Extended Lee Model with an Arbitrary Number of Possible Baryon States*

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(Received 8 May 1969)

The application of Tomonaga's intermediate-coupling approximation to an extended Lee model in which the single baryon may exist in k possible states is outlined. The intermediate-coupling solution of such a model is shown to be characterized by a class of polynomial functions which are defined by a three-term recurrence relation. The bound states correspond to the zeros of these polynomials, but there is no simple relation between the bound states of the model with k possible baryon states and the next "higher" Lee-type model with $k+1$ possible baryon states. Furthermore, an algebraic solution of the problem is not possible for $k \geq 5$, because the order of the secular equation is equal to the value of k. Some specific results are given for the $n=3$ sector of the extended Lee-type model with four possible baryon states. It is conjectured that all extended Lee models of this type will exhibit a strong-coupling "isobar" spectrum of the type found by North for the cases of the ordinary Lee model and the Bronzan-Lee model.

IN a footnote to his recent article on the Bronzan-Lee model,[1] North points out that several other versions of the Lee model may be approximately solved by using harmonic-oscillator wave functions. It has also been pointed out that Tomonaga's intermediate-coupling approximation[2] (ICA) is also applicable to the ordinary Lee model[3] or the Bronzan-Lee model,[4] and again leads to an approximate solution in terms of harmonic-oscillator wave functions. In fact, North's strong-coupling approximation (SCA) may be regarded as a special case of Tomonaga's ICA. As already noted in Refs. 3 and 4, the principal advantage of Tomonaga's ICA is that it eliminates the large source assumption inherent in North's treatment.

The object of the present paper is to indicate how Tomonaga's ICA may be used to solve an "extended Lee model" with k possible baryon states, where k is any integer greater than or equal to 2. This model will henceforth be referred to as ELM-k; for example, the ordinary Lee model[5] is ELM-2, and the Bronzan-Lee model is ELM-3. Although the extended Lee model is only a very simple mathematical model, it is nevertheless interesting to find that the ICA bound-state energy spectrum becomes much richer and much more complex as the value of the integer k increases.

The present article is organized as follows: First we discuss the ICA solution of the extended Lee model with four possible baryon states. Then the analysis is extended to the case of ELM-k by introducing a particular class of polynomial functions. The energy levels in the ICA actually correspond to the zeros of these polynomial functions. In principle, the determination of the ICA solution to the ELM-k problem is very straightforward. However, in order to determine the ICA solution explicitly, it appears to be necessary to use a computer, and such computer calculations do not appear to be advisable at this time in view of the unrealistic nature of the model. In this connection, however, it should be noted that the ELM-k problem is closely related (from a mathematical point of view) to the problem of a k-level atom interacting with an electromagnetic field. Thus, one cannot rule out the possibility that the polynomial functions defined here may also turn up in problems of physical interest.

SOLUTION OF THE MODEL WITH FOUR POSSIBLE BARYON STATES

A nonrelativistic, extended Lee model with four possible baryon states (henceforth referred to as the ELM-4) may be defined by the Hamiltonian

$$H=H_{\text{mes}}+H_{\text{int}}+H_{\text{baryon}}, \tag{1}$$

where[6]

$$H_{\text{mes}}=\sum_k \omega_k a_k^\dagger a_k, \tag{2}$$

$$\begin{aligned}H_{\text{int}}=&\{e[\tfrac{1}{2}(1+\tau_3)\times\tau_-]\sum_k u_k a_k+\text{H.c.}\}\\&+\{f(\tau_-\times\tau_+)\sum_k u_k a_k+\text{H.c.}\}\\&+\{g[\tfrac{1}{2}(1-\tau_3)\times\tau_-]\sum_k u_k a_k+\text{H.c.}\},\end{aligned} \tag{3}$$

$$\begin{aligned}H_{\text{baryon}}=&\tfrac{1}{4}[(1+\tau_3)\times(1-\tau_3)]\epsilon_N\\&+\tfrac{1}{4}[(1-\tau_3)\times(1+\tau_3)]\epsilon_U\\&+\tfrac{1}{4}[(1-\tau_3)\times(1-\tau_3)]\epsilon_V.\end{aligned} \tag{4}$$

$a_k^\dagger$ and a_k are the creation and annihilation operators of a meson with momentum $\mathbf{k}$, and

$$u_k=(2\pi)^{-3/2}\int U(\mathbf{r})e^{i\mathbf{k}\cdot\mathbf{r}}d^3r, \tag{5}$$

where $U(\mathbf{r})$ is the nucleon source function, which is

* Research supported by the project Special Research in Numerical Analysis for the U. S. Army Research Office, Durham, under Contract No. DA-31-124-AROD-13 at Duke University.

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[1] G. R. North, Phys. Rev. **168**, 1698 (1968).
[2] S. Tomonaga, Progr. Theoret. Phys. (Kyoto) **2**, 6 (1947).
[3] H. H. Nickle, Phys. Rev. **178**, 2382 (1969).
[4] H. H. Nickle, Phys. Rev. **182**, 1475 (1969).
[5] T. D. Lee, Phys. Rev. **95**, 1329 (1954).

[6] Here, for example, $(\tau_-\times\tau_+)$ means the "direct product" of the two 2×2 matrices τ_- and τ_+.

normalized according to

$$\int U(\mathbf{r})d^3r=1. \tag{6}$$

ω_k denotes the total energy of a nonrelativistic meson with momentum $\mathbf{k}$, $\omega_k=\mathbf{k}^2+\frac{1}{2}$, where energies are expressed in units of $2m$. Here e is the bare (π^-,P,N) coupling constant, f is the bare (π^-,N,U^-) coupling constant, g is the bare (π^-,U^-,V^{--}) coupling constant, and $\epsilon_{N,U,V}$ may be regarded as the bare N, U, V masses, respectively.

The only processes allowed by the Hamiltonian (1) are

$$P+\pi^-\leftrightarrow N, \tag{7a}$$

$$N+\pi^-\leftrightarrow U^-, \tag{7b}$$

$$U^-+\pi^-\leftrightarrow V^{--}, \tag{7c}$$

where the four possible baryon states are henceforth referred to as "proton" (P), "neutron" (N), U, and V. It is clear from the allowed processes (7) that the ELM-4 conserves the total electric charge of the hadron system. This corresponds to the fact that the operator of total charge

$$q=\tfrac{1}{4}[(1+\tau_3)\times(1+\tau_3)]-\tfrac{1}{4}[(1-\tau_3)\times(1+\tau_3)]-\tfrac{1}{2}[(1-\tau_3)\times(1-\tau_3)]-\sum_k a_k^\dagger a_k \tag{8}$$

commutes with the total Hamiltonian (1). Just as in the case of the ordinary Lee model (ELM-2) or the Bronzan-Lee model (ELM-3), the eigenvalues of the total-charge operator are restricted to the following integer values: $+1, 0, -1, -2, -3, \cdots, -\infty$.

Lee and Pines[7] have shown that Tomonaga's ICA is equivalent to the following substitution of reduced-space operators in the Hamiltonian:

$$a_k\rightarrow f_k a, \tag{9a}$$

$$a_k\rightarrow f_k a^\dagger. \tag{9b}$$

The trial function f_k is then chosen to minimize the *lowest* set of eigenvalues of the reduced-space Hamiltonian, which turns out to be

$$H_{\mathrm{ICA}}=\begin{Bmatrix}\omega a^\dagger a & eQa^\dagger & 0 & 0\\ eQa & \omega a^\dagger a+\epsilon_N & fQa^\dagger & 0\\ 0 & fQa & \omega a^\dagger a+\epsilon_U & gQa^\dagger\\ 0 & 0 & gQa & \omega a^\dagger a+\epsilon_V\end{Bmatrix}, \tag{10}$$

where

$$\omega\equiv\sum_k \omega_k f_k^2, \tag{11}$$

$$Q\equiv\sum_k u_k f_k. \tag{12}$$

We also note that the reduced-space operator of total charge is a diagonal 4×4 matrix whose elements are given by

$$(q_{\mathrm{ICA}})_{ij}=(2-i-a^\dagger a)\delta_{ij}, \tag{13}$$

where the normalization condition

$$\sum_k f_k^2=1 \tag{14}$$

has been taken into consideration.

It should also be noted that North's SCA[1,8] may be regarded as a special case of Tomonaga's ICA, that is, it corresponds to the specific choice

$$f_k=N^{-1/2}u_k, \tag{15}$$

where

$$N\equiv\sum_k u_k^2. \tag{16}$$

We therefore confine our attention to the ICA, henceforth omitting the subscripts ICA.

We now seek simultaneous eigenfunctions of H_{ICA} and q_{ICA} in the form

$$\phi_n=(1+b_n^2+c_n^2+d_n^2)^{-1/2}\begin{bmatrix}\psi_n\\ b_n\psi_{n-1}\\ c_n\psi_{n-2}\\ d_n\psi_{n-3}\end{bmatrix}, \tag{17}$$

where the harmonic-oscillator functions ψ_n are defined according to

$$a\psi_n=0, \tag{18a}$$

$$\psi_n=(n!)^{-1/2}(a^\dagger)^n\psi_0, \tag{18b}$$

$$a^\dagger a\psi_n=n\psi_n. \tag{18c}$$

The desired eigenfunctions ϕ_n satisfy

$$H_{\mathrm{ICA}}\phi_n=E_n\phi_n, \tag{19}$$

$$q_{\mathrm{ICA}}\phi_n=-(n-1)\phi_n. \tag{20}$$

One can easily verify that the secular equation which yields the energy eigenvalues of the matrix equation (19) is given by

$$\begin{vmatrix} n-y_n & e'(n)^{1/2} & 0 & 0\\ e'(n)^{1/2} & n-1+\epsilon_N'-y_n & f'(n-1)^{1/2} & 0\\ 0 & f'(n-1)^{1/2} & n-2+\epsilon_U'-y_n & g'(n-2)^{1/2}\\ 0 & 0 & g'(n-2)^{1/2} & n-3+\epsilon_V'-y_n\end{vmatrix}=0, \tag{21}$$

[7] T. D. Lee and D. Pines, Phys. Rev. **92**, 883 (1953).
[8] G. R. North, Phys. Rev. **164**, 2056 (1967).

where the notation has been simplified somewhat by introducing

$$y_n \equiv E_n/\omega, \tag{22a}$$

$$e' \equiv eQ/\omega, \tag{22b}$$

$$f' \equiv fQ/\omega, \tag{22c}$$

$$g' \equiv gQ/\omega, \tag{22d}$$

$$\epsilon'_{N,U,V} \equiv \epsilon_{N,U,V}\omega^{-1}. \tag{22e}$$

Just as in the case of the ordinary Lee model (ELM-2), ϵ_N is set equal to $e^2Q^2\omega^{-1}$ in order to make the lowest-energy eigenvalue of the sector $n=1$ vanish. Furthermore, just as in the case of the Bronzan-Lee model (ELM-3), ϵ_U is set equal to $f^2Q^2\omega^{-1}$ in order to make the lowest-energy eigenvalue of the sector $n=2$ vanish. As one would expect, it also turns out that the choice

$$e_V = g^2Q^2\omega^{-1} \tag{23}$$

will guarantee that the sector $n=3$ has one energy eigenvalue which vanishes, i.e., the value of the bare mass ϵ_V is adjusted so as to make the physical V particle have the same mass as the physical U particle, physical neutron, and proton.

Making these choices for the bare masses, the secular equation (21) can be rewritten in the following form:

$$S_4(y_n) = (n-3+g'^2-y_n)S_3(y_n) - (n-2)g'^2S_2(y_n) = 0, \tag{21$'$}$$

where

$$S_2(y_n) = \begin{vmatrix} n-y_n & e'\sqrt{n} \\ e'\sqrt{n} & n-1+e'^2-y_n \end{vmatrix}, \tag{24}$$

$$S_3(y_n) = \begin{vmatrix} n-y_n & e'\sqrt{n} & 0 \\ e'\sqrt{n} & n-1+e'^2-y_n & f'(n-1)^{1/2} \\ 0 & f'(n-1)^{1/2} & n-2+f'^2-y_n \end{vmatrix}. \tag{25}$$

Note that the secular equation for the ordinary Lee model (ELM-2) is simply

$$S_2(y_n) = 0, \tag{26}$$

and the secular equation for the Bronzan-Lee model (ELM-3) is given by

$$S_3(y_n) = 0. \tag{27}$$

It is clear from the secular equations (21′), (26), and (27) that for arbitrary values of the bare coupling constants e, f, and g, there is *no simple* relationship between the spectra of the ordinary Lee model (ELM-2), the Bronzan-Lee model (ELM-3), and the ELM-4. In the ICA, and in North's SCA as well, the energy eigenvalues for sector n correspond to the zeros of the corresponding polynomials $S_2(y_n)$, $S_3(y_n)$, and $S_4(y_n)$. Although there does not appear to be any simple relationship between the zeros of the polynomials $S_2(y_n)$, $S_3(y_n)$, and $S_4(y_n)$, there is a very simple relation between these three polynomials:

$$S_4(y_n) = (n-3+g'^2-y_n)S_3(y_n) - (n-2)g'^2S_2(y_n). \tag{28}$$

Furthermore, one can prove that the *three* energy eigenvalues of the Bronzan-Lee model for a given sector $n=1-q$ lie between the *four* energy eigenvalues of the ELM-4 for the same sector n, and also that the *two* energy eigenvalues of the ordinary Lee model for a given sector n will lie between the *three* eigenvalues of the Bronzan-Lee model for the same sector.[9]

Expanding the 4×4 determinant $S_4(y_n)$, one can rewrite the secular equation (21) in the form

$$y_n^4 + a_3y_n^3 + a_2y_n^2 + a_1y_n + a_0 = 0, \tag{29a}$$

where

$$a_0 \equiv n(n-1)(n-2)(n-3), \tag{29b}$$

$$\begin{aligned} a_1 \equiv\ & -4n^3+18n^2-22n+6-(n-2)(n-3)e'^2 \\ & -n(n-3)f'^2-n(n-1)g'^2-(n-3)(e'f')^2 \\ & -n(f'g')^2-(e'f'g')^2, \end{aligned} \tag{29c}$$

$$\begin{aligned} a_2 \equiv\ & 6n^2-18n+11+(2n-5)e'^2+(2n-3)f'^2 \\ & +(2n-1)g'^2+(e'f')^2+(e'g')^2+(f'g')^2, \end{aligned} \tag{29d}$$

$$a_3 \equiv -4n+6-e'^2-f'^2-g'^2. \tag{29e}$$

Using standard methods for the solution of quartic equations,[10] in principle one could obtain exact expressions for the four roots of Eq. (29).

Once the exact roots of the quartic equation have been determined, one proceeds in the same way as for the Bronzan-Lee model.[4] The first step is the ansatz

$$(\alpha_n\omega_k+\beta_n)f_k = u_k, \tag{30}$$

where the α_n and β_n are to be chosen so as to minimize the lowest-energy eigenvalue $\bar{E}_n$ for the sector n and simultaneously satisfy the normalization condition (14). The ansatz (30) implies the following relation between the values of ω and Q for the nth sector:

$$\alpha_n\omega_n+\beta_n = Q_n. \tag{31}$$

The lowest-energy eigenvalue $\bar{E}_n$ for any given sector n can be expressed as a function of ω_n and Q_n—call it $\bar{E}_n(\omega_n,Q_n)$. Then, using relation (31) between ω_n and Q_n, one can easily determine the total derivative of $\bar{E}_n(\omega_n,Q_n)$ with respect to ω_n: $d\bar{E}_n/d\omega_n = (\partial\bar{E}_n/\partial\omega_n) + \alpha_n(\partial\bar{E}_n/\partial Q_n)$. The minimization condition $d\bar{E}_n/d\omega_n = 0$ then yields one relation between α_n and β_n, and the normalization condition (14) yields a second

[9] This is a special case of the following theorem: If the k characteristic roots of a real symmetric matrix are distinct, they are in general separated by the $k-1$ characteristic roots of each diagonal submatrix of order $k-1$. A proof of this theorem is given, for example, in H. W. Turnbull and A. C. Aitken, *An Introduction to the Theory of Canonical Matrices* (Blackie & Son Ltd., London, 1932), pp. 101, 102.

[10] See, e.g., *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, edited by M. Abramowitz and I. A. Stegun (U. S. Government Printing Office, Washington, D. C. 20025, 1964), Sec. 3.8.3, pp. 17, 18.

relation. Hence the parameters α_n and β_n are uniquely determined for any sector $n\equiv 1-q$ and for arbitrary values of the coupling constants e, f, and g and of the momentum cutoff R^{-1} (except that the results cannot be extrapolated to the point-source limit $R=0$). Thus, in principle at least, the ICA solution of the ELM-4 is essentially no more difficult to obtain than the solution of a quartic equation. (For the sector $n=3$, one of the energy eigenvalues vanishes as a consequence of our choices for the bare masses. Hence the solution for the $n=3$ sector only involves the solution of a cubic equation. A strong-coupling solution for this particular sector is given in the Appendix.)

SOLUTION OF THE MODEL WITH k POSSIBLE BARYON STATES

It is now more or less obvious how to generalize or extend the ICA approximation to the case of an extended Lee model with k possible baryon states. Instead of e', f', g', etc., let us denote the corresponding parameters by g_i', with $i=1, 2, 3, \cdots, k-1$ (i.e., $g_1'=eQ\omega^{-1}$, $g_2'=fQ\omega^{-1}$, $g_3'=gQ\omega^{-1}$, etc.). If $S_1(y_n)$ is defined to be equal to unity, and $S_2(y_n)$ is defined by Eq. (24) with e' replaced by g_1', then one can easily verify that the three-term recurrence relation

$$S_i(y_n)=[n-(i-1)+g_{i-1}'^2-y_n]S_{i-1}(y_n) -g_{i-2}'^2[n-(i-2)]S_{i-2}(y_n) \quad (32)$$

can be used to determine the polynomial $S_k(y_n)$, whose zeros correspond to the energy eigenvalues of ELM-k. [Here we are completely ignoring the problem of choosing α_n and β_n to minimize the lowest-energy eigenvalue $\bar{E}_n=\omega\tilde{y}_n$, where $\tilde{y}_n$ denotes the smallest nonvanishing zero of the polynomial $S_k(y_n)$. It appears that, in general, this part of the problem will have to be solved numerically.]

In order to simplify the present discussion, we henceforth confine our attention to the case of "global coupling"; that is, all coupling constants are assumed to be identical. Then the recurrence relation (32) may be rewritten in the form

$$S_i(y_n)=[n-(i-1)+\lambda-y_n]S_{i-1}(y_n) -\lambda[n-(i-2)]S_{i-2}(y_n), \quad (33)$$

where

$$\lambda\equiv(gQ/\omega)^2>0. \quad (34)$$

We reiterate that the three-term recurrence relation (33) can be used to find $S_k(y_n)$ for arbitrary integer values of k, since S_1 was defined to be equal to unity and

$$S_2(y_n)\equiv n(n-1)-(2n-1+\lambda)y_n+y_n^2. \quad (24)$$

In an attempt to identify the polynomials defined by (33), let us introduce the generating function

$$G(x,t)\equiv\sum_{l=1}^{\infty} S_l(x)t^l, \quad (35)$$

where we have replaced y_n by x in order to simplify the notation. Then one can easily verify that the recurrence relation (33) implies

$$G=(n+\lambda-x)tG-\lambda nt^2G-t^2G'+\lambda t^3G' \quad (36)$$

or

$$-(G'/G)=t^{-2}+(x-n)t^{-1}+\lambda x(1-\lambda t)^{-1}. \quad (37)$$

One can easily integrate Eq. (37) to obtain

$$-\ln G=-t^{-1}+(x-n)\ln t-x\ln(1-\lambda t)+\text{const}, \quad (38)$$

or

$$G(x,t)=G_0(t_0/t)^{x-n}\left(\frac{1-\lambda t}{1-\lambda t_0}\right)^x \exp\left(\frac{1}{t}-\frac{1}{t_0}\right), \quad (39)$$

where

$$G_0\equiv\sum_{l=1}^{\infty} S_l(x)t_0^{\,l}. \quad (40)$$

A rather cursory examination of Chap. XIX of Vol. 3 of the Bateman Manuscript Project[11] has failed to identify $G(x,t)$ with any of the generating functions listed there.

In summary, one can easily use the recurrence relation (33) or the generating function (39) to determine $S_k(x)$. In order to determine completely the ICA solution for ELM-k, one must also determine the lowest zero $\tilde{x}$ of the polynomial $S_k(x)$ and then choose α_n and β_n to minimize $\tilde{x}$. Although we have not been able to obtain an explicit expression for the ICA eigenvalues for arbitrary values of k and for any arbitrary sector $n\equiv 1-q$, it is nevertheless clear from the present analysis that explicit solutions could be obtained by using a computer, at least for relatively small integers k.

APPENDIX: EXPLICIT SOLUTION OF ELM-4 FOR SECTOR $n=3$

For the sector $n=3$, Eq. (29) simplifies to

$$y_3\{y_3^3-(6+e'^2+f'^2+g'^2)y_3^2 +(11+e'^2+3f'^2+5g'^2+e'^2f'^2+e'^2g'^2+f'^2g'^2)y_3 -(6+6g'^2+3f'^2g'^2+e'^2f'^2g'^2)\}=0. \quad (A1)$$

The appearance of the zero root in the $n=3$ sector is due to our particular choice for the bare mass of the V particle, namely,

$$\epsilon_V=g^2Q^2\omega^{-1}. \quad (23)$$

The other three roots of Eq. (A1) may be written in the form

$$y_3^{\mathrm{I}}=2+\tfrac{1}{3}(e'^2+f'^2+g'^2)+2(\sqrt{c})\cos\tfrac{1}{3}\phi, \quad (A2a)$$

$$y_3^{\mathrm{II}}=2+\tfrac{1}{3}(e'^2+f'^2+g'^2)-(\sqrt{c})(\cos\tfrac{1}{3}\phi +\sqrt{3}\sin\tfrac{1}{3}\phi), \quad (A2b)$$

[11] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher Transcendental Functions* (McGraw-Hill Book Co., New York, 1955), Vol. 3, pp. 228–282.

$$y_3^{\mathrm{III}}=2+\tfrac{1}{3}(e'^2+f'^2+g'^2)-(\sqrt{c})(\cos\tfrac{1}{3}\phi-\sqrt{3}\sin\tfrac{1}{3}\phi), \quad \text{(A2c)}$$

where

$$c\equiv\tfrac{1}{3}+e'^2+\tfrac{1}{3}f'^2-\tfrac{1}{3}g'^2-\tfrac{1}{9}(e'^2f'^2+e'^2g'^2+f'^2g'^2)+\tfrac{1}{9}(e'^4+f'^4+g'^4), \quad \text{(A2d)}$$

$$\phi\equiv\cos^{-1}\left(\frac{-\tfrac{1}{2}b}{c\sqrt{c}}\right), \quad \text{(A2e)}$$

$$-\tfrac{1}{2}b=(7/6)e'^2-\tfrac{5}{6}f'^2+\tfrac{1}{6}g'^2-\tfrac{1}{3}(e'f')^2-\tfrac{2}{3}(e'g')^2+\tfrac{1}{2}(f'g')^2+\tfrac{1}{2}e'^4+\tfrac{1}{6}f'^4-\tfrac{1}{6}g'^4-(1/18)e'^4(f'^2+g'^2)-(1/18)f'^4(e'^2+g'^2)-(1/18)g'^4(e'^2+f'^2)+(1/27)(e'^6+f'^6+g'^6)+(2/9)(e'f'g')^2. \quad \text{(A2f)}$$

From now on we shall confine our attention to the special case of "global coupling," that is, we assume $e=f=g$. In this case, Eqs. (A2d) and (A2f) simplify to

$$c=\tfrac{1}{3}+g'^2, \quad \text{(A3a)}$$

$$-\tfrac{1}{2}b=\tfrac{1}{2}g'^2. \quad \text{(A3b)}$$

Hence

$$\cos\phi=\tfrac{1}{2}g'^2(\tfrac{1}{3}+g'^2)^{-3/2}=(1/2g')(1+\tfrac{1}{3}g'^{-2})^{-3/2}. \quad \text{(A3c)}$$

Expanding the right-hand side of Eq. (A3c) in powers of $\frac{1}{3}g'^2$, one finds

$$\cos\phi=(1/2g')[1-\tfrac{1}{2}g'^{-2}+(5/24)g'^{-4}-(35/432)g'^{-6}]+O(g'^{-9}). \quad \text{(A4)}$$

In other words, if $\frac{1}{2}(\omega/gQ)^2$ is very small compared to unity, then the angle ϕ will turn out to be slightly less than $\frac{1}{2}\pi$ rad, in which case the lowest-energy level is given by

$$\tilde{E}_3(\omega,Q)=E_3^{\mathrm{II}}=2\omega+g^2Q^2\omega^{-1}-gQ(1+\tfrac{1}{3}g'^{-2})^{1/2}\times(x-\sqrt{3}y), \quad \text{(A5a)}$$

where

$$x\equiv\cos\tfrac{1}{3}\phi, \quad \text{(A5b)}$$

$$y\equiv\sin\tfrac{1}{3}\phi. \quad \text{(A5c)}$$

We also note that the separations between level I or III and the lowest level II are given by

$$\Delta_3^{\mathrm{I,II}}\equiv E_3^{\mathrm{I}}-E_3^{\mathrm{II}}=gQ(3+g'^{-2})^{1/2}(\sqrt{3}x+y), \quad \text{(A6a)}$$

$$\Delta_3^{\mathrm{III,II}}\equiv E_3^{\mathrm{III}}-E_3^{\mathrm{II}}=2gQ(3+g'^{-2})^{1/2}y. \quad \text{(A6b)}$$

In the extreme strong-coupling limit, $\frac{1}{3}\phi\to\frac{1}{6}\pi$ rad, and

$$\Delta_3^{\mathrm{I,II}}=2\Delta_3^{\mathrm{III,II}}\cong2\sqrt{3}gQ, \quad \text{(A7)}$$

that is, the solutions E_3^{I} and E_3^{III} are each widely separated from E_3^{II} in the strong-coupling limit as $g\to\infty$.

We also note that $x\equiv\cos\frac{1}{3}\phi$ is related to $\cos\phi$ by the well-known trigonometric relation

$$4x^3-3x=\cos\phi, \quad \text{(A8)}$$

where $\cos\phi$ is given by Eq. (A3c). Using Eqs. (A4) and (A8), one can easily verify that

$$x=\tfrac{1}{2}\sqrt{3}+\tfrac{1}{12}(\omega/gQ)-(\sqrt{3}/144)(\omega/gQ)^2-(25/648)(\omega/gQ)^3+(397\sqrt{3}/2^8\times3^5)\times(\omega/gQ)^4+O(g'^{-5}), \quad \text{(A9a)}$$

$$y=\tfrac{1}{2}-\tfrac{1}{12}\sqrt{3}(\omega/gQ)-(1/144)(\omega/gQ)^2+(25\sqrt{3}/648)(\omega/gQ)^3+(397/2^8\times3^5)\times(\omega/gQ)^4+O(g'^{-5}). \quad \text{(A9b)}$$

After tedious but straightforward algebra, one finds

$$\tilde{E}_3=g^2Q^2\omega^{-1}-\sqrt{3}gQ+(7/6)\omega-(11\sqrt{3}/72)\times(\omega^2/gQ)+(1/36)(\omega^3/g^2Q^2)+(107\sqrt{3}/2^7\times3^5)\times(\omega^4/g^3Q^3)+O(\omega^5/g^4Q^4). \quad \text{(A10)}$$

At the corresponding stage in the ICA calculation, the lowest-energy levels for the $n=3$ sector of the ordinary Lee model and of the Bronzan-Lee model are given by

$$E_3^{\mathrm{Lee}}=\tfrac{5}{2}\omega+\tfrac{1}{2}g^2Q^2\omega^{-1}-\tfrac{1}{2}g^2Q^2\omega^{-1}[1+10(\omega/gQ)^2+(\omega/gQ)^4]^{1/2}$$
$$=6(\omega^3/g^2Q^2)-30(\omega^5/g^4Q^4)+O(\omega^7/g^6Q^6),$$
$$E_3^{\mathrm{Bronzan\text{-}Lee}}\cong46(\omega^3/g^2Q^2).$$

The results of North's strong-coupling treatment[8] applied to the $n=3$ sector of ELM-4 would be obtained by taking $f_k=N^{-1/2}u_k$. In other words, we should replace ω and Q according to the following rules:

$$\omega\to\Omega=N^{-1}\sum_k\omega_ku_k^2, \quad \text{(A11)}$$

$$Q\to N^{1/2}, \quad \text{(A12)}$$

where

$$N\equiv\sum_k u_k^2. \quad \text{(A13)}$$

In view of the similarity between $\tilde{E}_3$, E_3^{Lee}, and $E_3^{\mathrm{Bronzan\text{-}Lee}}$ (in particular, the fact that all three contain a term proportional to ω^3/g^2Q^2), we strongly suspect that the corresponding expansion of $\tilde{E}_n$ for ELM-4 will lead to the appearance of an "isobar" term $\sim n\omega^3/g^2Q^2$. However, because of the cumbersome nature of the algebraic solution of the quartic equation (29), this conjecture has not been verified. Furthermore, we believe that the same kind of isobar term (i.e., a term $\sim n\omega^3/g^2Q^2$) will also appear in the corresponding expansion of $\tilde{E}_n$ referring to the ICA solution for the nth sector of ELM-k. However, if $k>4$, an algebraic solution of the secular equation $S_k(y_n)=0$ is, in general, not possible.

Now let us return to our discussion of the ICA solution for the $n=3$ sector of ELM-4. Using Eq. (31) to replace Q by $\alpha\omega+\beta$, the lowest-energy eigenvalue becomes

$$\tilde{E}_3=2\omega+(g\alpha)^2\omega+2\alpha\beta g^2+(g\beta)^2\omega^{-1}+2\{[\tfrac{1}{3}+(g\alpha)^2]\omega^2+2\alpha\beta g^2\omega+(g\beta)^2\}^{1/2}\cos(\tfrac{1}{3}\phi+\tfrac{2}{3}\pi), \quad \text{(A14a)}$$

where

$$\phi\equiv\cos^{-1}\left(\frac{g^2(\alpha+\beta\omega^{-1})^2}{2[\frac{1}{3}+g^2(\alpha+\beta\omega^{-1})^2]^{3/2}}\right). \quad \text{(A14b)}$$

Hence $\tilde{E}_3$ has been expressed as a function of ω only. The minimization condition

$$d\tilde{E}_3/d\omega=0 \quad \text{(A15)}$$

then leads to one relation between α and β, and the normalization condition (14) leads to a second relation, namely,

$$\sum_k \frac{u_k{}^2}{(\alpha\omega_k+\beta)^2}=1. \quad (14)$$

By choosing a specific model for the momentum cutoff and then resorting to a computer solution, it is clear that one can obtain numerical values of $\tilde{E}_3$ for arbitrary values of the bare coupling constants e, f, and g. Also, we wish to emphasize that since the quartic equation (29) can be solved exactly for arbitrary integer values of n, the analogous program can also be pushed through for any sector of the ELM-4. Thus, the ICA solution of the ELM-4 is well understood, even though it is relatively cumbersome to obtain explicit results. It should also be noted that the problem becomes more complicated for ELM-k (with $k\geqslant 5$) because one can no longer determine $\tilde{E}_n(\omega,Q)$ by algebraic means, i.e., one apparently has to resort to a computer at an earlier stage in the ICA solution.

PHYSICAL REVIEW VOLUME 185, NUMBER 5 25 SEPTEMBER 1969

ω Independence of Internal Regge Couplings at Zero Momentum Transfer*

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(Received 29 April 1969)

Based on the analyticity properties of production amplitudes, we prove that the Reggeon-Reggeon-particle coupling $K(t_1,t_2,\omega)$ is independent of the ω angle when either t_1 or t_2 vanishes. This phenomenon is connected with the fact that the surfaces $t_1=0$ and $t_2=0$ are asymptotes of physical boundaries on which $\cos\omega$ is not defined. Our finding allows a considerable simplification of phenomenological analyses of multiparticle production data.

WITH the accumulation of high-energy production data for hadron collisions, it becomes increasingly interesting to study further the models which have been designed to describe such phenomena. One such model is the multi-Regge model[1] (hereafter referred to as MRM). This model is designed to describe the multiperipheral events where the invariant subenergies of the final particles are much greater than the corresponding momentum transfers. It is now known[2,3] that in MRM, in addition to the adjacent momentum transfers squared, it is very natural on group-theoretical ground to introduce an additional angle ω to describe the internal Regge couplings. There have been several attempts[3,4] to determine the dependence of the internal coupling on the angle ω. It is the purpose of this paper to point out that the internal Regge coupling associated with the leading asymptotic power term is independent of the ω angle if any one or both of the momentum transfers squared associated with this vertex are zero. [In this paper we refer to the asymptotic power term $s_1{}^{\alpha_1(t_1)}s_2{}^{\alpha_2(t_2)}$ as the leading power term if $\alpha_1(t_1)$ and $\alpha_2(t_2)$ are both parent trajectories.] The internal coupling associated with lower powers can be dependent on ω. This is a consequence of analyticity properties of the production amplitudes and has a simple geometrical interpretation.

From factorization of the leading power term, the leading internal Regge coupling which appears in three-particle production is the same as those which appear in n-particle production. Therefore, for the purpose of the present paper, it suffices to discuss the production amplitudes with only three final particles. We define the notation as follows (see Fig. 1):

$$t_1=(p-q)^2,\quad t_2=(p'-q'')^2,$$
$$s_1=(q+q')^2,\quad s_2=(q'+q''^2),\quad s=(p+p')^2.$$

The ω angle is defined to be the spatial angle between the plane formed by the three-vectors $\mathbf{p}'$ and $\mathbf{q}''$ and the plane formed by $\mathbf{p}$ and $\mathbf{q}$ in the rest frame of q'.

* Work supported by the U. S. Air Force Office of Scientific Research, under Contract No. AF 49(638)-1545.

[1] For a list of references about MRM, see Chang Hong-Mo, rapporteur talk, in *Proceedings of the Fourteenth International Conference on High-Energy Physics, Vienna, 1968* (CERN, Geneva, 1968), p. 391.

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