Regge Cuts and Iterative Procedures in Diffraction Scattering*†

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Freund and O'Donovan's dynamical model for Regge cuts is extended to proton-proton diffraction scattering. The input Pomeranchukon is given a phenomenological representation corresponding to a flat trajectory. The question of which part of the total amplitude to identify with the input Pomeranchukon is examined in the context of a generating-function formalism. Consistency with the above-mentioned model and other physical requirements corresponds to restrictions on this generating function. We consider some new and some previously proposed iterative models for diffraction scattering, to wit: (1) a rescattering model (Van Hove), (2) a mixed rescattering-absorption model, (3) an eikonal (Chou-Yang, Frautschi-Margolis) model, and (4) a K-matrix model. Models (2)–(4) are consistent with our requirements and exhibit reasonable (and very similar) asymptotic differential cross sections. Possible modifications for nonasymptotic energies are discussed.

1. INTRODUCTION

THEORETICAL arguments for the existence of Regge-cut contributions are persuasive and well known.[1,2] In a quantitative approach to these contributions, a number of authors[3,4] have proposed dynamical approximations which may be viewed physically as multiple Regge-pole exchange. In the present paper, we extend one of these approaches, that of Freund and O'Donovan (FO),[3] to nonforward elastic scattering and also consider some previously proposed approaches. The results of FO and other authors[5] have suggested that the "cut corrections" (or multiple-exchange contributions) are substantial for elastic scattering but smaller for inelastic processes.

The Pomeranchukon remains something of a mystery. Whether it is itself a pole or a cut or whether it has zero or nonzero slope is not yet clear. We consider here the simplest ansatz under the circumstances and take the P to be a fixed pole in the angular momentum plane. With this fixed-pole ansatz, the "corrected" amplitude which we obtain will have a different t dependence but will still correspond to a fixed pole in the l plane at $l=1$. Had we assumed a nonzero slope for P, the nth-order correction would contain a factor $(\ln s)^{1-n}$ and would thus represent an l-plane cut.

We are aware of the conflict with t-channel unitarity presented by a fixed pole in the absence of a special family of shielding cuts,[6] and the lack of an obvious mechanism for the generation of these cuts at $l=1$.[7] We feel that this is not a serious problem, since (1) our ansatz is a phenomenological one and is only meant to approximately describe the amplitude near $t=0$, and (2) the fraction of the amplitude contributed by shielding cuts decreases with increasing s.[6]

We consider proton-proton scattering at (asymptotically) high energies where we need only consider the spin-nonflip amplitude. We also omit consideration of lower-lying Regge trajectories.

We use the following normalizations:

$$S=1+2iT, \tag{1}$$

$$\operatorname{Im}T(s,0)=s\sigma_{\text{total}}(s). \tag{2}$$

We parametrize the Pomeranchukon as follows:

$$P(s,t)=i\beta(0)e^{at}(s/s_0), \tag{3}$$

thus putting all the t dependence into the residue function which, assuming (3) to dominate the forward peak in pp scattering, we must take to be of the form $\beta(0)e^{at}$, with $a=5$ BeV^{-2} and $\beta(0)=120$. Note that with P written in this form, the parameter a does not have an s dependence, as we mentioned before. For s_0, we take the usual value of 1 BeV^2 and are left with no parameters to fit.

The model of Ref. 3 yields a contribution given by

$$A_c(s,t)=\frac{i}{16\pi^2 s}\int_{-s}^{0}dt'\int_{-s}^{0}dt''P(s,t')P^*(s,t'')\frac{\theta(K)}{\sqrt{K}}, \tag{4}$$

where

$$K(t,t',t'')=-(t^2+t'^2+t''^2)+2(tt'+t't''+tt'')+4tt't''/s,$$

and P is the pole amplitude. Physically, this would correspond to the contribution of the PP cut, even though with the specific parametrization (3), Eq. (4) is but a correction to the residue of a fixed pole at $l=1$.

High-energy pp elastic scattering displays a strong forward peak, indicating the contribution of many

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[1] D. Amati, S. Fubini, and A. Stanghellini, Nuovo Cimento **26**, 896 (1962).
[2] S. Mandelstam, Nuovo Cimento **30**, 1127 (1963); **30**, 1148 (1963).
[3] P. G. O. Freund and P. O'Donovan, Phys. Rev. Letters **20**, 1329 (1968).
[4] S. Frautschi and B. Margolis, Nuovo Cimento **56A**, 1155 (1968).
[5] E. J. Squires, Phys. Letters **26B**, 461 (1968).
[6] R. Oehme, Phys. Rev. Letters **18**, 1222 (1967).
[7] J. Finkelstein and C. I. Tan, Phys. Rev. Letters **19**, 1061 (1967).

partial waves and reminiscent of "diffraction" scattering in optics. It has been suggested that elastic scattering be viewed as the "shadow" scattering produced by absorption into many open inelastic channels. Then, with an ansatz for the "overlap" function, the elastic scattering could be calculated.[8] In the multiperipheral Amati-Fubini-Stanghellini model,[1] the overlap function exhibits a Regge behavior,[9] and it is tempting to identify it with a Regge pole in inelastic scattering or with the Pomeranchukon in elastic scattering. This is what we call rescattering. This, however, would produce a second-order correction having the same sign as the first-order for an imaginary amplitude, as was pointed out by Finkelstein and Jacob,[10] who argue that, experimentally, the corrections should have opposite signs (i.e., the second-order correction should be absorptive). Frautschi and Margolis[4] have used a different approach by parametrizing the elastic S matrix in the Glauber fashion and identifying the Pomeranchukon (which they take to have nonzero slope) with the first Born term, thus producing a secondary structure to $(d\sigma/dt)_{pp}$ beyond the forward peak, in rough agreement with what one might expect for asymptotically high s. The "diffraction minima" they predict are not observed at present energies, but might well be filled in by lower-lying trajectories, the real part of the amplitude, spin-flip amplitudes, etc.[11] We should mention that this Glauber-type iteration was used earlier by Chou and Yang,[12] whose "input," obtained from a geometrical picture of interpenetrating hadron matter densities, would correspond to a Pomeranchukon of zero slope and with t dependence taken proportional to the square of the proton electric form factor. They obtain the same sort of diffraction-minimum structure. Yet another approach is that of absorptive corrections. This is essentially equivalent to identifying the Regge pole with the amplitude on the second sheet, reached by going through the elastic cut in the s plane.[10] These approaches are examples of procedures which generate a secondary structure in $(d\sigma/dt)_{pp}$ beyond the forward peak with the Pomeranchukon (assumed to dominate the forward peak) used as the input.

We may view these approaches as different attempts to answer two questions: (A) What form does P, the input Pomeranchukon take? and (B) What part of the amplitude does the input Pomeranchukon represent?

Of course this is a somewhat oversimplified statement of the problem, since these questions are inextricably related to one another and, besides, the phrasing of (A) and (B) represents a sort of parametrization of our view of the problem.

[8] L. Van Hove, Rev. Mod. Phys. **36**, 655 (1964).
[9] D. Amati, M. Cini, and A. Stanghellini, Nuovo Cimento **30**, 193 (1963).
[10] J. Finkelstein and M. Jacob, Nuovo Cimento **56A**, 681 (1968).
[11] L. Durand and R. Lipes, Phys. Rev. Letters **20**, 637 (1968).
[12] T. T. Chou and C. N. Yang, Phys. Rev. Letters **20**, 1213 (1968).

When one has answered (A) and (B), one can then proceed to the total amplitude via iterative procedures. We propose to examine several possibilities for (B) under out ansatz (3) for (A). We recall that Eq. (3), or any other ansatz for (A), should satisfy the requirement that P dominates the forward peak. On the other hand, an answer to (B) makes little difference for $-0.5\ \text{BeV}^2 \lesssim t \leqslant 0$ (the forward peak), but becomes important for larger values of $|t|$. Indeed, while we do not, in the present paper, consider the amplitude at very large values of $|t|$, where the behavior $T \sim \exp[-\text{const}\sqrt{(-t)}]$ is expected[13] and, to some extent, observed, we wish to emphasize that, unless a different mechanism is responsible for the behavior of the amplitude in this region, the large-angle behavior is crucially dependent upon the answer to (B).

We now proceed to examine several approaches to (B) with the ansatz (3) for (A). We will find it expedient to discuss these in the context of a generating-function formalism which is particularly suited to our zero-slope ansatz for P.

2. RESCATTERING MODEL

For completeness, and to introduce our generating-function formalism, we consider rescattering as an approach to question (B).

Let us write the s-channel unitarity equation somewhat symbolically as

$$\text{Im}T_{ii} = \sum_{k\neq i} \int T_{ik}^{\dagger}\Gamma_k T_{ki} + \int T_{ii}^{\dagger}\Gamma_i T_{ii}, \tag{5}$$

where i is a two-proton state, k is summed over many-particle states, and Γ_k is the phase-space factor.

As a first attempt to answer question (B), let us identify the "overlap function," i.e., the first term on the right-hand side of (5), with $P(s,t)$ the input Pomeranchukon. With this assumption, we arrive at the model of Van Hove.[8] We shall call this the rescattering ansatz. We then obtain the rescattering equation

$$\text{Im}T_R = \text{Im}P + \int T_R^{\dagger}\Gamma T_R. \tag{6}$$

Iteration of this equation yields (for T, P imaginary)

$$T_R = P + i\int P^{\dagger}\Gamma P + i\int P^{\dagger}\Gamma\left(i\int P^{\dagger}\Gamma P\right) + i\int\left(i\int P^{\dagger}\Gamma P\right)^{\dagger}\Gamma P + \cdots. \tag{7}$$

It is easy to verify that this corresponds to

$$T_R(s,t) = \sum_{N=1}^{\infty} T_R^{(N)}(s,t), \tag{8}$$

[13] A. A. Anselm and I. T. Dyatlov, Phys. Letters **24B**, 479 (1967).

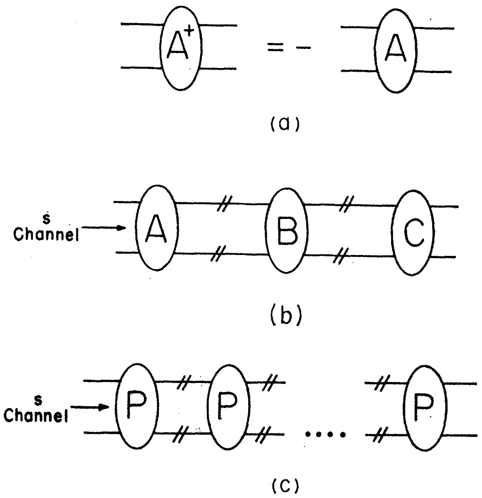

FIG. 1. (a) Diagrammatic relation holding for a purely imaginary amplitude. (b) Diagrammatic representation of the amplitude obtained by combining amplitudes A, B, and C with two-particle on-mass-shell intermediate states. (c) The amplitude [Eq. (14)] obtained by combining the amplitude P successively n times with two-particle on-mass-shell intermediate states.

where $T_1(s,t)\equiv P(s,t)$ and, for $N\geqslant 2$,

$$T_R{}^{(N)}(s,t)=\frac{i}{16\pi^2 s}\sum_{j=1}^{N-1}\int_{-s}^{0}dt'\int_{-s}^{0}dt''T_j(s,t')\times T_{N-j}{}^*(s,t'')\frac{\theta(K)}{\sqrt{K}}. \quad (9)$$

After some manipulation, (9) becomes (see Appendix A)

$$T_R{}^{(N)}(s,t)=(i/N)K_N\beta(0)(s/s_0)G^{N-1}e^{at/N}, \quad (10)$$

where $G\equiv\beta(0)/16\pi a s_0$ and K_N obeys the recursion relation

$$K_N=\sum_{j=1}^{N-1}K_jK_{N-j} \quad \text{for} \quad N\geqslant 2, \quad (11)$$

with $K_1=1$. A combinatorial solution of this recursion relation is presented in Appendix B, where it is shown that

$$K_N=(2N-2)!/N[(N-1)!]^2. \quad (12)$$

We finally obtain that iteration of Eq. (6) with our input (3) for $P(s,t)$ yields

$$T_R(s,t)=i\beta(0)\left(\frac{s}{s_0}\right)\sum_{N=1}^{\infty}G^{N-1}\frac{(2N-2)!}{(N!)^2}e^{at/N}. \quad (13)$$

We now choose to write this in a somewhat different form. Let us call attention to two things. First, for a purely imaginary amplitude $A(s,t)$, $A^\dagger(s,t)=-A(s,t)$, which we depict in Fig. 1(a). Second, the iteration represented by Eq. (7) and those which we will consider below involve multiple scatterings with two-body on-mass-shell intermediate states, as in Fig. 1(b). It follows from these two observations that the total amplitude resulting from any such iteration will be expressible as a sum over graphs of the type shown in Fig. 1(c), each graph entering the sum with a certain coefficient determined from the iteration procedure.

Let us symbolically denote by $(\hat{P})^n$ the amplitude obtained by calculating the graph of Fig. 1(c) [similarly, we would write $\hat{A}\cdot\hat{B}\cdot\hat{C}$ for the graph of Fig. 1(b)]. We can write this, for $n\geqslant 2$, as

$$(\hat{P})^n=\frac{1}{(16\pi^2 s)^{n-1}}\int dt_1\cdots dt_n\mathcal{T}^{(n)}(t;t_1\cdots t_n)\times\prod_{j=1}^{n}P(s,t_j), \quad (14)$$

where $\mathcal{T}^{(n)}(t;t_1\cdots t_n)$ is defined in Appendix C (we use here the notation of Martinis[14]).

Using Eq. (3), we get

$$(\hat{P})^N=i\beta(0)(S/S_0)(G^{N-1}/N)e^{at/N}. \quad (15)$$

We may associate, with each of the iterations we consider, a generating function $g(x)$ such that

$$iT(s,t)=g(0)+g'(0)i\hat{P}+(1/2!)g''(0)(i\hat{P})\cdot(i\hat{P})+\cdots\equiv g(i\hat{P}). \quad (16)$$

In the case of the rescattering iteration, Eq. (6), we have

$$g_R(x)=\tfrac{1}{2}[-1+(1+4x)^{1/2}]. \quad (17)$$

Note that $g_R(x)$ is simply related to the function $f(x)$, obtained in Appendix B, by $g_R(x)=-f(-x)$. We can view Eq. (17) as a solution to Eq. (6) in the following fashion:

In our symbolic notation, Eq. (6) becomes

$$\hat{T}_R=\hat{P}+i(\hat{T}_R{}^\dagger)\cdot\hat{T}_R. \quad (18)$$

Since P and T are, under our assumptions, purely imaginary, this becomes

$$i\hat{T}_R=i\hat{P}-(i\hat{T})^2. \quad (19)$$

Solving this equation as a simple quadratic, we get

$$i\hat{T}_R=\tfrac{1}{2}[-1\pm(1+4i\hat{P})^{1/2}]. \quad (20)$$

Choosing the sign corresponding to $\hat{T}_R=\hat{P}+\cdots$, we arrive at Eqs. (16) and (17).

3. IMPACT-PARAMETER REPRESENTATION

The formulas of this section will enable us in the following to sum up the various iterative expressions in closed form.

The partial-wave expansion, with our normalization, reads (at high s)

$$T(s,t)=16\pi\sum_{l=0}^{\infty}f_l(s)P_l(z), \quad (21)$$

[14] M. Martinis, Imperial College report, 1968 (unpublished).

which, in the impact-parameter representation, becomes

$$T(s,t)=8\pi s\int_0^\infty bdb\ f(s,b)J_0(b\sqrt{-t})\,. \qquad (22)$$

The relation between $g(x)$ and the impact-parameter representation is given by (see Appendix C)

$$if(s,b)=g\left(\frac{i}{8\pi s}\int_0^\infty xdx\ P(s,\ -x^2)J_0(bx)\right). \qquad (23)$$

With our ansatz (3),

$$\frac{i}{8\pi s}\int_0^\infty xdx\ P(s,\ -x^2)J_0(bx)=-Ge^{-b^2/4a}, \qquad (24)$$

thus

$$iT(s,t)=8\pi s\int_0^\infty bdb\ J_0(x\sqrt{-t})g(-Ge^{-b^2/4a}), \qquad (25)$$

and we may express the total cross section as

$$\sigma_{\mathrm{tot}}(s\to\infty)=-8\pi\int_0^\infty xdx\ g(-Ge^{-x^2/4a}). \qquad (26)$$

If we define $S(s,b)$ by

$$f(s,b)\equiv[S(s,b)-1]/2i, \qquad (27)$$

we can then express the unitarity condition on $T(s,t)$ as

$$|S(s,b)|\leqslant 1 \quad \text{for} \quad 0\leqslant b<\infty\,, \quad \text{as} \quad s\to\infty\,. \qquad (28)$$

From Eqs. (22) and (23),

$$S(s,b)=1+2g(-Ge^{-b^2/4a})\,. \qquad (29)$$

4. MIXED RESCATTERING-ABSORPTION MODEL

If the iteration (7) were correct, we would have to identify the leading-cut correction (hereafter referred to as the second-order term) with the diagram of Fig. 2(a). As is well known,[2] perturbation theory suggests that this is not a *bona fide* cut contribution.[15] Further, if P is taken to be purely imaginary, the second term in (7) has the same sign as the first term. As we mentioned previously, Finkelstein and Jacob[10] have shown that, in the cases they consider, experiment favors a relative minus sign between these two terms. Also, for the parameters we have specified for Eq. (3), the sum for $T(s,t)$ in Eq. (13) is divergent.

We now consider approaches to question (B) that are consistent with the model of Ref. 3. The first case we consider is a modification of the rescattering method motivated by this model.

[15] For a sum of ladders in a Φ^3 theory of scalar particles, for instance, the moving-cut contribution of the two-particle discontinuity represented by Fig. 2(a) is cancelled at high s by the contribution of inelastic intermediate states and the cut behavior only appears on the second sheet reached by continuing through the s-plane elastic cut.

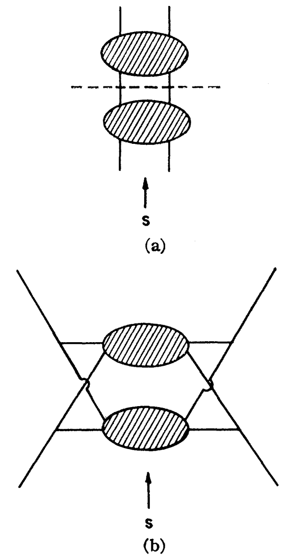

FIG. 2. (a) Diagram representing the two-particle discontinuity corresponding to the leading-cut correction in the rescattering model of Sec. 2. The shaded parts represent the Pomeranchukon amplitude. (b) Mandelstam-type diagram giving a cut contribution on the physical sheet of the s plane, discussed in Ref. 3 in connection with the diagram of Fig. 2(a).

The diagram shown in Fig. 2(b) is thought to give a cut contribution on the physical sheet, unlike the diagram of Fig. 1(a). We discussed these diagrams in Ref. 3 and suggested that the physical-sheet behavior of Fig. 2(b) might be approximated by the second-sheet behavior of Fig. 2(a). Since the discontinuity on the second sheet is the negative of that on the first sheet, our model takes Fig. 2(a) with a minus sign as the leading cut-contribution.

We now suggest that a possible modification of Eq. (6), consistent with this model, is

$$\mathrm{Im}T_M=\mathrm{Im}P-\int T_M{}^\dagger\Gamma T_M\,. \qquad (30)$$

That is, we identify P not with the overlap function, but with the overlap function plus twice the elastic discontinuity. It is easy to check, using our previous results for rescattering, that the expression for T corresponding to Eq. (30) is just

$$T_M(s,t)=i\beta(0)\left(\frac{s}{s_0}\right)\sum_{N=1}^{\infty}(-1)^{N+1}G^{N-1}\times\frac{(2N-2)!}{(N!)^2}e^{at/N}, \qquad (31)$$

i.e., the same as our rescattering expression, Eq. (13), except for the factor $(-1)^{N+1}$. This alteration of signs is consistent with the results of Gribov[16] and is thought to be a general feature to be expected in such multiple-scattering approaches. It is this alternation of signs which makes the existence of diffraction minima possible.

The generating function corresponding to Eq. (30) is

$$g_M(x)=\tfrac{1}{2}[1-(1-4x)^{1/2}]. \qquad (32)$$

[16] V. N. Gribov, Zh. Eksperim. i Teor. Fiz. **53**, 654 (1967) [English transl.: Soviet Phys.—JETP **26**, 414 (1968)].

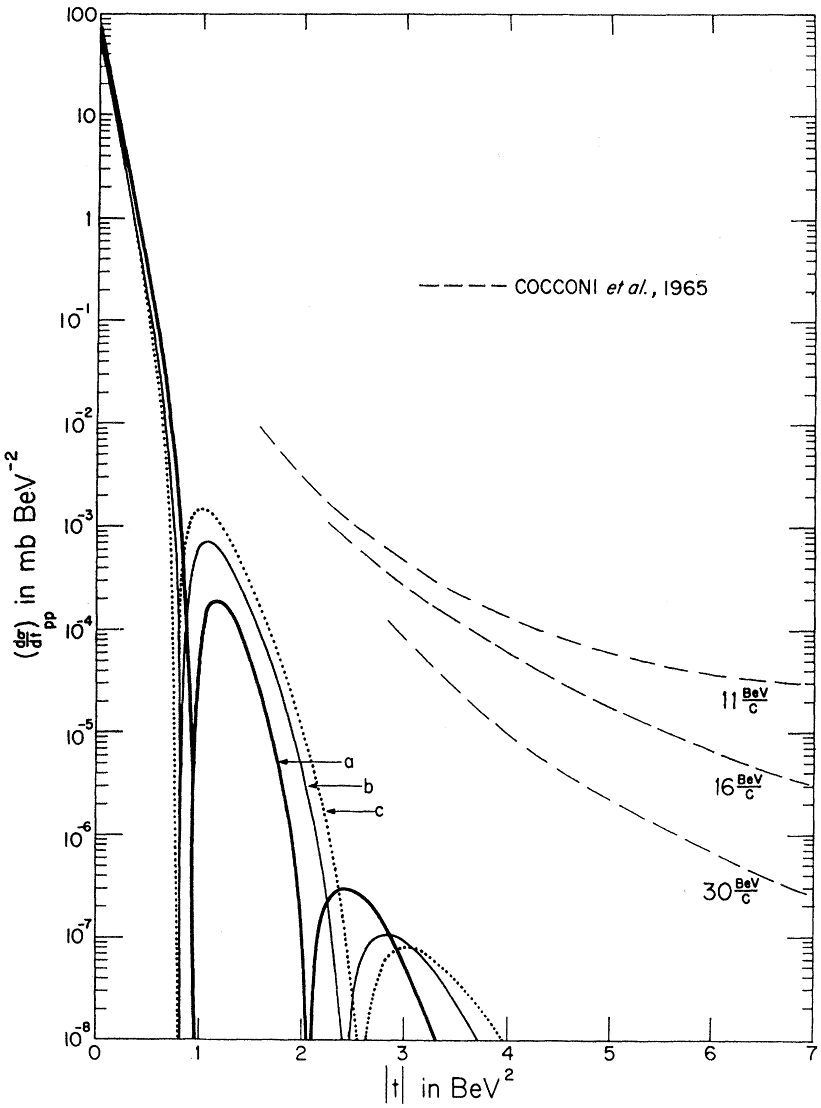

FIG. 3. Proton-proton asymptotic differential cross section according to (a) the mixed rescattering-absorption model, (b) the K-matrix model, and (c) the eikonal model. The data shown are those of Cocconi *et al.* [Phys. Rev. **138**, B165 (1965)]. For $-0.8\ \text{BeV}^2 < t \leqslant 0$ (the forward peak), experimental data correspond closely with the curves and are not shown.

We may translate our requirements into conditions on the generating function $g(x)$ as follows:

(a) "$\hat{T}=0$ if $\hat{P}=0$" means that $g(0)=0$.

(b) "The Pomeranchukon corresponds to the leading (first-order) term" means that $g'(0)=1$.

(c) "The *FO* model yields the second-order term" means that $g''(0)=2$.

(d) "The successive terms alternate in sign" means that

$$g^{(n)}(0)>0 \quad \text{for all } n. \tag{33}$$

(e) "Unitarity is satisfied asymptotically" means that $-1\leqslant g(x)\leqslant 0$ for $-G\leqslant x\leqslant 0$.

(f) "The total cross section is finite" means [from Eq. (26)] that

$$0<\int_{-G}^{0}\frac{dx}{x}g(x)<\infty .$$

We see that the physical requirements which we impose do not completely specify the iterative procedure, but rather serve to limit the possibilities to those procedures corresponding to generating functions satisfying conditions (33). Our rescattering expression is thus not acceptable, since it fails to satisfy several of these conditions. $g_M(x)$, however, is an example of a generating function which satisfies all the conditions (33).

With our normalizations (1) and (2), we have

$$\frac{d\sigma}{dt}=\frac{|T(s,t)|^2}{16\pi s^2}. \tag{34}$$

However, we cannot apply this expression to the series (31) for $T_M(s,t)$, since this series is divergent. Unlike the other cases we will consider, in which the corresponding series are convergent and well behaved, the present case is analogous to trying to utilize a power-series expansion outside its radius of convergence. To clarify this, let us note that formally summing the series (31) (using the method of Appendix C) leads us to the expression [see Eqs. (25) and (32)]

$$iT_M(s,t)=8\pi s\int_0^\infty bdb\,J_0(b\sqrt{-t})\times\tfrac{1}{2}[1-(1+4Ge^{-b^2/4a})^{1/2}]. \quad (35)$$

Conversely, we can view the series (31) as the result of expanding $[1-(1+4Ge^{-b^2/4a})^{1/2}]$ in a power series in $Ge^{-b^2/4a}$ and then performing the integration in b. This is clearly not a valid expansion, since $4G>1$. To calculate $(d\sigma/dt)_M$, we must use the expression (35) for $T_M(s,t)$. The Romberg method of numerical integration yields the results shown in Fig. 3, curve a.

The total cross section is given by

$$\sigma_M{}^{\text{total}}(s\to\infty)=16\pi a\{(1+4G)^{1/2}-1-\ln[\tfrac{1}{2}+\tfrac{1}{2}(1+4G)^{1/2}]\}=41\text{ mb}. \quad (36)$$

We may now inquire whether we can clear up the divergence encountered in the Van Hove rescattering expression [Eq. (13)] in the same fashion. The answer is that we cannot. The equation for $iT_R(s,t)$ corresponding to Eq. (35) involves the factor $(1-4Ge^{-b^2/4a})^{1/2}$ in the integrand. Since $4G>1$, this is an imaginary number for small b and thus gives a nonsensical result. For πp scattering, the magnitude of the forward peak would require that $4G\simeq0.91$, which is less than 1, and in that case the Van Hove expression is convergent. Thus, as has been noted previously,[17] an unmodified Van Hove model may be applied to πp diffraction scattering but is unable to accommodate pp diffraction scattering. None of the acceptable models we consider suffers from this difficulty.

5. DIFFRACTION MINIMA

We wish to point out that our conditions (33), and condition (d) in particular, do not necessarily imply the existence of diffraction minima (actually, zeros). Such minima occur only when

$$|\sum_{N\text{ even}}T^{(N)}|-|\sum_{N\text{ odd}}T^{(N)}|$$

changes sign. The cases we consider consistent with (33) do display these minima.

We expect the first minimum to occur when, roughly speaking, the amplitude passes from single- to double-exchange dominance. We can write a simple relation for the approximate value of t (call it t_0) for which we expect this first minimum, and t_0 is approximately determined by the conditions (33).

Since $T(s,t)\simeq P(s,t)$ for small t, we have

$$\begin{aligned}\operatorname{Im}T^{(2)}(s,t)&\simeq-\sigma_{\text{el}}se^{\frac{1}{2}at},\\ \operatorname{Im}T^{(1)}(s,t)&\simeq\sigma_{\text{tot}}se^{at},\end{aligned} \quad (37)$$

so that t_0 is given by

$$t_0\simeq-(2/a)\ln(\sigma_{\text{tot}}{}^{pp}/\sigma_{\text{el}}{}^{pp})\simeq-0.6\text{ BeV}^2, \quad (38)$$

where σ_{tot} and σ_{el} are the asymptotic pp total and elastic scattering cross sections. The models we consider exhibit t_0 in the range -0.8 to -0.95 BeV^2, and $(t_0)_M=-0.95$ BeV^2.

The iterative procedures which satisfy (33) are all equivalent to second order and it is the higher-order terms which distinguish them.

6. EIKONAL MODEL

The eikonal method of Arnold[18] corresponds to writing the S matrix in the impact-parameter representation as

$$S=e^{2i\chi}. \quad (39)$$

The answer to question (B) is provided by identifying χ with the Fourier-Bessel transform of the P amplitude. This corresponds in our notation to

$$\hat{S}=\exp(2i\hat{P}). \quad (40)$$

This is essentially the procedure used by Chou and Yang,[12] who take $P(s,t)\propto|\text{proton electric form factor}|^2\times(s/s_0)$, corresponding to a flat Pomeranchukon trajectory, and by Frautschi and Margolis,[4] who take a Regge pole with nonzero slope for $P(s,t)$.

In terms of T, (40) reads

$$i\hat{T}=\tfrac{1}{2}[\exp(2i\hat{P})-1]. \quad (41)$$

The generating function $g_E(x)$ is thus

$$g_E(x)=\tfrac{1}{2}(e^{2x}-1), \quad (42)$$

which satisfies all of the conditions (25). The total amplitude is

$$T_E(s,t)=i\beta(0)\left(\frac{s}{s_0}\right)\sum_{N=1}^{\infty}(-1)^{N+1}\frac{(2G)^{N-1}}{N(N!)}e^{at/N}. \quad (43)$$

Unlike Eq. (32), this sum is convergent and the resulting $(d\sigma/dt)_E$ is shown in Fig. 3, curve c, as

$$\begin{aligned}\sigma_E{}^{\text{tot}}(s\to\infty)&=38.2\text{ mb},\\ (t_0)_E&=-0.8\text{ BeV}^2.\end{aligned} \quad (44)$$

7. K-MATRIX MODEL

In the model we now consider, we answer question (B) by parametrizing the elastic S matrix as

$$\hat{S}=(1+i\hat{P})/(1-i\hat{P}). \quad (45)$$

This is analogous to the single-channel K-matrix parametrization of the S matrix, with P playing the

[17] W. N. Cottingham and R. F. Peierls, Phys. Rev. **137**, B147 (1964); R. Henzi, Nuovo Cimento **52A**, 772 (1967); **57A**, 301 (1968).

[18] R. C. Arnold, Phys. Rev. **153**, 1523 (1967).

role of the K matrix. In terms of $\hat{T}$, this is

$$i\hat{T}=i\hat{P}/(1-i\hat{P}). \tag{46}$$

The generating function $g_K(x)$ then becomes

$$g_K(x)=x/(1-x). \tag{47}$$

$g_K(x)$ satisfies the conditions (33) and is automatically consistent with our application of the model of Ref. 3. $T_K(s,t)$ is given by

$$T_K(s,t)=i\beta(0)\left(\frac{s}{s_0}\right)\sum_{N=1}^{\infty}(-1)^{N+1}\frac{G^{N-1}}{N}e^{at/N}. \tag{48}$$

As in the eikonal case, this sum is convergent. The differential cross section $(d\sigma/dt)_K$ is plotted in Fig. 3, curve b, and has a behavior very similar to that of $(d\sigma/dt)_E$ and $(d\sigma/dt)_M$:

$$\sigma_K{}^{\text{total}}(s\to\infty)=\frac{\beta(0)}{s_0}\frac{\ln(1+G)}{G}=33.7\text{ mb}, \qquad (t_0)_K=-0.8\text{ BeV}^2. \tag{49}$$

8. NONASYMPTOTIC ENERGIES

At nonasymptotic energies, we might ask questions similar to (A) and (B) for lower-lying proper Regge trajectories. For a proper Regge-pole amplitude contributing to elastic scattering (call it R), we have answered (A) once we know the residue, the trajectory, and the signature. For (B), there is a wide range of possibilities. Two simple possibilities are

$$i\hat{T}=g(i\hat{P})+i\hat{R}, \tag{50a}$$

$$i\hat{T}=g(i(\hat{P}+\hat{R})); \tag{50b}$$

i.e., we could consider putting R directly into the amplitude or, alternatively, into the single-scattering term. As Chiu and Finkelstein[19] have argued in connection with their model, the most natural choice seems to be (50b), since the total amplitude then contains all of the cuts in which the proper trajectories participate. Equation (50a) clearly does not contain these cuts.

9. CONCLUSIONS

We have seen that the diffraction behavior of $(d\sigma/dt)_{pp}$ $(s\to\infty)$ can be obtained in a variety of ways depending (A) upon the form of the Pomeranchukon contribution, and (B) upon which part of the total amplitude it represents.

We have formulated the various possibilities for (B) in terms of a generating function $g(x)$. Consistency with the model of Ref. 3 and other physical requirements translate into restrictions on $g(x)$ which in turn define a class of acceptable iterative procedures. We examine three of these procedures, one of which (eikonal) has had extensive previous use, and we note that (1) they are all equivalent to second order [a consequence of the restrictions on $g(x)$], (2) they all display diffraction minima, and (3) they all have a t dependence compatible with experiment (see Fig. 3). Additional conditions on $g(x)$ resulting from the imposition of further physical requirements might serve to further narrow the class of acceptable iterative procedures. The most natural way to proceed to nonasymptotic energies seems to be to put proper Regge trajectories into the single-scattering term.[19]

[19] C. B. Chiu and J. Finkelstein, Nuovo Cimento **57A**, 649 (1968).

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APPENDIX A

In this appendix, we indicate the derivation of Eq. (10). In the steps connecting Eqs. (9) and (10), it is necessary to employ the integral

$$\begin{aligned} I_{j,N-j}(t)&\equiv\int_{-s}^{0}dt'\int_{-s}^{0}dt''\exp(a_jt'+a_{N-j}t'')\frac{\theta(K)}{\sqrt{K}}\\ &=\frac{\pi}{(a_j^2+2a_ja_{N-j}\cos\theta+a_2^2)^{1/2}}\Bigg\{\exp\Bigg[\frac{t}{2\sin^2(\frac{1}{2}\theta)}\\ &\quad\times\Big((a_j+a_{N-j})-(a_j^2+2a_ja_{N-j}\cos\theta+a_{N-j}^2)^{1/2}\Big)\Bigg]\\ &\quad-\exp\Bigg[\frac{t}{2\sin^2(\frac{1}{2}\theta)}\Big((a_j+a_{N-j})\\ &\qquad+(a_j^2+2a_ja_{N-j}\cos\theta+a_{N-j}^2)^{1/2}\Big)\Bigg]\Bigg\}, \end{aligned} \tag{A1}$$

where

$$\cos\theta=1+t/2p^2\sim1+2t/s.$$

To order $1/s$, $\cos\theta\to1$ and

$$I_{j,N-j}(t)=\frac{\pi}{a_j+a_{N-j}}\exp\left(\frac{a_ja_{N-j}}{a_j+a_{N-j}}t\right). \tag{A2}$$

With the form (3) for $T^{(1)}(s,t)$, repeated application of (9) shows that $T^{(N)}(s,t)$ must be of the form

$$T^{(N)}(s,t)=i\beta_N(0)(s/s_0)(-1)^{N+1}e^{(a/N)t}. \tag{A3}$$

Equation (A2) then becomes

$$I_{j,N-j}(t)=[\pi j(N-j)/aN]e^{(a/N)t}, \tag{A4}$$

and the relation (9) becomes

$$T_R{}^{(N)}(s,t)=\frac{i(-1)^{N+1}}{16\pi s_0aN}\left(\frac{s}{s_0}\right)\times\left[\sum_{j=1}^{N-1}\beta_j(0)\beta_{N-j}(0)j(N-j)\right]e^{(a/N)t}. \tag{A5}$$

This provides a recursion relation for $\beta_N(0)$:

$$\beta_N(0)=\frac{1}{16\pi s_0 aN}\sum_{j=1}^{N-1}\beta_j(0)\beta_{N-j}(0)j(N-j)\,, \quad \text{(A6)}$$

and with the definitions

$$G\equiv\frac{\beta_1(0)}{16\pi s_0 a}\,,\quad K_N\equiv\frac{\beta_N(0)N}{\beta_1(0)G^{N-1}}\,, \quad \text{(A7)}$$

we obtain the recursion relation, Eq. (11).

APPENDIX B

In this appendix, we present a combinatorial solution of the recursion relation

$$K_N=\sum_{j=0}^{N-1}K_jK_{N-j}\,,\quad N\geqslant 2\,,\quad K_1=1\,. \quad \text{(11)}$$

Assuming a generating function for the series K_0, K_1, $\cdots$ (with $K_0=0$), we have

$$f(x)=K_0+K_1x+\cdots+K_Nx^N+\cdots\,. \quad \text{(B1)}$$

(In this appendix, use the term generating function in the usual sense of combinatorial mathematics; it has a different meaning in the text of the paper.)

Multiplication of generating functions

$$C(x)=A(x)B(x) \quad \text{(B2)}$$

implies

$$C_i=\sum_{j=0}^{i}a_jb_{i-j}\,, \quad \text{(B3)}$$

where $C(x)=C_0+C_1x+\cdots$, etc.

From Eq. (11), we have

$$\sum_{N=2}^{\infty}K_Nx^N=\sum_{N=2}^{\infty}\Big(\sum_{j=0}^{N}K_jK_{N-j}\Big)x^N\,. \quad \text{(B4)}$$

Substituting $f(x)$ for $A(x)$ and $B(x)$ in (B2), we get

$$[f(x)]^2=\sum_{N=0}^{\infty}\Big(\sum_{j=0}^{N}K_jK_{N-j}\Big)x^N\,. \quad \text{(B5)}$$

Combining (B4) and (B5), we get

$$f(x)-K_0-K_1x=[f(x)]^2-K_0^2-(K_1K_0+K_0K_1)x\,,$$

or

$$[f(x)]^2-f(x)+x=0\,, \quad \text{(B6)}$$

the solution of which is

$$f(x)=\tfrac{1}{2}[1\pm(1-4x)^{1/2}]\,. \quad \text{(B7)}$$

Expanding the two expressions for $f(x)$ in Taylor's series, we see that we must choose the minus sign to ensure $K_N\geqslant 0$ for all N. We get

$$f(x)=\tfrac{1}{2}[1-(1-4x)^{1/2}]\,. \quad \text{(B8)}$$

The expansion is

$$f(x)=\sum_{N=1}^{\infty}\frac{(2N-2)!}{N[(N-1)!]^2}x^N\,. \quad \text{(B9)}$$

The solution to Eq. (11) is thus

$$K_N=\frac{(2N-2)!}{N[(N-1)!]^2}\,. \quad \text{(B10)}$$

APPENDIX C

In this appendix, we present some background for the impact-parameter representation (Sec. 3).

In Eq. (14), we use the same notation as Martinis,[14] although our normalization (1) differs slightly from his. The factor $\mathcal{T}^{(n)}(t;t_1\cdots t_n)$ is given by

$$\mathcal{T}^{(n)}(t;t_1\cdots t_n)=\frac{\pi^{n-1}}{4p^2}\sum_{l=0}^{\infty}(2l+1)P_l(z)\prod_{i=1}^{n}P_l(z_i)\,, \quad \text{(C1)}$$

where $t_i=-2p^2(1+z_i)$.

At high s and fixed t, t_1, t_2, $\cdots$, t_n,

$$\mathcal{T}^{(n)}(t;t_1\cdots t_n)\approx\tfrac{1}{2}\pi^{n-1}\int_0^{\infty}bdb\,J_0(b\sqrt{-t})\times\prod_{i=1}^{n}J_0(b\sqrt{-t_i})\,. \quad \text{(C2)}$$

Combining these relations with Eqs. (14) and (16), and using the Fourier-Bessel relation

$$F(x)=\int_0^{\infty}J_0(xy)y\Big[\int_0^{\infty}F(x')J_0(yx')x'dx'\Big]dy\,, \quad \text{(C3)}$$

we obtain Eq. (22), with $f(s,b)$ given by Eq. (23).