise well defined. Also, one needs to specify the values at equal times by introducing δ functions and derivatives thereof. In momentum space, these "seagulls" go over into a polynomial in q_0. By dropping all polynomials in q_0 from $T_{AB}^*(q)$, we may unambiguously isolate the ordinary T product

$$T_{AB}(q)=\int d^4x\, e^{iqx}\langle\alpha|TA(x)B(0)|\beta\rangle. \tag{2.2}$$

Performing an integration by parts on x_0 [9] in (2.2) and dropping surface terms now gives

$$T_{AB}(q)=\frac{i}{q_0}\int d^3x\, e^{-i\mathbf{q}\cdot\mathbf{x}}\langle\alpha|[A(0,\mathbf{x}),B(0)]|\beta\rangle +\frac{i}{q_0}\int d^4x\, e^{iqx}\langle\alpha|T\dot{A}(x)B(0)|\beta\rangle. \tag{2.3}$$

The dot indicates time differentiation. The BJL theorem now follows if we assert that $\langle\alpha|T\dot{A}(x)B(0)|\beta\rangle$ is well defined. It will then, as a function of x_0, be at most discontinuous, but possess no δ functions in x_0. Under this assumption,

$$T_{A_tB}(q)\equiv\int d^4x\, e^{iqx}\langle\alpha|T\dot{A}(x)B(0)|\beta\rangle \tag{2.4}$$

decreases with large q_0, and one finds

$$\lim_{q_0\to\infty} q_0T_{AB}(q_0,\mathbf{q}) =i\int d^3x\, e^{-i\mathbf{q}\cdot\mathbf{x}}\langle\alpha|[A(0,\mathbf{x}),B(0)]|\beta\rangle. \tag{2.5}$$

In Eq. (2.4), and subsequently, the subscript t, appearing on a subscripted variable, indicates time differentiation. Clearly, a prerequisite for the above is the existence of

$$C_{AB}(\mathbf{x})\equiv\langle\alpha|[A(0,\mathbf{x}),B(0)]|\beta\rangle. \tag{2.6}$$

However, if $T_{A_tB}(q)$ does not exist, one cannot conclude from the above that $T_{AB}(q)$ satisfies (2.5). Indeed, all the examples of failures of the BJL theorem which have been encountered have the property that $T_{A_tB}(q)$ is not well defined, although $C_{AB}(\mathbf{x})$ may be finite.

Consider now the case when A is the time component of a current, $A=j^0$. Evidently $\dot{A}=\partial^ij^i+\partial_\mu j^\mu$. Although we assume that $T^\mu(q)\equiv T_{j^\mu B}$ is well defined, $C_{j^0B}(\mathbf{x})$ will, in general, contain diverging Schwinger terms. However, we may remain with finite quantities if we set $\mathbf{q}$ equal to zero. From (2.3) it follows then that

$$T^0(q_0,\mathbf{0})=(i/q_0)\langle\alpha|[Q,B(0)]|\beta\rangle +(i/q_0)T_{j_t^0B}(q_0,\mathbf{0}), \tag{2.7a}$$

$$Q\equiv\int d^3x\, j^0(x), \tag{2.7b}$$

$$T_{j_t^0B}(q_0,0)=\int d^4x\, e^{iq_0x_0}\langle\alpha|T\partial_\mu j^\mu(x)B(0)|\beta\rangle. \tag{2.7c}$$

For conserved currents $T_{j_t^0B}$ vanishes, and the BJL theorem holds. For PCAC currents, we may interpret PCAC as the statement that $\partial_\mu j^\mu$ is a gentle operator, such that its matrix elements are finite. Then $T_{j_t^0B}$ again exists and the BJL theorem holds.

The validity of the BJL theorem for time components, established above, can be easily related to the constraints placed by conservation and/or PCAC. Consider again

$$T^{*\mu}(q)=\int d^4x\, e^{iqx}\langle\alpha|T^*j^\mu(x)B(0)|\beta\rangle. \tag{2.8a}$$

By virtue of the gauge conditions of the theory, which lead to conservation and/or PCAC, $T^{*\mu}$ satisfies the formal divergence condition (Ward identity)

$$q_\mu T^{*\mu}(q)=i\int d^3x\, e^{-i\mathbf{q}\cdot\mathbf{x}}C_{j^\mu B}(\mathbf{x}) +i\int d^4x\, e^{iqx}\langle\alpha|T^*\partial_\mu j^\mu(x)B(0)|\beta\rangle, \tag{2.8b}$$

where Schwinger terms and divergences of seagulls necessarily cancel by virtue of the gauge condition. Now it is not *a priori* certain that these formal (naive) Ward identities will survive the vicissitudes of perturbation theory. Indeed, there exist Ward identities which *cannot* be maintained in the usual perturbative solution. These, however, have been catalogued elsewhere,[10] and for the

[9] This approach to the BJL limit was developed in conversations with Dr. I. Gerstein.

[10] I. Gerstein and R. Jackiw, Phys. Rev. 181, 1955 (1969).

present we shall assume the validity of (2.8b). It then follows, upon evaluation of (2.8b) in the q rest frame, that (2.7a) is true for the covariant T^* product, and the BJL theorem is satisfied, *a fortiori*, for the T product, when $\mathbf{q}$ is zero.

Let us now consider A to be j^i, $\dot{A}=\dot{j}^i$. Even though $\int d^3x\, C_{\dot{j}^iB}(\mathbf{x})$ may be well defined, $\langle\alpha|T\dot{j}^i(x)B(0)|\beta\rangle$ will not, in general, have finite matrix elements, since $\dot{j}^i$, unlike $\dot{j}^0$, is not, in general, a gentle operator. For example, when

$$j^i=\bar{\psi}\gamma^i\psi\,, \tag{2.9a}$$

$$i\gamma_\mu\partial^\mu\psi=O\psi\,, \tag{2.9b}$$

where O is some local operator, then

$$\dot{j}^i=(\partial^j\bar{\psi})\gamma^j\gamma^0\gamma^i\psi+\bar{\psi}\gamma^i\gamma^0\gamma^j(\partial^j\psi) +i\bar{\psi}(O\gamma^0\gamma^i-\gamma^i\gamma^0O)\psi\,. \tag{2.9c}$$

It is seen that the expression for $\dot{j}^i$ involves operators which are not, in general, finite in renormalized perturbation theory, in sharp contrast to the case for $\dot{j}^0$. Thus we do not expect the BJL theorem to hold in this instance. It is also seen that one cannot construct Ward identities to yield useful information about $\dot{j}^i$.

III. APPLICATIONS OF BJL THEOREM IN PERTURBATION THEORY

The failures of the BJL theorem, which have been the focus of recent attention, were exhibited for the Compton amplitude in second-order perturbation theory[4,5,7] However, we shall now demonstrate that very familiar examples of failure exist even in simpler matrix elements. We shall first discuss these, and then turn to those of the Compton amplitude. It will be seen that the latter are closely related to the former.

Our investigation will involve second-order perturbation theory in a theory of fermions interacting with a massive vector meson B_μ through the conserved current $j_\mu=\bar{\psi}\gamma_\mu\psi$. We deal here only with one type of quark, ψ; the extension of our discussion to the case of an internal SU_3 space is completely straightforward. The relevant canonical anticommutator is

$$[\psi(x),\bar{\psi}(y)]_+\delta(x_0-y_0)=\gamma^0\delta^4(x-y)\,. \tag{3.1}$$

Other canonical commutators which follow from (3.1) are

$$[\psi(x),j^\mu(y)]\delta(x_0-y_0)=\gamma^0\gamma^\mu\psi(x)\delta^4(x-y)\,, \tag{3.2}$$

$$[j^0(x),j^\mu(y)]\delta(x_0-y_0)=0\,, \tag{3.3}$$

$$[j^i(x),j^j(y)]\delta(x_0-y_0)=i\epsilon^{ijk}j_5{}^k(x)\delta^4(x-y)\,. \tag{3.4}$$

The axial current $j_5{}^\mu=i\bar{\psi}\gamma^\mu\gamma^5\psi$ has been introduced. It also satisfies the commutators

$$[\psi(x),j_5{}^\mu(y)]\delta(x_0-y_0)=i\gamma^0\gamma^\mu\gamma^5\psi(x)\delta^4(x-y)\,, \tag{3.5}$$

$$[j_5{}^0(x),j_5{}^\mu(x)]\delta(x_0-y_0)=0\,, \tag{3.6}$$

$$[j_5{}^i(x),j_5{}^j(y)]\delta(x_0-y_0)=i\epsilon^{ijk}j_5{}^k(x)\delta^4(x-y)\,, \tag{3.7}$$

$$[j_5{}^0(x),j^\mu(y)]\delta(x_0-y_0)=0\,, \tag{3.8}$$

$$[j_5{}^i(x),j^j(y)]\delta(x_0-y_0)=i\epsilon^{ijk}j^k(x)\delta^4(x-y)\,. \tag{3.9}$$

In order to simplify the discussion and to remove unnecessary divergences, we shall frequently set the fermion and boson masses equal to each other and to zero. Thus we will not need to perform mass renormalization to obtain finite results. Also, we shall always work in the Landau gauge,[11] in which the renormalization constants $Z_1=Z_2\equiv Z$ are finite, so that the unrenormalized expressions which we calculate are equally finite. It is seen that our model possesses chiral symmetry and, within the context of our calculations, the axial current is conserved. The field equation of motion is

$$i\gamma_\mu\partial^\mu\psi=-g\gamma_\mu B^\mu\psi\,. \tag{3.10}$$

In this model we shall need the expressions for Z and various unrenormalized and renormalized quantities. The latter are indicated by a tilde. The expressions can be easily obtained from the relevant formulas of quantum electrodynamics, for example, those given by Bogoliubov and Shirkov.[12] The formulas we need are[13]

$$Z=1-3g^2/32\pi^2\,, \tag{3.11}$$

$$\Sigma(p)=-(3g^2/32\pi^2)\gamma_\mu p^\mu\,, \tag{3.12a}$$

$$\tilde{\Sigma}(p)=0\,, \tag{3.12b}$$

$$G(p)=Zi/\gamma_\mu p^\mu\,, \tag{3.12c}$$

$$\tilde{G}(p)=i/\gamma_\mu p^\mu\,, \tag{3.12d}$$

$$\Gamma^\mu(p,q)=Z^{-1}[\gamma^\mu+\tilde{\Lambda}^\mu(p,q)]\,, \tag{3.13a}$$

$$\tilde{\Lambda}^\mu(p,q)\xrightarrow[q\to\infty,\ p\text{ fixed}]{}-\frac{g^2}{8\pi^2}\left(g^{\mu\nu}-\frac{q^\mu q^\nu}{q^2}\right)\gamma_\nu\,, \tag{3.13b}$$

$$\Gamma_5{}^\mu(p,q)=i\gamma^5\Gamma^\mu(p,q)\,, \tag{3.14a}$$

$$\Gamma_5{}^\mu(p,q)=Z^{-1}[i\gamma^5\gamma^\mu+\tilde{\Lambda}_5{}^\mu(p,q)]\,. \tag{3.14b}$$

We now enumerate examples of the BJL theorem.

A. Fermion Propagator

Consider the unrenormalized fermion propagator

$$G(p)=\int d^4x\, e^{ipx}\langle\Omega|T\psi(x)\bar{\psi}(0)|\Omega\rangle\,. \tag{3.15}$$

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[11] It should be clear that the Landau gauge is available even for the massive vector-boson theory. The reason for this is that the $k_\mu k_\nu$ portion of the boson propagator will have no observable consequences, since the boson couples to a conserved current.

[12] N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields* (Wiley-Interscience Inc., New York, 1959).

[13] In the massive case, the formula which we exhibit for $\Sigma(p)$ is exact at large p, while the formula for $\Lambda^\mu(p,q)$ at large q is unmodified.

According to the BJL theorem, we have from (3.1)

$$G(p_0,\mathbf{p}) \xrightarrow[p_0\to\infty]{} \frac{i\gamma_0}{p_0}. \tag{3.16a}$$

Since G is Lorentz-covariant and depends only on the vector p, the above must be expressible covariantly as

$$G(p) \xrightarrow[p\to\infty]{} \frac{i}{\gamma_\mu p^\mu}, \tag{3.16b}$$

which is just the well-known formal result about the asymptotic behavior of unrenormalized propagators. The above obviously fails in general gauges since $G(p)$ does not exist. Also, the renormalized propagator $\tilde{G}(p)$, which according to (3.16b) should be asymptotically proportional to $(\gamma_\mu p^\mu)^{-1}$, decreases in perturbation theory in general gauges only as $(\ln p^2)/\gamma_\mu p^\mu$. Hence, here too the BJL theorem fails. However, this failure is not directly related to divergences, as is seen by comparing (3.16b) to the *finite* result in the Landau gauge (3.12c). Since according to (3.11) $Z\neq 1$, the BJL theorem fails even though the asymptotic limit is finite.

To exhibit the reason for the failure, we return to (2.3). By use of the equation of motion (3.10), we have

$$G(p_0,\mathbf{0}) = \frac{i\gamma^0}{p_0} - \frac{g\gamma^0}{p_0}\int d^4x e^{ip_0x_0}\langle\Omega|T\gamma_\mu B^\mu(x)\psi(x)\bar{\psi}(0)|\Omega\rangle. \tag{3.17a}$$

The remainder integral is *superficially* linearly divergent, and this infinity spoils the BJL theorem in the present application. Detailed analysis shows that in fact the integral is finite since

$$g\int d^4x\, e^{ipx}\langle\Omega|T\gamma_\mu B^\mu(x)\psi(x)\bar{\psi}(0)|\Omega\rangle = -i\Sigma(p)G(p), \tag{3.17b}$$

which according to (3.12a) and (3.12c) contains no divergences. Although one would expect $i\Sigma(p)G(p)$ to decrease with p, since it is a Fourier transform of a T product, which formally has only step-function singularities in x_0, in fact, according to (3.12a) and (3.12c), this quantity is constant. Evidently, the superficial linear divergence asserts itself in this subtle fashion and makes the remainder term of the BJL theorem exactly comparable to the formally dominant term. Indeed, from (3.17a), (3.17b), and (3.12a), it follows that to second order in g,

$$G(p_0,\mathbf{0}) = \frac{i\gamma^0}{p_0}[1+\Sigma(p)S(p)] = \frac{i\gamma^0}{p_0}\left(1-\frac{3g^2}{32\pi^2}\right), \tag{3.18a}$$

$$G(p) = (1-3g^2/32\pi^2)i/\gamma_\mu p^\mu. \tag{3.18b}$$

This is precisely the correct value (3.12c), and the remainder term restores the full asymptotic form of $G(p)$.

We may also understand how these problems arise, even though everything is finite. Recall that $\Sigma(p)$ is finite in the Landau gauge only by *formal convention*. Indeed, in the evaluation of $\Sigma(p)$ one encounters as the dominant part of the integrand $\int d^4r\, \gamma_\mu r^\mu/r^4$. This linear divergence is zero only by the *convention* of symmetric integration. (The logarithmically divergent subdominant part of the integrand is absent.) This convention about linearly divergent integrals invalidates the formal results. The relevance of linear divergences to the failure of the BJL theorem will be exhibited again in Sec. IV.[14]

B. Three-Point Function

We define the (improper) unrenormalized three-point function

$$F^\mu(p,q) = \int d^4x\, d^4y \times e^{ipx}e^{-ipy}\langle\Omega|T\psi(x)\bar{\psi}(y)j^\mu(0)|\Omega\rangle, \tag{3.19a}$$

$$F^\mu(p,q) = G(p)\Gamma^\mu(p,q)G(q), \tag{3.19b}$$

which satisfies the Ward identity

$$i(p_\mu - q_\mu)F^\mu(p,q) = G(p) - G(q), \tag{3.20a}$$

$$iF^\mu(p,p) = \partial^\mu G(p). \tag{3.20b}$$

It is well known that this Ward identity can be maintained in perturbation theory. The technique of integration by parts, introduced in connection with the general two-point function, (2.3) can be readily extended to three-point functions. The BJL limit now asserts that

$$F^\mu(p,q) \xrightarrow[p_0\to\infty,\ q\ \text{fixed}]{} \frac{i\gamma^0\gamma^\mu}{p_0}G(q), \tag{3.21a}$$

$$F^\mu(p,q) \xrightarrow[q_0\to\infty,\ p\ \text{fixed}]{} \frac{i\gamma^\mu\gamma^0}{q_0}G(p). \tag{3.21b}$$

We now inquire whether Eqs. (3.21) are verified in perturbation theory. We have for (3.21b)

$$F^\mu(p,q) = G(p)\Gamma^\mu(p,q)G(q) = G(p)\tilde{\Gamma}^\mu(p,q)\tilde{G}(q) \xrightarrow[q_0\to\infty,\ p\ \text{fixed}]{} G(p)\left[\gamma^\mu - \frac{g^2}{8\pi^2}\left(g^{\mu\nu}-\frac{q^\mu q^\nu}{q^2}\right)\gamma_\nu\right]\frac{i\gamma^0}{q_0} \to G(p)[\gamma^\mu-(g^2/8\pi^2)(\gamma^\mu-g^{\mu0}\gamma^0)]i\gamma^0/q_0. \tag{3.22a}$$

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[14] This is in striking similarity to the anomalies of the axial current which have been recently exhibited. There, too, a linear divergence was responsible for noncanonical behavior; see J. S. Bell and R. Jackiw, Nuovo Cimento **60**, 47 (1969). The similarity is not accidental; the axial-current anomaly can be cast in the form of an anomalous, noncanonical current commutator; see R. Jackiw and K. Johnson, Phys. Rev. **182**, 1459 (1969).

Use has been made of (3.12d), (3.13a), and (3.13b). It is seen from a comparison of (3.21b) with (3.22a) that the BJL theorem is satisfied for the time component, and it fails for the space component, though again everything is finite:

$$F^i(q,p) \xrightarrow[q_0\to\infty,\ p\text{ fixed}]{} \left(1-\frac{g^2}{8\pi^2}\right)\frac{iG(p)\gamma^i\gamma^0}{q^0}. \quad (3.22b)$$

The validity for the time component is related to the Ward identity (3.20a). In the q rest frame, (3.20a) demands

$$F^0(p,q_0\mathbf{0}) = \frac{iG(p)}{q_0} - \frac{iG(q)}{q_0} - \frac{p_\mu F^\mu(p,q_0\mathbf{0})}{q_0}. \quad (3.22c)$$

Since $G(q)$ and $F^\mu(p,q_0\mathbf{0})$ are themselves $O(1/q_0)$, (3.22c) becomes equivalent to (3.21b) at $\mathbf{q}=0$.

No further information is obtained from (3.21a). (Completely analogous results hold for the axial three-point function.) The remainder terms in this three-point function application of the BJL theorem are considerably more complicated than those of the two-point function, and we do not pursue the analysis further.

C. Compton Amplitude

We now turn to the matrix element which contemporary investigations[4,5,7] have shown to violate the BJL limit and invalidate many of its applications.[6,8] Consider[15]

$$T^{\mu\nu}(q) = \int d^4x\, e^{iqx}\langle p|Tj^\mu(x)j^\nu(0)|p'\rangle. \quad (3.23)$$

The BJL prediction for this amplitude is

$$T^{\mu\nu}(q) \xrightarrow[q_0\to\infty]{} \frac{i}{q_0}\int d^3x \times e^{-i\mathbf{q}\cdot\mathbf{x}}\langle p|[j^\mu(0,\mathbf{x})j^\nu(0)]|p'\rangle. \quad (3.24)$$

From (3.24), (3.3), and (3.4) it follows that

$$\mu=0 \quad\text{or}\quad \nu=0,\quad T^{\mu\nu}(q) \xrightarrow[q_0\to\infty]{} \frac{1}{q_0}\times 0, \quad (3.25a)$$

$$T^{ij}(q) \xrightarrow[q_0\to\infty]{} \frac{\epsilon^{ijk}}{q_0}\langle p|j_5{}^k(0)|p'\rangle. \quad (3.25b)$$

[15] Strictly speaking, our massless theory does not possess matrix elements which can be evaluated on the (zero) mass shell. One may ignore this difficulty by, for example, considering the external states to have nonzero mass. In any case, the off-mass-shell amplitude $M^{\mu\nu}$, introduced in (3.26a), is all we are interested in, and it exists.

We now attempt to verify (3.25a) and (3.25b) in perturbation theory. Our approach, which will expose the fact that (3.25b) is not verified, is closely related to that of Vainshtein and Ioffe,[4] and even more so to that of Adler and Tung.[7] We present it here nevertheless, to exhibit the relation of this violation to the previous ones, which is particularly transparent in the Landau gauge. According to standard perturbation theory, we have

$$T^{\mu\nu}(q) = \bar{u}(p)M^{\mu\nu}(q)u(p'), \quad (3.26a)$$

$$M^{\mu\nu}(q) = M_0{}^{\mu\nu} + M_1{}^{\mu\nu} + M_2{}^{\mu\nu}, \quad (3.26b)$$

$$M_0{}^{\mu\nu}(q) = \gamma^\mu S(p+q)\gamma^\nu + \gamma^\nu S(p'-q)\gamma^\mu, \quad (3.26c)$$

$$\begin{aligned} M_1{}^{\mu\nu}(q) = {}& \gamma^\mu S(p+q)\tilde{\Sigma}(p+q)S(p+q)\gamma^\nu \\ & + \gamma^\nu S(p'-q)\tilde{\Sigma}(p'-q)S(p'-q) \\ & + \tilde{\Lambda}^\mu(p,\, p+q)S(p+q)\gamma^\nu \\ & + \tilde{\Lambda}^\nu(p,\, p'-q)S(p'-q)\gamma^\mu \\ & + \gamma^\mu S(p+q)\tilde{\Lambda}^\nu(p+q,\, p') \\ & + \gamma^\nu S(p'-q)\tilde{\Lambda}^\mu(p'-q,\, p'), \end{aligned} \quad (3.26d)$$

$$\begin{aligned} M_2{}^{\mu\nu}(q) = {}& g^2 \times \int \frac{d^4r}{(2\pi)^4}\gamma^\alpha S(p-r)\gamma^\mu S(p+q-r)\gamma^\nu S(p'-r)\gamma^\beta D_{\alpha\beta}(r) \\ & + g^2\int \frac{d^4r}{(2\pi)^4}\gamma^\alpha S(p-r)\gamma^\nu S(p'-q-r)\gamma^\mu S(p'-r)\gamma^\beta D_{\alpha\beta}(r). \end{aligned} \quad (3.26e)$$

The perturbative expression for the right-hand side of (3.25a) is zero, while for (3.25b) we get

$$\langle p|j_5{}^k(0)|p'\rangle = \bar{u}(p)C^k(p,p')u(p'), \quad (3.27a)$$

$$\begin{aligned} C^\mu(p,p') &= i\gamma^5\gamma^\mu + \tilde{\Lambda}_5{}^\mu(p,p') \\ &= Z[i\gamma^5\gamma^\mu + \Lambda_5{}^\mu(p,p')] \\ &= (1-3g^2/32\pi^2)i\gamma^5\gamma^\mu + \Lambda_5{}^\mu(p,p'). \end{aligned} \quad (3.27b)$$

$M_0{}^{\mu\nu}$ is trivially calculated, while the asymptotic form of the expression with insertions, $M_1{}^{\mu\nu}$, is obtained from the formulas (3.12b) and (3.13b). Lastly, we need to consider the asymptotic form of (3.26e). At large q_0, one would like to replace $S(p+q-r)$ and $S(p'-q-r)$ by $\pm i\gamma^0/q^0$. However, this replacement renders the remaining r integral superficially logarithmically divergent since one loses in this process one power of r in the denominator. Thus a more careful analysis is required. Consider the first of the two integrals in (3.26e), and define the function $S(r) = ir_\mu\gamma^\mu/(r^2-M^2)$. The integral may now be rewritten as

$$g^2\int\frac{d^4r}{(2\pi)^4}\gamma^\alpha S(p-r)\gamma^\mu S(p+q-r)\gamma^\nu S(p'-r)\gamma^\beta D_{\alpha\beta}(r)$$

$$=g^2\int\frac{d^4r}{(2\pi)^4}\gamma^\alpha[S(p-r)-S'(-r)]\gamma^\mu S(p+q-r)\gamma^\nu S(p'-r)\gamma^\beta D_{\alpha\beta}(r)$$
$$+g^2\int\frac{d^4r}{(2\pi)^4}\gamma^\alpha S'(-r)\gamma^\mu S(p+q-r)\gamma^\nu[S(p'-r)-S'(-r)]\gamma^\beta D_{\alpha\beta}(r)$$
$$+g^2\int\frac{d^4r}{(2\pi)^4}\gamma^\alpha S'(-r)\gamma^\mu S(p+q-r)\gamma^\nu S'(-r)\gamma^\beta D_{\alpha\beta}(r).\quad(3.28\text{a})$$

In the first two terms on the right-hand side of (3.28a), one may now replace $S(p+q-r)$ by $i\gamma^0/q_0$, since the remaining integral is now convergent. In the last term we do not make this replacement. Thus we find that the asymptotic form of (3.28a) is, after some recombination,

$$\frac{i}{q_0}g^2\int\frac{d^4r}{(2\pi)^4}\gamma^\alpha S(p-r)\gamma^\mu\gamma^0\gamma^\nu S(p'-r)\gamma^\beta D_{\alpha\beta}(r)$$
$$-\frac{i}{q_0}g^2\int\frac{d^4r}{(2\pi)^4}\gamma^\alpha S'(r)\gamma^\mu\gamma^0\gamma^\nu S'(r)\gamma^\beta D_{\alpha\beta}(r)$$
$$+g^2\int\frac{d^4r}{(2\pi)^4}\gamma^\alpha S'(r)\gamma^\mu S(q+r)\gamma^\nu S'(r)\gamma^\beta D_{\alpha\beta}(r).\quad(3.28\text{b})$$

The second term in (3.28b) is easily seen to be zero in the Landau gauge. One finally gets for the asymptotic behavior of $M_2^{\mu\nu}$

$$M_2^{\mu\nu}(q)\xrightarrow[q_0\to\infty]{}$$
$$\frac{i}{q_0}g^2\int\frac{d^4r}{(2\pi)^4}\gamma^\alpha S(p-r)[\gamma^\mu\gamma^0\gamma^\nu-\gamma^\nu\gamma^0\gamma^\mu]S(p'-r)\gamma_\beta D_{\alpha\beta}(r)$$
$$+g^2\int\frac{d^4r}{(2\pi)^4}\gamma^\alpha S'(r)[\gamma^\mu S(q+r)\gamma^\nu-\gamma^\nu S(q+r)\gamma^\mu]$$
$$\times S'(r)\gamma^\beta D_{\alpha\beta}(r).\quad(3.28\text{c})$$

The first term is zero when $\mu=0$ or $\nu=0$, while for $\mu=i$, $\nu=j$ it is $-(1/q_0)\epsilon^{ijk}\Lambda_5{}^k(p,p')$. The second term can be easily evaluated and its $1/q_0$ asymptotic part extracted. One finds

$$M_2^{\mu\nu}(q)\xrightarrow[q_0\to\infty]{}\left\{\begin{array}{ll}0 & \text{for }\ \mu=0\ \text{ or }\ \nu=0\\ -(\epsilon^{ijk}/q_0)\Lambda_5{}^k(p,p') & \text{otherwise}\end{array}\right\}$$
$$+\frac{i}{q_0}\frac{-g^2}{32\pi^2}(\gamma^\mu\gamma^0\gamma^\nu-\gamma^\nu\gamma^0\gamma^\mu).\quad(3.28\text{d})$$

Combining the asymptotic expressions gives finally

$$M^{\mu\nu}(q)\xrightarrow[q_0\to\infty]{}\frac{1}{q_0}\times 0,\quad\text{for}\quad\mu=0\quad\text{or}\quad\nu=0,$$
$$M^{ij}(q)\xrightarrow[q_0\to\infty]{}\frac{-\epsilon^{ijk}}{q_0}[i\gamma^5\gamma^k+\Lambda_5{}^k(p,p')]$$
$$-\frac{\epsilon^{ijk}}{q_0}\left(-\frac{9g^2}{32\pi^2}i\gamma^5\gamma^k\right).\quad(3.29)$$

Comparison with the formal predictions (3.25b) and (3.27b) shows that

$$M^{ij}(q)\xrightarrow[q_0\to\infty]{}\frac{-\epsilon^{ijk}}{q_0}\left[C^k(p,p')-\frac{3g^2}{16\pi^2}i\gamma^5\gamma^k\right].\quad(3.30)$$

The result agrees with that of Adler and Tung[7] in the Abelian case. It is seen that for time components the BJL theorem is satisfied, as is to be expected from the Ward identity, while it is violated for the space components. This state of affairs clearly follows the general outlines presented in Sec. II.

We also mention for completeness the violation of the BJL theorem for the *forward* spin-averaged Compton amplitude $\bar{T}^{\mu\nu}(p,p)$.[5,7] In this case, the $1/q_0$ term is necessarily absent by crossing symmetry, and the BJL theorem asserts that

$$\bar{T}^{kl}(p,p)\xrightarrow[q_0\to\infty]{}-\frac{1}{q_0{}^2}\int e^{-i\mathbf{q}\cdot\mathbf{x}}\langle p|[j^k(x),j^l(0)]|p\rangle.\quad(3.31)$$

In our model, the commutator may be evaluated from the canonical definition of the current, the equation of motion for ψ, and the canonical commutators. The result has a structure of the form[5,6]

$$\bar{T}^{ij}(p,p)\xrightarrow[q_0\to\infty]{}\frac{-1}{q_0{}^2}[(\delta^{ij}\mathbf{p}^2-p^ip^j)A+\delta^{ij}B].\quad(3.32)$$

Here A and B are Lorentz scalars, hence independent of $\mathbf{p}$. However, an explicit examination of the high-energy behavior of the amplitude calculated in perturbation theory indicates that the asymptotic form of $\bar{T}^{ij}(p,p)$

possesses, in addition to the structure (3.32), a term of the form $\delta^{ij}\mathbf{p}^2B'$, where B' is a scalar. We do not present the details of this calculation as they have been discussed at length elsewhere.[5]

The three examples we have examined—fermion propagator, three-point function, and Compton amplitude—exhibit the fact that as the matrix element becomes more complicated, a new anomaly crops up, and the old anomalies combine with it. Thus the three-point function has the anomalous behavior (3.22b), which according to (3.22a) is a consequence of the anomalies in the fermion propagator G, as well as of those of the proper vertex Γ^μ. Similarly, the anomaly of the Compton amplitude (3.30) is a consequence of the anomalies in G and Γ^μ, as is seen from (3.26d), as well as those of the irreducible structure $M_2^{\mu\nu}$.

D. Graphical Analysis of the BJL Theorem

In Sec. II, we gave general agruments which led to the conclusion that the BJL theorem at zero three-momentum should be valid for time components of conserved and PCAC currents, but not for space components. We shall now, relying on the examples given by the above three calculations, give a different, diagrammatic characterization of the violation of the BJL theorem for arbitrary operators. To develop this characterization consider the diagrammatic representation of the second-order Compton amplitude, given in Fig. 1. In the first three diagrams there is one fermion propagator carrying the momentum q; in the next two, there are two propagators; while the last diagram possesses three such propagators. If one could replace these propagator denominators by $i\gamma^0/q_0$, one would obtain for the $O(1/q_0)$ part of the Compton amplitude the BJL result, diagrammed in Fig. 2. However, when such a replacement is performed, one loses denominators involving the large momenta. In the case of the first diagram the suppression of such a denominator leads to the disappearance of one denominator carrying the variable, and a divergent integral is encountered in spite of the fact that the original diagram is convergent. This means that the high-energy behavior of the diagram need not be given by this manipulation, and the BJL theorem may be violated for the total amplitude.

Evidently this analysis can be performed for any amplitude, and the following criterion for the validity of the BJL limit may be enunciated: If in the diagrammatic representation for the amplitude the above-described replacement procedures leads to finite integrals, the BJL theorem will be valid. If divergent integrals are encountered, an application of the BJL theorem is suspect. It is clear that "finite" and "divergent" here refer to the superficial characteristics of the integral. If a superficially divergent integral is rendered finite by any of the usual tricks, e.g., symmetric integration, gauge-invariant integration, etc; it must, nevertheless, be considered to be divergent.[3]

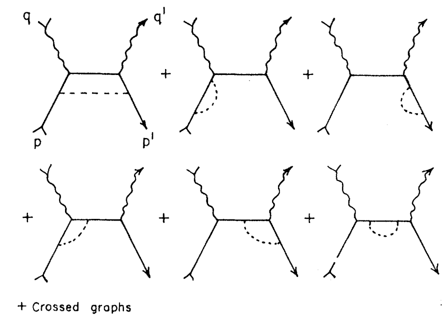

FIG. 1. Second-order Compton amplitude.

E. Vacuum Polarization

We conclude this section with an examination of the vacuum polarization amplitude

$$\Pi^{\mu\nu}(q)=\int d^4x\, e^{ipx}\langle\Omega|T^*j^\mu(x)j^\nu(0)|\Omega\rangle. \quad (3.33)$$

Although this object does not behave in a way comparable to the previously discussed three cases, it does possess interesting properties which we wish to expose, and which will guide us in the subsequent discussion of anomalous commutators.

In order to calculate $\Pi^{\mu\nu}$, we modify our model somewhat in that we introduce a fermion mass so that an energy scale exists. According to (3.3) and the BJL theorem, the associated T product should have no $1/q_0$ part. We now turn to a perturbative calculation of $\Pi^{\mu\nu}$. Unfortunately, a difficulty is encountered in that the relevant Feynman rules lead to a diverging, non-unique, gauge-noninvariant expression. However, it is recognized that all these ambiguities can be collected into a polynomial in q, hence, according to the discussion in Sec. II, should be interpreted as the seagull term which is neglected in considerations of the T product. From the explicit expression for the finite part of $\Pi^{\mu\nu}$, we find therefore, apart from the seagull term,[12]

$$\Pi^{0i}(q)=\frac{1}{(2\pi)^2}q_0q_i\int_0^1 dx\, x(1-x) \times\ln\left|\frac{m^2-x(1-x)q^2}{x(1-x)m^2}\right|. \quad (3.34)$$

It is seen, therefore, that the BJL theorem holds, at zero $\mathbf{q}$, since $\Pi^{0i}(q)$ vanishes then. This, of course, is just a consequence of the divergence condition which is satisfied, and which, as has been repeatedly stated, validates

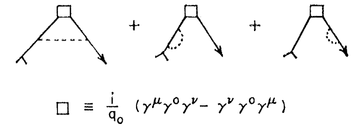

FIG. 2. Formal BJL limit of the second-order Compton amplitude.

the BJL theorem at zero three-momentum. For finite $\mathbf{q}$, however, Π^{0i} diverges for large q_0. Indeed, the coefficient of the $1/q_0$ term behaves asymptotically as

$$\Pi^{0i}(q) \xrightarrow[q_0\to\infty]{} \frac{1}{q_0}\frac{1}{(2\pi)^2}\tfrac{1}{3}q_i q_0^2 \ln\left|\frac{q_0}{m}\right|; \quad (3.35)$$

i.e., it diverges quadratically (up to logarithmic terms).[16] Thus we conclude that the BJL theorem is violated.

This violation is very easy to understand. The presence of the q_i term in the $1/q_0$ asymptote indicates a derivative of the δ function and it is well known that the canonical commutator

$$[j^0(x), j^i(0)]\delta(x_0) = 0 \quad (3.36)$$

is inconsistent with other properties of the theory; this is the famous Schwinger term.[17] Thus the BJL theorem can be reestablished by modifying the canonical commutator by a term proportional to $\partial^i\delta(x)$.[16] Now, also, the quadratic divergence $q_0^2 \ln(q_0/m)$ can be understood: The Schwinger term is quadratically divergent. Of course, compelling reasons, other than the BJL theorem, exist for modifying the canonical commutator: uniqueness of the vacuum, Lorentz invariance, and positivity.[17] But we may take the point of view that the BJL theorem *defines* the commutator, and we shall pursue in Sec. IV this approach for the other three anomalies discussed before.

Note also that although the complete vacuum-polarization tensor (seagull plus T product) diverges in the BJL limit, the divergences cannot be collected into a polynomial seagull. The residual divergence, which is not a polynomial, is interpreted as a divergence in the commutator, which by definition governs the BJL limit. For a differing emphasis, see Adler and Boulware.[18]

IV. ANOMALOUS COMMUTATORS

The high-energy behavior of the vacuum polarization shows that the commutator $[j^0, j^i]$ is different from its canonical value. We shall give now a list of commutators which reproduce, to second order in the interaction, the high-energy behavior of the relevant amplitudes. We want to stress that such expressions may only hold for the particular matrix elements we have studied.

$$[\psi(x), \bar\psi(y)]_+ \delta(x_0 - y_0) = (1 - 3g^2/32\pi^2)\gamma^0\delta^4(x-y), \quad (4.1)$$

$$[\psi(x), j^0(y)]\delta(x_0 - y_0) = \psi(x)\delta^4(x-y), \quad (4.2a)$$

$$[\psi(x), j^i(y)]\delta(x_0 - y_0) = (1 - g^2/8\pi^2) \times\gamma^0\gamma^i\psi(x)\delta^4(x-y), \quad (4.2b)$$

$$[j^0(x), j^0(y)]\delta(x_0 - y_0) = 0, \quad (4.3a)$$

$$[j^0(x), j^i(y)]\delta(x_0 - y_0) = iS\partial^i\delta^4(x-y), \quad (4.3b)$$

$$[j^i(x), j^j(y)]\delta(x_0 - y_0) = i\epsilon^{ijk}(1 - 3g^2/16\pi^2) \times j_5^k(x)\delta^4(x-y). \quad (4.3c)$$

In the above, S must be a quadratically diverging object.[16] We shall assume that it is a c number, an assumption based on the fact that the $0i$ component of the Compton amplitude has no anomalies. Also it is known that the high-energy behavior of the photon-photon scattering amplitude gives no indication of the presence of a q-number Schwinger term.[19]

One may verify that (4.2a) and (4.2b) are consistent with Lorentz covariance. Furthermore, (4.3a)–(4.3c) are consistent with Lorentz covariance.[20]

The fact that (4.2a) and (4.3a) have no modifications proportional to the δ function is again seen to be a consequence of the underlying gauge principles: Since $Q = \int d^3x\, j^0(x)$ is a generator of the gauge transformation, it has fixed commutations with other operators.

We shall now inquire whether there exist reasons other than the BJL theorem which would indicate that the modifications in (4.1)–(4.3) should indeed be made. It will be seen that one can give support to these modifications. We present two methods, which are known to be able to exhibit the extra term in the $[j^0, j^i]$ commutator, and which will give rise to the other anomalies. The methods are both due to Schwinger; they make use of (a) external gauge fields[21] and (b) point splitting.[17] For completeness, we first apply them to a classical calculation of the $[j^0, j^i]$ commutator. These methods are not sufficiently delicate to give a complete calculation of the anomaly. They do, however, permit one to recognize the existence of these anomalies.

A. External Gauge Fields

The theory is extended to include a coupling with an external classical gauge field A_μ through a Lagrange density $j_\mu A^\mu$, and at the end of the calculation A_μ is set to zero. At all stages, invariance is maintained under the transformations

$$\psi \to e^{i\lambda}, \quad A_\mu \to A_\mu + \partial_\mu\lambda. \quad (4.4)$$

Equal-time commutators then reflect the response of the theory to variations of A_μ. Specifically, for any operator O,

$$\int d^3x [O(t,\mathbf{x}), \delta\mathcal{L}(t,\mathbf{x}')] = i[\partial_0\delta O(t,\mathbf{x}) - \delta\dot O(t,\mathbf{x})]. \quad (4.5)$$

[16] If one continues the expansion of $q_0\pi^{0i}(q)$ at high q_0, beyond the leading term (3.35), one encounters an expression proportional to $\mathbf{q}^2 q^i$. In position space this corresponds to a Schwinger term involving *three* derivatives of a δ function. It can be shown that this object is a finite c number. We co not pursue the study of this triple-derivative anomaly here; an investigation can be found in D. G. Boulware and R. Jackiw, Phys. Rev. **186**, 1442 (1969). In that paper it is shown how the techniques of Secs. IV A and IV B can be extended to expose this structure, which properly speaking is present, and should occur on the right-hand side of (4.3b).

[17] J. Schwinger, Phys. Rev. Letters **3**, 296 (1959).

[18] S. L. Adler and D. Boulware, Phys. Rev. **184**, 1740 (1969).

[19] K. Johnson (unpublished); T. Nagylaki, Phys. Rev. **158**, 1534 (1967); D. G. Boulware and R. Jackiw, *ibid.* **186**, 1442 (1969).

[20] D. Gross and R. Jackiw, Phys. Rev. **163**, 1688 (1967).

[21] J. Schwinger, Phys. Rev. **130**, 406 (1963).

First we take O to be j^0 and vary A^i:

$$\delta\mathcal{L}(t,\mathbf{x}')/\delta A_i(t,\mathbf{y}) = j^i(y)\delta(\mathbf{x}'-\mathbf{y}).$$

From (4.5) it follows that

$$[j^0(t,\mathbf{x}), j^i(t,\mathbf{y})] = i\left[\partial^0 \frac{\delta j^0(t,\mathbf{x})}{\delta A_i(t,\mathbf{y})} - \frac{\delta \dot{j}^0(t,\mathbf{x})}{\delta A_i(t,\mathbf{y})}\right]. \quad (4.6)$$

We assume that j^i does not depend on *time* derivatives of A^μ. This implies, since $\dot{j}^0 = \partial^i j^i$, that j^0 does not depend on A^μ. Hence the first term in (4.6) vanishes. To evaluate the second term we define the current $j^\mu = \bar{\psi}\gamma^\mu\psi$ with split points and, to preserve gauge invariance, we introduce an A^μ-dependent exponential. However, since j^0 must not depend on A^μ, we do not split points there. Thus

$$j^0 = \bar{\psi}(x)\gamma^\mu\psi(x), \quad (4.7a)$$

$$j^i = \bar{\psi}(x+\tfrac{1}{2}\epsilon)\gamma^i\psi(x-\tfrac{1}{2}\epsilon)\ \exp i\int_{\mathbf{x}-\epsilon/2}^{\mathbf{x}+\epsilon/2} A_j(y)dy^j. \quad (4.7b)$$

In the above, ϵ has no time component and is to be taken to zero in a symmetric fashion at the end of the calculation. To first order in ϵ and A^μ, (4.7b) becomes

$$j^i = \bar{\psi}(x+\tfrac{1}{2}\epsilon)\gamma^i\psi(x-\tfrac{1}{2}\epsilon)[1+i\epsilon_j A^j(x) + O(\epsilon^2) + O(A^2)]. \quad (4.7c)$$

We finally obtain[16]

$$\frac{\delta \dot{j}^0(t,\mathbf{x})}{\delta A_i(t,\mathbf{y})} = -\frac{\partial}{\partial x^j}\frac{\delta j^j(t,\mathbf{x})}{\delta A_i(t,\mathbf{y})}$$

$$= -i\epsilon^i \frac{\partial}{\partial x^j}[\bar{\psi}(x+\tfrac{1}{2}\epsilon)\gamma^j\psi(x-\tfrac{1}{2}\epsilon)\delta(\mathbf{x}-\mathbf{y})] \quad (4.8)$$

and

$$[j^0(t,\mathbf{x}), j^i(t,\mathbf{y})] = -\epsilon^i\bar{\psi}(y+\tfrac{1}{2}\epsilon)\gamma^j\psi(y-\tfrac{1}{2}\epsilon)\partial_j\delta(\mathbf{x}-\mathbf{y}). \quad (4.9)$$

This verifies (4.3b) when it is assumed that the singular part of $\psi(y+\frac{1}{2}\epsilon)\gamma^j\psi(y-\frac{1}{2}\epsilon)$ is a c number, i.e.,

$$\epsilon^i\bar{\psi}(y+\tfrac{1}{2}\epsilon)\gamma^j\psi(y-\tfrac{1}{2}\epsilon) = \epsilon^i\langle 0|\bar{\psi}(y+\tfrac{1}{2}\epsilon)\gamma^j\psi(y-\tfrac{1}{2}\epsilon)|0\rangle$$
$$= \epsilon^i\langle 0|\bar{\psi}(\epsilon)\gamma^j\psi(0)|0\rangle$$
$$= i\epsilon^i\epsilon^j f(\epsilon^2) = \tfrac{1}{3}i\delta^{ij}f(\epsilon^2)\epsilon^2 \quad (4.10a)$$

and

$$S = -\tfrac{1}{3}f(\epsilon^2)\epsilon^2. \quad (4.10b)$$

Next we take O to be ψ and vary A^i again. From (4.5), we get

$$[\psi(t,\mathbf{x}), j^i(t,\mathbf{y})] = i\left[\partial_0\frac{\delta\psi(t,\mathbf{x})}{\delta A_i(t,\mathbf{y})} - \frac{\delta\partial^0\psi(t,\mathbf{x})}{\delta A_i(t,\mathbf{y})}\right]. \quad (4.11)$$

Since ψ obviously does not involve A_i, the first term in the brackets is zero. For the second term, we need the equation of motion for ψ. The naive equation is

$$\partial^0\psi = -\gamma^0\gamma^i\partial_i\psi + i\gamma^0\gamma^\mu A_\mu\psi + ig\gamma^0\gamma^\mu B_\mu\psi, \quad (4.12a)$$

FIG. 3. Linearly divergent contribution to $\gamma^0\langle\Omega|T\gamma_\mu B^\mu(x-\eta')\psi(x-\eta)\bar{\psi}(0)|\Omega\rangle$.

which, because of the products occurring on the right-hand side of (4.12a), is ill defined. To give a regularized definition, we again need to split points. The interaction with the external field needs no such point splitting since A^μ is a c-number variable. On the other hand, the strong interaction must be modified since B^μ is a q-number variable. We replace $B_\mu(x)\psi(x)$ by $B_\mu(x-\eta')\psi(x-\eta)$, where η and η' have no time component. To maintain gauge covariance of the equation of motion an exponential must be added. Thus the regulated equation becomes

$$\partial^0\psi(x) = -\gamma^0\gamma^i\partial_i\psi(x) + i\gamma^0\gamma^\mu A_\mu(x)\psi(x) + ig\gamma^0\gamma^\mu B_\mu(x-\eta')\psi(x-\eta)\ \exp i\int_{\mathbf{x}-\eta}^{\mathbf{x}} A_i(y)dy^i \quad (4.12b)$$

or, to first order in η and A^μ,

$$\partial^0\psi(x) = -\gamma^0\gamma^i\partial_i\psi(x) + i\gamma^0\gamma^\mu A_\mu(x)\psi(x) + ig\gamma^0\gamma^\mu B_\mu(x-\eta')\psi(x-\eta) \times[1+i\eta_j A^j(x) + O(\eta^2) + O(A^2)], \quad (4.12c)$$

$$\partial^0\bar{\psi}(x) = -\partial_i\bar{\psi}(x)\gamma^i\gamma^0 - i\bar{\psi}(x)\gamma^\mu\gamma^0 A_\mu(x) - ig\bar{\psi}(x+\eta)B_\mu(x+\eta')\gamma^\mu\gamma^0 \times[1+i\eta_j A^j(x) + O(\eta^2) + O(A^2)]. \quad (4.12d)$$

Therefore from (4.11) and (4.12c), we get

$$[\psi(t,\mathbf{x}), j^i(t,\mathbf{y})] = \{\gamma^0\gamma^i\psi(x) + i\eta^i g\gamma^0\gamma_\mu B^\mu(x-\eta')\psi(x-\eta)\} \times\delta(\mathbf{x}-\mathbf{y}). \quad (4.13)$$

The first term in the curly brackets is seen to be the naive term. The second would be absent if η could be set to zero with impunity. This cannot be done if $g\gamma^0\gamma_\mu B^\mu(x-\eta')\psi(x-\eta)$ possesses linearly divergent matrix elements as $\eta, \eta' \to 0$. That such divergences are indeed present is seen from the lowest-order matrix element $g\gamma^0\langle\Omega|T\gamma_\mu B^\mu(x-\eta')\psi(x-\eta)\bar{\psi}(0)|\Omega\rangle$, which has the diagrammatic representation of Fig. 3.

The precise value of this anomaly seems to depend on the relation between η and η' as they approach zero. We do not pursue this question here any further beyond calling attention to the fact that linear divergences were also found to be responsible for the anomalies in the T products; compare (3.17). Thus we have exhibited the $[j^i,\psi]$ anomaly, though with the present technique we have not succeeded in evaluating it uniquely. Of course, the value for the anomaly determined by the T-product method of Sec. III is unique.

Lastly, we take O to be j^i and vary A^j. Again (4.5) implies that

$$[j^i(t,\mathbf{x}), j^j(t,\mathbf{y})] = i\left[\partial^0\frac{\delta j^i(t,\mathbf{x})}{\delta A_i(t,\mathbf{y})} - \frac{\delta \dot{j}^i(t,\mathbf{x})}{\delta A_j(t,\mathbf{y})}\right]. \quad (4.14)$$

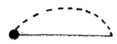

FIG. 4. Linearly divergent contribution to $\langle p|\bar{\psi}(x+\frac{1}{2}\epsilon)\gamma^i\gamma^0\gamma^\mu B_\mu(x-\eta'-\frac{1}{2}\epsilon)\psi(x-\eta-\frac{1}{2}\epsilon)|p'\rangle$.

The first expression in the right-hand bracket is a time derivative of the Schwinger term. Since we have assumed the Schwinger term to be a c number, it is time-independent and that term vanishes. Use of (4.7c) for $\dot{j}^i$ implies that

$$\frac{\delta \dot{j}^i(t,\mathbf{x})}{\delta A_j(t,\mathbf{y})}=i\partial^0[\bar{\psi}(x+\tfrac{1}{2}\epsilon)\gamma^i\psi(x-\tfrac{1}{2}\epsilon)]\epsilon^j+\frac{\delta\partial^0\bar{\psi}(t,\mathbf{x}+\frac{1}{2}\epsilon)}{\delta A_j(t,\mathbf{y})}\gamma^i\psi(x-\tfrac{1}{2}\epsilon)+\bar{\psi}(x+\tfrac{1}{2}\epsilon)\gamma^i\frac{\delta\partial^0\psi(t,\mathbf{x}-\frac{1}{2}\epsilon)}{\delta A_j(t,\mathbf{y})}. \quad (4.15)$$

The term in the bracket is again the usual Schwinger term and vanishes upon time differentiation. [Alternatively, it cancels against the Schwinger term contribution to (4.14).] The remaining variations may be evaluated from (4.12c) and (4.12d). The result is

$$\begin{aligned}[j^i(t,\mathbf{x}),j^j(t,\mathbf{y})]&=i\epsilon^{ijk}j_5{}^k(t,\mathbf{x})\delta(\mathbf{x}-\mathbf{y})\\&+ig\eta^j[\bar{\psi}(x+\tfrac{1}{2}\epsilon)\gamma^i\gamma^0\gamma^\mu B_\mu(x-\eta'-\tfrac{1}{2}\epsilon)\psi(x-\eta-\tfrac{1}{2}\epsilon)\\&-\bar{\psi}(x+\tfrac{1}{2}\epsilon+\eta)B_\mu(x+\eta'+\tfrac{1}{2}\epsilon)\gamma^\mu\gamma^0\gamma^i\psi(x-\tfrac{1}{2}\epsilon)],\end{aligned} \quad (4.16)$$

where $j_5{}^k$ is now definied with an ϵ separation. Thus, in addition to the naive commutator, there exists a further term which is interaction-dependent and which contributes if there are linear divergences present. Again we do not pursue the specific value of this term, beyond noting that linearly diverging matrix elements exist. Consider the contribution to

$$\langle p|\bar{\psi}(x+\tfrac{1}{2}\epsilon)\gamma^i\gamma^0\gamma^\mu B_\mu(x-\eta'-\tfrac{1}{2}\epsilon)\psi(x-\eta-\tfrac{1}{2}\epsilon)|p'\rangle$$

diagrammed in Fig. 4.

Obviously, an external guage field derivation of $[\bar{\psi},\psi]_+$ does not exist. We shall treat this case separately in Sec. IV C.

It should be emphasized that the present argument does not depend on the details of vector-meson (B^μ) coupling. Analogous results will hold for scalar and pseudoscalar mesons coupled to the fermions.

B. Split-Point Technique

Rather than turn on external fields, it is also possible to substantiate the anomalies by a careful application of *reliable* canonical commutators. Let us recall how current commutators are conventionally derived. One decomposes the two currents into the constituent fields and uses canonical field commutators to evaluate the current commutator. However, we must reject this method since we have seen that field commutators are anomalous. One can avoid use of field commutators if only one current is decomposed into fields and use is made of field-current commutators. This can be reliable as long as one uses only commutations of the time component of the current. Space-component commutators cannot be used since they, too, are unreliable. We shall now derive most of the anomalous commutators using only the reliable commutator

$$[j^0(t,\mathbf{x}),\psi(t,\mathbf{y})]=-\psi(x)\delta(\mathbf{x}-\mathbf{y}). \quad (4.17)$$

We begin, as before, with the classical example—the Schwinger term. Use of (4.17), together with the split-point definition for j^i (4.7b) (A^μ is now zero, of course), easily is seen to give

$$\begin{aligned}[j^0(t,\mathbf{x}),j^i(t,\mathbf{y})]&=-\psi(y+\tfrac{1}{2}\epsilon)\gamma^i\psi(y-\tfrac{1}{2}\epsilon)[\delta(\mathbf{x}-\mathbf{y}-\tfrac{1}{2}\epsilon)-\delta(\mathbf{x}-\mathbf{y}-\tfrac{1}{2}\epsilon)]\\&=-j^i(y)[\epsilon^j\partial_j\delta(x-y)+O(\epsilon^2)].\end{aligned} \quad (4.18)$$

This agrees with (4.9) when $\epsilon^i\epsilon^j$ is averaged to $\frac{1}{3}\delta^{ij}\epsilon^2$.

Next consider ψ with j^i. As we wish to determine only the term in the commutator proportional to the δ function, we take

$$\begin{aligned}\int d^3y[\psi(t,\mathbf{x}),j^i(t,\mathbf{y})]&=\int d^3y\ y^i[\partial_j j^j(t,\mathbf{y}),\psi(t,\mathbf{x})]\\&=\int d^3y\ y^i[\psi(t,\mathbf{x}),\dot{j}^0(t,\mathbf{y})]\\&=\int d^3y\ y^i\partial_0[\psi(t,\mathbf{x}),j^0(x,\mathbf{y})]-\int d^3y\ y^i[\partial^0\psi(t,\mathbf{x}),j^0(t,\mathbf{y})]\\&=x^i\partial^0\psi(x)-\int d^3y\ y^i[\partial^0\psi(t,\mathbf{x}),j^0(t,\mathbf{y})].\end{aligned} \quad (4.19a)$$

Use is now made of the regulated equation of motion for $\partial^0\psi$, (4.12c) (with A^μ set to zero), to evaluate the commutator on the right of (4.19a). One finds

$$\int d^3y[\psi(t,\mathbf{x}),j^i(t,\mathbf{y})]=x^i\partial^0\psi(x)+\gamma^0\gamma^j\frac{\partial}{\partial x_j}(x^i\psi(x))-ig(x-\eta)^i\gamma^0\gamma^\mu B_\mu(x-\eta')\psi(x-\eta). \quad (4.19b)$$

Finally, (4.12c) is substituted for $\partial^0\psi$, with the result that

$$\int d^3y[\psi(t,\mathbf{x}),j^i(t,\mathbf{y})]=\gamma^0\gamma^i\psi(x)+i\eta^i g\gamma^0\gamma^\mu B_\mu(x-\eta')\psi(x-\eta). \quad (4.19c)$$

This agrees with (4.13).

For the last application of this technique, we examine

$$\int d^3y[j^i(t,\mathbf{x}),j^i(t,\mathbf{y})]=\int d^3y\; y^j\partial_0[j^i(t,\mathbf{x}),j^0(t,\mathbf{y})]-\int d^3y\; y^j[\dot{j}^i(t,\mathbf{x}),j^0(t,\mathbf{y})]. \tag{4.20a}$$

The first term involves a time derivative of the Schwinger term; we drop it. The second term is evaluated with the help of the split point definition of j^i and the equations of motion for the fields. These give

$$\begin{aligned}\int d^3y[j^i(t,\mathbf{x}),j^j(t,\mathbf{y})]&=\epsilon^j\partial_0[\bar\psi(x+\tfrac{1}{2}\epsilon)\gamma^i\psi(x-\tfrac{1}{2}\epsilon)]\\
&-x^j[\partial^0\bar\psi(x+\tfrac{1}{2}\epsilon)\gamma^i\psi(x-\tfrac{1}{2}\epsilon)-\bar\psi(x+\tfrac{1}{2}\epsilon)\gamma^i\partial^0\psi(x-\tfrac{1}{2}\epsilon)]-x^j\{[\partial_l\bar\psi(x+\tfrac{1}{2}\epsilon)]\gamma^l\gamma^0\gamma^i\psi(x-\tfrac{1}{2}\epsilon)-\bar\psi(x+\tfrac{1}{2}\epsilon)\gamma^i\gamma^0\gamma^l\partial_l\psi(x-\tfrac{1}{2}\epsilon)\\
&+ig\bar\psi(x+\eta+\tfrac{1}{2}\epsilon)B_\mu(x+\eta'+\tfrac{1}{2}\epsilon)\gamma^\mu\gamma^0\gamma^i\psi(x-\tfrac{1}{2}\epsilon)+ig\bar\psi(x-\tfrac{1}{2}\epsilon)\gamma^i\gamma^0\gamma^\mu B_\mu(x-\eta'-\tfrac{1}{2}\epsilon)\psi(x-\eta-\tfrac{1}{2}\epsilon)\}\\
&-\bar\psi(x+\tfrac{1}{2}\epsilon)(\gamma^j\gamma^0\gamma^i-\gamma^i\gamma^0\gamma^j)\psi(x-\tfrac{1}{2}\epsilon)-\tfrac{1}{2}\epsilon^j\{[\partial_l\bar\psi(x+\tfrac{1}{2}\epsilon)]\gamma^l\gamma^0\gamma^i\psi(x-\tfrac{1}{2}\epsilon)+\bar\psi(x+\tfrac{1}{2}\epsilon)\gamma^i\gamma^0\gamma^l\partial_l\psi(x-\tfrac{1}{2}\epsilon)\\
&+ig\bar\psi(x+\eta-\tfrac{1}{2}\epsilon)B_\mu(x+\eta'+\tfrac{1}{2}\epsilon)\gamma^\mu\gamma^0\gamma^i\psi(x-\tfrac{1}{2}\epsilon)-ig\bar\psi(x+\tfrac{1}{2}\epsilon)\gamma^i\gamma^0\gamma^\mu B_\mu(x-\eta'-\tfrac{1}{2}\epsilon)\psi(x-\eta-\tfrac{1}{2}\epsilon)\}\\
&-ig\eta^j[\bar\psi(x+\eta-\tfrac{1}{2}\epsilon)B_\mu(x+\eta'+\tfrac{1}{2}\epsilon)\gamma^\mu\gamma^0\gamma^i\psi(x-\tfrac{1}{2}\epsilon)-\bar\psi(x+\tfrac{1}{2}\epsilon)\gamma^i\gamma^0\gamma^\mu B_\mu(x-\eta'-\tfrac{1}{2}\epsilon)\psi(x-\eta-\tfrac{1}{2}\epsilon)].\end{aligned} \tag{4.20b}$$

When use is made of the equation of motion, it is found that in addition to the time derivative of the Schwinger term, which we drop, there is left

$$\begin{aligned}\int d^3y[j^i(t,x),j^j(t,y)]&\\
=\bar\psi(x+\tfrac{1}{2}\epsilon)[\gamma^i\gamma^0\gamma^j-\gamma^j\gamma^0\gamma^i]\psi(x-\tfrac{1}{2}\epsilon)&\\
+ig\eta^j[\bar\psi(x+\tfrac{1}{2}\epsilon)\gamma^i\gamma^0\gamma^\mu B_\mu(x-\eta'-\tfrac{1}{2}\epsilon)\psi(x-\eta-\tfrac{1}{2}\epsilon)&\\
-\bar\psi(x+\eta+\tfrac{1}{2}\epsilon)B_\mu(x+\eta'+\tfrac{1}{2}\epsilon)\gamma^\mu\gamma^0\gamma^i\psi(x-\tfrac{1}{2}\epsilon)].&\end{aligned} \tag{4.20c}$$

This agrees with (4.16).

Again we cannot use the present technique to study the canonical field commutator. We now examine this object.

C. Field Commutator

Although it has been possible to substantiate the existence of anomalies in commutators involving currents by the above two techniques, the field (anti) commutator $[\psi,\bar\psi]_+$ obviously cannot be handled by those methods; there are no points to split, nor can one vary the external photon field to generate this commutator. However, our direct calculation of $G(p)$ may be used to justify the modification of the equal-time commutator of $[\psi,\bar\psi]_+$. We observe that the Fourier transform of the commutator is proportional to the discontinuity of $G(p)$. Since the latter is modified from its naive value by the factor $(1-3g^2/32\pi^2)$, so must the former, even at equal times.

V. CONCLUSION

We have demonstrated the failure of the BJL theorem in many applications, making contact with previous work and exhibiting some new examples. It has been shown that one may interpret this as evidence for modification of canonical commutators. Additional reasons for modifying the commutators were exhibited.

It should be clear that these failures of the BJL theorem, or alternatively of the canonical commutators, invalidate most of the applications of these techniques, as long as these applications rely on the unreliable commutators. Such applications are typically high-energy theorems of one sort or another. The low-energy theorems are not put into question by the present results, since the latter theorems are mainly a consequence of the gauge-transformation properties of the theory which are maintained in perturbation theory (except for well-defined violations).

It would be of great value to study further anomalies in order to determine whether they are simply of the form given in (4.1)–(4.3c), or whether they are more complicated, and merely reduce to the present formulas for the matrix elements we considered. Also one would like to know whether or not higher-order perturbation theory modifies the anomalies.

We may give a partial answer to the last question—in the affirmative.

It has been pointed out[22] that when the propagator is calculated to fourth order in the gauge which renders it finite to that order,

$$D_{\alpha\beta}(k)=\left(g_{\alpha\beta}-\frac{k_\alpha k_\beta}{k^2}\right)\frac{1}{k^2}+\frac{3g^2}{32\pi^2}\frac{k_\alpha k_\beta}{k_4}, \tag{5.1}$$

then its asymptotic behavior is

$$G(p)\underset{p\to\infty}{\longrightarrow}[1-3g^2/32\pi^2+O(g^4)]i/\gamma_\mu p^\mu. \tag{5.2}$$

Hence, the canonical field commutators are modified to fourth order. We would expect therefore that this is also true for all the commutators which we calculated,

[22] K. Johnson (unpublished).

since it was seen that the anomaly in $G(p)$ affects the anomaly in $\Gamma^\mu(p,q)$, and also in the Compton amplitude. We must also make explicit mention of the fact that our results involving field commutators are gauge-dependent. In other gauges, no result could be given since the unrenormalized fields do not exist. Of course, the results for the Compton amplitude are gauge-invariant.

It should be noticed that the anomalous commutator algebra (4.1)–(4.3c) is nondistributive. That is, even if we accept the definition $j^i=\bar{\psi}(x+\frac{1}{2}\epsilon)\gamma^i\psi(x-\frac{1}{2}\epsilon)$, we cannot calculate the anomalous $[\psi,j^i]$ or the $[j^i,j^k]$ commutators from the anomalous $[\bar{\psi},\psi]_+$ commutator. This is analogous to the breakdown of the Jacobi identity discovered by Johnson and Low.[2]

ACKNOWLEDGMENT

It is a pleasure to acknowledge many helpful and stimulating conversations with Professor K. Johnson.

PHYSICAL REVIEW VOLUME 185, NUMBER 5 25 SEPTEMBER 1969

Sum Rules from Local Current Algebra

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(Received 20 March 1969; revised manuscript received 27 June 1969)

By combining the local current algebra with the commutation relations between the current densities and the Lorentz boost operator, we derive low-energy theorems as well as sum rules which involve neutrino and electron scattering form factors.

1. INTRODUCTION

A GREAT deal of recent activity has been based on the equal-time commutation relations proposed by Gell-Mann[1]:

$$[j_0{}^\alpha(x),j_0{}^\beta(y)]=if^{\alpha\beta\gamma}j_0{}^\gamma(x)\delta(\mathbf{x}-\mathbf{y}) \text{ at } x_0=y_0, \quad (1a)$$

$$[j_0{}^\alpha(x),j_{05}{}^\beta(y)]=if^{\alpha\beta\gamma}j_{05}{}^\gamma(x)\delta(\mathbf{x}-\mathbf{y}) \text{ at } x_0=y_0, \quad (1b)$$

$$[j_{05}{}^\alpha(x),j_{05}{}^\beta(y)]=if^{\alpha\beta\gamma}j_0{}^\gamma(x)\delta(\mathbf{x}-\mathbf{y}) \text{ at } x_0=y_0. \quad (1c)$$

Here $j_0{}^\alpha$ and $j_{05}{}^\alpha$ are the time components of the hadronic vector and axial current densities and α is the $SU(3)$ index.

Adler[2] has derived sum rules involving high-energy neutrino reactions which can provide tests of Eq. (1). A sum rule involving photoabsorption has been derived by Cabbibo and Radicati[3] and an inequality for high-energy electron scattering has been obtained by Bjorken.[4] Our purpose in this paper is to derive more sum rules from Eq. (1).

Our procedure is essentially the same as that of Adler[2] except for one crucial difference: We do not use Eq. (1) directly. Instead, we commute both sides of Eq. (1) with the Lorentz boost operator K_i (i=1, 2, 3) and use

$$i[K_i,j_0{}^\alpha(x)]=j_i{}^\alpha(x)-x_i\partial_0 j_0{}^\alpha(x) \quad \text{at} \quad x_0=0. \quad (2)$$

Defining

$$\tilde{j}_i{}^\alpha(x)\equiv j_i{}^\alpha(x)-x_i\partial_0 j_0{}^\alpha(x) \quad \text{(for all } x_0) \quad (3a)$$

$$\tilde{j}_{i5}{}^\alpha(x)\equiv j_{i5}{}^\alpha(x)-x_i\partial_0 j_0{}^\alpha(x) \quad \text{(for all } x_0), \quad (3b)$$

we find the following commutation relations:

$$[j_0{}^\alpha(x),\tilde{j}_i{}^\beta(y)]+[\tilde{j}_i{}^\alpha(x),j_0{}^\beta(y)] =if^{\alpha\beta\gamma}\tilde{j}_i{}^\gamma(x)\delta(\mathbf{x}-\mathbf{y}) \quad \text{at} \quad x_0=y_0=0, \quad (4a)$$

$$[j_0{}^\alpha(x),\tilde{j}_{i5}{}^\beta(y)]+[\tilde{j}_i{}^\alpha(x),j_{05}{}^\beta(y)] =if^{\alpha\beta\gamma}\tilde{j}_{i5}{}^\gamma\delta(\mathbf{x}-\mathbf{y}) \quad \text{at} \quad x_0=y_0=0, \quad (4b)$$

$$[j_{05}{}^\alpha(x),\tilde{j}_{i5}{}^\beta(y)]+[\tilde{j}_{i5}{}^\alpha(x),j_{05}{}^\beta(y)] =if^{\alpha\beta\gamma}\tilde{j}_i{}^\gamma\delta(\mathbf{x}-\mathbf{y}) \quad \text{at} \quad x_0=y_0=0. \quad (4c)$$

The equal-time commutation relations in Eq. (4) form the basis of the results to be obtained in the present paper. It may be pointed out that, in spite of the appearance of the space components of the currents, Eq. (4) is an exact consequence of Eq. (1) and Lorentz covariance. Moreover, Eq. (4) is free from the unknown Schwinger term which has to be present in the commutation relations between the time and space components of currents. Actually, Eq. (4) is a constraint on the commutation relations between the time and space components of the currents.[5]

In Sec. 2, we derive low-energy theorems based on Eq. (4). These are exact results following from the assumed local current commutation relations. We then convert these theorems into sum rules involving weak and elec-

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