Covariant Phase-Space Calculations of n-Body Decay and Production Processes

RAJENDRA KUMAR
Department of Physics and Astrophysics, University of Delhi, Delhi-7, India

(Received 16 September 1968; revised manuscript received 20 February 1969)

Lorentz-invariant phase-space integrals for decay and production processes involving n particles in the final state—with integrand containing arbitrary invariant functions of momenta of particles—are transformed into simple definite integrals over Mandelstam-like variables. Given the T-matrix element squared as a function of scalar products of initial- and final-state particle momenta, the results may be used for the computation of the production cross section, decay rate, energy and momentum spectra, invariant mass spectra, and angular correlations.

I. INTRODUCTION

THE experimental information about elementary-particle interactions comes mainly from the study of decay and collision processes. In theory, these processes are described by the S matrix. The experimental consequences of the theory are obtained by integrating the squared S-matrix element over the available final states. The T-matrix element squared being a Lorentz-invariant quantity, its dependence on the initial- and final-state particle momenta can appear only in the form of scalar products of four-momenta. It is therefore desirable that the integration over the final-state particle momenta (constrained by the condition of energy-momentum conservation) be transformed into definite integrals over the independent scalar products of four-momenta. In the present note, we discuss such transformations of phase-space integrals[1] for decay and production processes involving n particles in the final state. Our aim is not merely to evaluate the phase-space factor[2]; in other words, the T-matrix element is not assumed to be momentum-independent. In fact, the constant-matrix-element approximation[3] cannot be expected to be good since dynamics plays an important role and is of primary interest in the study of elementary particle physics.

The number of scalar products of the type $P_i\cdot P_j$ $(i\neq j)$ which can be formed from the four-momenta of initial- and final-state particles in a decay or production process is $\frac{1}{2}N(N-1)$, where N denotes the total number of particles participating in the process. However, only $(3N-10)$ of these are independent. Hence, in general, the T-matrix element squared describing a decay or scattering process may depend on $(3N-10)$-independent Lorentz-invariant kinematical variables. Our aim is to transform the integration of squared T-matrix element over the final-state particle momenta into integration over these $(3N-10)$ variables. The transformation is conveniently done by making a judicious choice of $(3N-10)$-independent Mandelstam-like variables and artificially introducing Dirac δ functions, the arguments of which define the Mandelstam-like variables. The limits of integration can be obtained in a straightforward manner without depending on involved geometrical considerations. In the particular case, when the T-matrix element is a constant, the phase-space factor is obtained as an integral of rank $(n-2)$.

Given the T-matrix element, the results may be used for the computation of decay rate or cross section, invariant mass spectra of desired particles, and energy (or momentum) spectrum of any of the final-state particles in the c.m. system. With a little modification, angular correlations in the c.m. system can also be computed. Thus, the formulas given here may be useful in making a comparison of theoretical predictions with experiment in order to test the basic assumptions of the theory and may also be helpful in making spin and parity assignments of resonances. The transformations of phase-space integrals are discussed in Sec. II. Some applications of the results are illustrated in the Appendix.

II. TRANSFORMATIONS OF PHASE-SPACE INTEGRALS

A. Production Processes

The phase-space integral to be evaluated for the scattering process,

$$A_1(q_1)+A_2(q_2)\to\sum_{i=1}^{n}a_i(p_i)\,, \tag{1}$$

is of the form

$$\mathcal{P}_n=\left[\prod_{i=1}^{n}\int d^4p_i\,\delta(p_i^2+m_i^2)\right]\times\delta^4(Q-\sum_{i=1}^{n}p_i)F(q_1,q_2;\,p_i)\,, \tag{2}$$

where $Q=q_1+q_2$, $q_1^2=-M_1^2$, $q_2^2=-M_2^2$, and

$$F(q_1,q_2;\,p_i)$$

[1] An introduction to phase-space techniques may be found in R. Hagedorn, *Relativistic Kinematics* (W. A. Benjamin, Inc., New York, 1963); G. Källén, *Elementary Particle Physics* (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1964); and J. D. Jackson, *1962 Brandeis Lectures* (W. A. Benjamin, Inc., New York, 1963), Vol. I.

[2] A recurrence formula relating the phase-space factors for n- and $(n-1)$-particle final states was first given by P. P. Srivastava and E. C. G. Sudarshan, Phys. Rev. **110**, 765 (1958).

[3] For a review of the statistical model and its applications see, e.g., M. Kretzschmar, Ann. Rev. Nucl. Sci. **11**, 1 (1961). See also the critical review by R. Hagedorn, CERN **61-62**, 183 (1963).

is the T-matrix element squared, summed over the final-spin states and averaged over the initial spin states.

The $(3n-4)$-independent Lorentz-invariant kinematical variables which uniquely specify a point in the phase space and in terms of which all of $\frac{1}{2}(n+1)(n+2)$ scalar products of the type $P_j \cdot P_k$ $(j\neq k)$ formed from the four-momenta q_1, q_2, and p_i can be expressed, may be chosen as follows:

$$s\equiv s_0=-Q^2,\quad t_0=-(q_1-p_1)^2,$$

$$s_r=-(Q-\sum_{i=1}^{r} p_i)^2,\quad u_r=-(Q-p_{r+1})^2,$$

$$t_r=-(q_1-p_{r+1})^2,\quad 1\leqslant r\leqslant n-2,\tag{3}$$

where p_r denotes the four-momentum of the rth particle. The symbols u_0 and s_{n-1} used in the following expressions have the following meaning:

$$u_0\equiv s_1 \quad\text{and}\quad s_{n-1}\equiv m_n^2.\tag{4}$$

The phase-space integral $\mathcal{P}_n$ is then transformed into a definite integral of rank $(3n-5)$:

$$\mathcal{P}_n=\tfrac{1}{2}\pi\{\lambda(s,M_1^2,M_2^2)\}^{-1/2}\{s[\lambda(s,M_1^2,M_2^2)]^{-1/2}\}^{n-2}\prod_{r=1}^{n-2}\int_{s_{r-}}^{s_{r+}} ds_r \prod_{r=1}^{n-2}\left[\int_{u_{r-}}^{u_{r+}} du_r[\lambda(s,s_r,s_r')\lambda(s,m_{r+1}^2,u_r)]^{-1/2}\right]$$
$$\times\int_{t_{0-}}^{t_{0+}} dt_0 \prod_{r=1}^{n-2}\left[\int_{t_{r-}}^{t_{r+}} dt_r[(1-\xi_r^2)(1-\eta_r^2)(1-\zeta_r^2)]^{-1/2}\right]F(s_r;\,u_r;\,t_r),\tag{5}$$

where

$$s_r'=-(\sum_{i=1}^{r} p_i)^2=s_r+(r-1)s+\sum_{i=1}^{r} m_i^2-\sum_{j=1}^{r} u_{j-1},\tag{6}$$

$$\xi_r=[(s+M_1^2-M_2^2)(s+s_r'-s_r)-2s(rM_1^2+\sum_{i=1}^{r} m_i^2-\sum_{j=1}^{r} t_{j-1})][\lambda(s,M_1^2,M_2^2)\lambda(s,s_r,s_r')]^{-1/2},\tag{7}$$

$$\eta_r=[2s(s_r+m_{r+1}^2-s_{r+1})-(s+m_{r+1}^2-u_r)(s+s_r-s_r')][\lambda(s,m_{r+1}^2,u_r)\lambda(s,s_r,s_r')]^{-1/2},\tag{8}$$

$$\zeta_r=(\omega_r-\xi_r\eta_r)[(1-\xi_r^2)(1-\eta_r^2)]^{-1/2},\tag{9}$$

$$\omega_r=[(s+M_1^2-M_2^2)(s+m_{r+1}^2-u_r)-2s(M_1^2+m_{r+1}^2-t_r)][\lambda(s,M_1^2,M_2^2)\lambda(s,m_{r+1}^2,u_r)]^{-1/2},\tag{10}$$

and $\lambda(a,b,c)$ stands for $(a^2+b^2+c^2-2ab-2bc-2ca)$. The limits of integration of variables s_r, u_r, and t_r are as follows:

$$s_{r-}=(\sum_{i=r+1}^{n} m_i)^2,\quad s_{r+}=(\sqrt{s_{r-1}}-m_r)^2,\quad 1\leqslant r\leqslant n-2\tag{11}$$

$$u_{r\pm}=s+m_{r+1}^2-\frac{(s_r+m_{r+1}^2-s_{r+1})(s+s_r-s_r')}{2s_r}\pm\frac{[\lambda(s_r,m_{r+1}^2,s_{r+1})\lambda(s,s_r,s_r')]^{1/2}}{2s_r},\quad 1\leqslant r\leqslant n-2\tag{12}$$

$$t_{r\pm}=M_1^2+m_{r+1}^2-\frac{(s+M_1^2-M_2^2)(s+m_{r+1}^2-u_r)}{2s}+\frac{[\lambda(s,M_1^2,M_2^2)\lambda(s,m_{r+1}^2,u_r)]^{1/2}}{2s}X_{r\pm},\quad 0\leqslant r\leqslant n-2\tag{13}$$

where

$$X_{r\pm}=\xi_r\eta_r\pm[(1-\xi_r^2)(1-\eta_r^2)]^{1/2},\quad\text{for } r>0$$
$$=\pm1,\qquad \text{for } r=0.\tag{14}$$

It should be noticed that the limits of integration of s_r depend only on the preceding s_i variable (i.e., s_{r-1}) and the limits of integration of variables u_r depend on the variables s_i and the preceding u_j $(j<r)$, whereas the limits of the t_r integration depend on the variables s_i, u_i, and the preceding t_j $(j<r)$.

In deriving the formula (5), a judicious choice of $(3n-4)$ Mandelstam-like variables is important. Once this has been done, the rest is quite simple. Starting from the phase-space integral (2), the integration over d^4p_n is done using the δ function which expresses the energy-momentum conservation. δ functions in the variables s_r, u_r, and t_r are then introduced and correspondingly integrations are done over these variables, so that we have

$$\mathcal{P}_n=\prod_{r=1}^{n-2}\left[\int\int\int ds_r du_r dt_r\right]\int dt_0\, F(s_r;\,u_r;\,t_r)$$
$$\times\int d^4p_1\,\delta(p_1^2+m_1^2)\delta((Q-p_1)^2+s_1)\delta((q_1-p_1)^2+t_0)$$
$$\times\prod_{r=1}^{n-2}\left[\int d^4p_{r+1}\,\delta(p_{r+1}^2+m_{r+1}^2)\delta((Q-\sum_{i=1}^{r+1} p_i)^2+s_{r+1})\right.$$
$$\left.\times\delta((Q-p_{r+1})^2+u_r)\delta((q_1-p_{r+1})^2+t_r)\right].\tag{15}$$

Integrations over the four-momenta of all particles are done in the c.m. system characterized by $\mathbf{Q}=0$. However, coordinate reference systems with different orientations in three-dimensional space are employed for integration over the momenta of different particles. Integration over d^4p_1 is done in an obvious manner. Integration over d^4p_{r+1} $(1\leqslant r\leqslant n-2)$ is done using a coordinate system of reference in which the vector $\sum_{i=1}^r \mathbf{p}_i$ points along the z axis. The x axis is chosen such that $\mathbf{q}_1$ lies in the zx plane, making an angle $\cos^{-1}\xi_r$ with the z axis. In such a coordinate system the vector $\mathbf{p}_{r+1}$ points in a direction $(\theta_{r+1},\phi_{r+1})$ given by $\cos\theta_{r+1}=\eta_r$ and $\cos\phi_{r+1}=\zeta_r$. Integrations over $|\mathbf{p}_{r+1}|$, $\cos\theta_{r+1}$, E_{r+1}, and ϕ_{r+1} are done in this reference system using the first, second, third, and fourth δ functions, respectively, in the square bracket of expression (15). Now $(Q-\sum_{i=1}^r p_i)$ is a timelike four-vector, and the variables s_r can be given the maximum freedom so that the fact that the minimum value of the scalar product $(-p_i\cdot p_j)$ is m_im_j, gives the minimum value of variables like $-(p_i+p_j)^2$ and the maximum value of variables like $-(p_i-p_j)^2$:

$$[-(p_i+p_j)^2]_{\min}=(m_i+m_j)^2,\qquad [-(p_i-p_j)^2]_{\max}=(m_i-m_j)^2, \tag{16}$$

where $m_i^2=-p_i^2$ and $m_j^2=-p_j^2$. Hence the limits of integration of variables s_r given by (11) are obvious. While integrating over $\cos\theta_{r+1}$ using the second δ function in expression (15), the condition $\cos^2\theta_{r+1}\leqslant 1$ gives a quadratic inequality in E_{r+1} which in turn is related to u_r by virtue of the third δ function. This inequality can be written in the form $(u_r-u_{r+})(u_r-u_{r-})\leqslant 0$, which determines the limits of integration[4] of u_r. Finally, the limits of integration of variables t_r are determined from the fourth δ function itself, using the fact that $\cos\phi_{r+1}$ lies between -1 and $+1$.

The scattering or production cross section is given by (5) apart from a multiplicative factor:

$$\sigma(s)=c(s)\mathcal{P}_n,$$

where $c(s)$ is a constant which depends on the c.m. energy $(\sqrt{s})$. The invariant mass spectrum of desired number of particles in the final state may be obtained by suitably changing the order of integration of variables s_r. In order to get the invariant mass spectrum of $(n-r)$ particles $(\sum_{i=r+1}^n a_i)$, the integration over the variables s_i may be performed in the following order:

$$\prod_{i=1}^{n-2}\int ds_i\rightarrow\int ds_r\prod_{\alpha=1}^{r-1}\int ds_\alpha\prod_{\beta=r+1}^{n-2}\int ds_\beta.$$

The limits of integrations are now changed in the following way:

$$s_{r-}=(\sum_{i=r+1}^n m_i)^2,\qquad s_{r+}=(\sqrt{s}-\sum_{i=1}^r m_i)^2,$$
$$s_{\alpha-}=(\sum_{i=\alpha+1}^r m_i+\sqrt{s_r})^2,\quad s_{\alpha+}=(\sqrt{s_{\alpha-1}}-m_\alpha)^2, \tag{17}$$
$$s_{\beta-}=(\sum_{i=\beta+1}^n m_i)^2,\qquad s_{\beta+}=(\sqrt{s_{\beta-1}}-m_\beta)^2.$$

$d\sigma/ds_r$ is then obtained by dropping the integration over s_r. By suitably identifying the particle momenta, invariant mass spectrum of *any* $(n-r)$ particles can be computed. Energy spectrum of particle a_1 is obtained in the c.m. system by dropping the integration over ds_1 in (5) and multiplying by $2\sqrt{s}$. The energy spectrum is given by parametric equations

$$\frac{d\sigma}{dE_1}=2(\sqrt{s})c(s)\frac{d\mathcal{P}_n}{ds_1},\qquad E_1=(s+m_1^2-s_1)/2\sqrt{s}. \tag{18}$$

The parameter s_1 takes values in the range given by (11). Similarly, the momentum spectrum of particle a_1 in the c.m. system is given by the parametric equations

$$\frac{d\sigma}{d|\mathbf{p}_1|}=(2\sqrt{s})c(s)\frac{[\lambda(s,m_1^2,s_1)]^{1/2}}{s+m_1^2-s_1}\frac{d\mathcal{P}_n}{ds_1},\qquad |\mathbf{p}_1|=[\lambda(s,m_1^2,s_1)]^{1/2}/2\sqrt{s}. \tag{19}$$

The energy or momentum spectra of all particles in the final state may be computed in succession by suitably identifying the particle momenta.

Angular correlation between an initial-state particle and a final-state particle in the c.m. system is easily obtained. Thus the angular correlation between particles A_1 and a_1 may be computed by replacing the t_0 integration by integration over $\zeta_0=\mathbf{p}_1\cdot\mathbf{q}_1/|\mathbf{p}_1||\mathbf{q}_1|$. (It may be noticed that $\xi_0=\eta_0=0$, and, therefore, $\zeta_0=\omega_0$.) Making this transformation, we have

$$\mathcal{P}_n=(\pi/4s)\{s[\lambda(s,M_1^2,M_2^2)]^{-1/2}\}^{n-2}\times\int_{-1}^{+1}d\zeta_0\prod_{r=1}^{n-2}\left[\int_{s_{1-}}^{s_{r+}}ds_r\right]\int_{u_{1-}}^{u_{1+}}du_1[\lambda(s,m_2^2,u_1)]^{-1/2}$$
$$\times\prod_{r=2}^{n-2}\left[\int_{u_{r-}}^{u_{r+}}du_r\{\lambda(s,s_r,s_r')\lambda(s,m_{r+1}^2,u_r)\}^{-1/2}\right]$$
$$\times\prod_{r=1}^{n-2}\left[\int_{t_{r-}}^{t_{r+}}dt_r\{(1-\xi_r^2)(1-\eta_r^2)(1-\zeta_r^2)\}^{-1/2}\right]\times F(s_r;u_r;t_r). \tag{20}$$

[4] An illustration of the determination of limits of integration in this way may be found in R. Hagedorn, Ref. 1, p. 110.

The variable t_0 occurring in $F(s_r;u_r;t_r)$ or elsewhere now stands for the following:

$$t_0=M_1^2+m_1^2-\frac{(s+m_1^2-s_1)(s+M_1^2-M_2^2)}{2s}+\zeta_0\frac{[\lambda(s,m_1^2,s_1)\lambda(s,M_1^2,M_2^2)]^{1/2}}{2s}. \quad (21)$$

From (20), the angular correlation $d\sigma/d\zeta_0$ is obtained in an obvious manner.

Angular correlation between two final-state particles is a bit difficult. From the definition of η_r, we know that in the c.m. system,

$$\eta_r=\mathbf{p}_{r+1}\cdot\sum_{i=1}^{r}\mathbf{p}_i/|\mathbf{p}_{r+1}|\,|\sum_{i=1}^{r}\mathbf{p}_i|. \quad (22)$$

Hence, $\eta_1=\mathbf{p}_1\cdot\mathbf{p}_2/|\mathbf{p}_1|\,|\mathbf{p}_2|$ is the cosine of angle between the momentum vectors of a_1 and a_2. Using (8), integration over u_1 can be changed into integration over η_1 and we can solve for u_1 in terms of η_1, s, s_1, and s_2. The limits of s_1 and s_2 integrations become more restricted since they now depend on η_1. However, the limits of integration as given in (11) may still be used subject to the condition that u_1 (which is now a function of η_1, s, s_1, and s_2) lies between u_{1-} and u_{1+} given by (12). We, therefore, have[5]

$$\begin{aligned}\mathcal{P}_n={}&\tfrac{1}{2}\pi[\lambda(s,M_1^2,M_2^2)]^{-1/2}\{s[\lambda(s,M_1^2,M_2^2)]^{-1/2}\}^{n-2}\\&\times\int_{-1}^{+1}d\eta_1\int_{s_{1-}}^{s_{1+}}\int_{s_{2-}}^{s_{2+}}\frac{ds_1ds_2\,\theta(u_1-u_{1-})\theta(u_{1+}-u_1)}{\{(s+s_1-m_1^2)+\eta_1(s+m_2^2-u_1)[\lambda(s,m_1^2,s_1)/\lambda(s,m_2^2,u_1)]^{1/2}\}}\prod_{r=3}^{n-2}\left[\int_{s_{r-}}^{s_{r+}}ds_r\right]\\&\times\prod_{r=2}^{n-2}\left[\int_{u_{r-}}^{u_{r+}}\frac{du_r}{[\lambda(s,s_r,s_r')\lambda(s,m_{r+1}^2,u_r)]^{1/2}}\right]\int_{t_{0-}}^{t_{0+}}dt_0\prod_{r=1}^{n-2}\left[\int_{t_{r-}}^{t_{r+}}\frac{dt_r}{[(1-\xi_r^2)(1-\eta_r^2)(1-\zeta_r^2)]^{1/2}}\right]F(s_r;u_r;t_r), \quad (23)\end{aligned}$$

where

$$\begin{aligned}u_1={}&s+m_2^2-\tfrac{1}{2}\{(s+s_1-m_1^2)(s_1+m_2^2-s_2)\\&-\eta_1[\lambda(s,s_1,m_1^2)]^{1/2}[\lambda(s_1,m_2^2,s_2)-(m_2^2/s)(1-\eta_1^2)\lambda(s,s_1,m_1^2)]^{1/2}\}[s_1+\lambda(s,m_1^2,s_1)(1-\eta_1^2)/4s]^{-1} \quad (24)\end{aligned}$$

and

$$u_{1\pm}=s+m_2^2-\frac{(s_1+m_2^2-s_2)(s+s_1-m_1^2)}{2s_1}\pm\frac{[\lambda(s_1,m_2^2,s_2)\lambda(s,s_1,m_1^2)]^{1/2}}{2s_1}. \quad (25)$$

$d\mathcal{P}_n/d\eta_1$ describes the angular correlation between particles a_1 and a_2. Thus, by suitably identifying the momenta of particles, angular correlation between any two particles in the final state may be computed.

Finally, the angular correlation between certain planes defined by the particle momenta in the c.m. system can be obtained. By definition, ζ_r $(1\leqslant r\leqslant n-2)$ is the cosine of the angle between the two planes: (i) the plane defined by the vectors $\mathbf{q}_1$ and $\sum_{i=1}^{r}\mathbf{p}_i$, and (ii) the plane defined by the vectors $\mathbf{p}_{r+1}$ and $\sum_{i=1}^{r}\mathbf{p}_i$ (or $\sum_{i=r+2}^{n}\mathbf{p}_i$). Using the relation between the variables ζ_r and t_r given by (9) and (10), t_r integration may be changed into integration over ζ_r with limits of integration of ζ_r from -1 to $+1$ for all values of r. Hence the integration over the variables ζ_r may be done in any order and $d\sigma/d\zeta_r$ then describes the angular correlation between the two planes defined above. We have

$$\begin{aligned}\mathcal{P}_n={}&2^{-n}\frac{\pi}{s}\int_{-1}^{+1}d\zeta_0\prod_{r=1}^{n-2}\left[\int_{-1}^{+1}\frac{d\zeta_r}{(1-\zeta_r^2)^{1/2}}\right]\prod_{r=1}^{n-2}\left[\int_{s_{r-}}^{s_{r+}}ds_r\right]\\&\times\int_{u_{1-}}^{u_{1+}}du_1\prod_{r=2}^{n-2}\left[\int_{u_{r-}}^{u_{r+}}\frac{du_r}{[\lambda(s,s_r,s_r')]^{1/2}}\right]\\&\times F(s_r;u_r;t_r(\zeta_r)), \quad (26)\end{aligned}$$

where the variables t_r occurring in $F(s_r;u_r;t_r)$ can be expressed in terms of ζ_r by using (9) and (10).

In the particular cases when the function F is simpler, $\mathcal{P}_n$ can be reduced to lower-rank integrals. If, for example, it does not depend on any of the variables t_r, integrating the expression (26) over all ζ_r gives[6]

$$\mathcal{P}_n=\frac{1}{s}(\tfrac{1}{2}\pi)^{n-1}\prod_{r=1}^{n-2}\left[\int_{s_{r-}}^{s_{r+}}ds_r\right]\int_{u_{1-}}^{u_{1+}}du_1\times\prod_{r=2}^{n-2}\left[\int_{u_{r-}}^{u_{r+}}\frac{du_r}{[\lambda(s,s_r,s_r')]^{1/2}}\right]F(s_r;u_r). \quad (27)$$

Further, if F does not depend on any of the variables u_r as well, integration over all u_r gives

$$\mathcal{P}_n=\frac{1}{s}(\tfrac{1}{2}\pi)^{n-1}\prod_{r=1}^{n-2}\left[\int_{s_{r-}}^{s_{r+}}\frac{ds_r}{s_r}\{\lambda(s_{r-1},s_r,m_r^2)\}^{1/2}\right]\times[\lambda(s_{n-2},m_{n-1}^2,m_n^2)]^{1/2}F(s_r). \quad (28)$$

[5] For $n=3$, $s_2=s_{n-1}=m_n^2$. It is understood that there is no integration over s_2 in this case.

[6] The result could have been obtained by removing

$$\prod_{r=0}^{n-2}\left[\int dt_r\,\delta(t_r+(q_1-p_{r+1})^2)\right]$$

from (15), in which case integration over $d\phi_{r+1}$ gives just a factor of 2π.

In the particular case when $F(s_r)=1$, the expression (28) gives just the phase-space factor.[7]

The order of integration of variables u_r and t_r cannot be interchanged in (5), since the limits of integration of variables t_r depend on u_i. Furthermore, integrations over the four-momenta of all final-state particles were done in the c.m. system in obtaining (5). Making use of Lorentz invariance, it is possible to integrate over the four-momenta of different particles in different inertial frames. If we choose to integrate over d^4p_{r+1} in a frame of reference in which $\mathbf{Q}-\sum_{i=1}^r \mathbf{p}_i=0$, it is more convenient to choose the variables t_r in a slightly different manner. With variables s_r and u_r still defined by (3), the variables t_r are now defined as[8]

$$t_r=-(q_1-\sum_{i=1}^{r+1} p_i)^2, \quad 0\leqslant r\leqslant n-2. \tag{29}$$

$\mathcal{P}_n$ can now be written in two different forms which differ in the order of integrations over the variables u_r and t_r. If the integration over the variables t_r is to be done first, $\mathcal{P}_n$ should be transformed in such a way that the limits of integration of t_r do not depend on u_i. In that case, we have

$$\mathcal{P}_n=\tfrac{1}{2}\pi[\lambda(s,M_1^2,M_2^2)]^{-1/2}\prod_{r=1}^{n-2}\left[\int_{s_{r-}}^{s_{r+}}\frac{s_r ds_r}{[\lambda(s_r,s_{r+1},m_{r+1}^2)]^{1/2}}\right]\int_{t_{0-}}^{t_{0+}}dt_0\prod_{r=1}^{n-2}\left[\int_{t_{r-}}^{t_{r+}}\frac{dt_r}{[\lambda(s_r,t_{r-1},M_2^2)]^{1/2}}\right]$$
$$\times\prod_{r=1}^{n-2}\left[\int_{u_{r-}}^{u_{r+}}\frac{du_r}{[\lambda(s,s_r,s_r')]^{1/2}[(1-\xi_r^2)(1-\eta_r^2)(1-\zeta_r^2)]^{1/2}}\right]F(s_r;u_r;t_r), \tag{30}$$

where s_r' is given by (6) and

$$\xi_r=[2s_r(s+M_2^2-M_1^2)-(s+s_r-s_r')(s_r+M_2^2-t_{r-1})]\times[\lambda(s,s_r,s_r')\lambda(s_r,t_{r-1},M_2^2)]^{-1/2}, \tag{31}$$

$$\eta_r=[2s_r(t_r-t_{r-1}-m_{r+1}^2)+(s_r+m_{r+1}^2-s_{r+1})(s_r+t_{r-1}-M_2^2)]\times[\lambda(s_r,m_{r+1}^2,s_{r+1})\lambda(s_r,t_{r-1},M_2^2)]^{-1/2}, \tag{32}$$

$$\zeta_r=(\omega_r-\xi_r\eta_r)[(1-\xi_r^2)(1-\eta_r^2)]^{-1/2}, \tag{33}$$

$$\omega_r=[(s+s_r-s_r')(s_r+m_{r+1}^2-s_{r+1})-2s_r(s+m_{r+1}^2-u_r)]\times[\lambda(s,s_r,s_r')\lambda(s_r,m_{r+1}^2,s_{r+1})]^{-1/2}. \tag{34}$$

The limits of integrations of variables s_r are given by (11) and

$$t_{r\pm}=t_{r-1}+m_{r+1}^2-\frac{(s_r+m_{r+1}^2-s_{r+1})(s_r+t_{r-1}-M_2^2)}{2s_r}\pm\frac{[\lambda(s_r,m_{r+1}^2,s_{r+1})\lambda(s_r,t_{r-1},M_2^2)]^{1/2}}{2s_r},$$
$$0\leqslant r\leqslant n-2;\ t_{-1}\equiv M_1^2 \tag{35}$$

$$u_{r\pm}=s+m_{r+1}^2-\frac{(s+s_r-s_r')(s_r+m_{r+1}^2-s_{r+1})}{2s_r}+\frac{[\lambda(s,s_r,s_r')\lambda(s_r,m_{r+1}^2,s_{r+1})]^{1/2}}{2s_r}$$
$$\times\{\xi_r\eta_r\pm[(1-\xi_r^2)(1-\eta_r^2)]^{1/2}\},\quad 1\leqslant r\leqslant n-2. \tag{36}$$

ξ_r, η_r, and ζ_r given by expressions (31)–(33) have the following geometrical meaning in the reference frames characterized by $\mathbf{Q}-\sum_{i=1}^r \mathbf{p}_i=0$:

$$\begin{aligned}\xi_r&=\mathbf{Q}\cdot(\mathbf{q}_1-\sum_{i=1}^r \mathbf{p}_i)/|\mathbf{Q}|\,|\mathbf{q}_1-\sum_{i=1}^r \mathbf{p}_i|\\&=-\mathbf{q}_2\cdot\mathbf{Q}/|\mathbf{q}_2|\,|\mathbf{Q}|,\\\eta_r&=(\mathbf{q}_1-\sum_{i=1}^r \mathbf{p}_i)\cdot\mathbf{p}_{r+1}/|\mathbf{q}_1-\sum_{i=1}^r \mathbf{p}_i|\,|\mathbf{p}_{r+1}|,\\\zeta_r&=\cos\phi_{r+1},\end{aligned} \tag{37}$$

where ϕ_{r+1} is the angle between the following two planes: (i) the plane defined by the vectors $\mathbf{Q}$ and $(\sum_{i=1}^r \mathbf{p}_i)$ (or $\mathbf{q}_1-\sum_{i=1}^r \mathbf{p}_i$) and (ii) the plane defined by the vectors $\mathbf{p}_{r+1}$ and $(\mathbf{q}_1-\sum_{i=1}^r \mathbf{p}_i)$. The limits of integration[9] of variables t_r and u_r have been obtained from their definition using the fact that η_r and ζ_r can take values only in the range -1 to $+1$.

In the particular case when F does not depend on any of the variables u_r, we have

$$\mathcal{P}_n=(\tfrac{1}{2}\pi)^{n-1}\prod_{r=1}^{n-2}\left[\int_{s_{r-}}^{s_{r+}}ds_r\right]\times\prod_{r=0}^{n-2}\left[\int_{t_{r-}}^{t_{r+}}\frac{dt_r}{[\lambda(s_r,t_{r-1},M_2^2)]^{1/2}}\right]F(s_r;t_r). \tag{38}$$

Further, if the function F does not depend on any of the variables t_r also, the expression (38) can be reduced to (28). Scattering or production cross section, invariant mass spectra, and energy spectra (in the c.m. system) of final-state particles may be computed as discussed

[7] A similar expression for the phase-space factor has been obtained by B. Almgren, University of Lund report (unpublished). The author is thankful to Dr. J. S. Vaishya for drawing his attention to this report.

[8] It may be seen that with this definition of variables t_r, the integration over the four-momenta of all particles can be conveniently done in the c.m. system also.

[9] It may be noticed that the limits of integration of u_r in the formula (5) and the limits of integration of t_r in (30) are not obtained in an identical fashion.

earlier. Angular correlation between particles A_1 and a_1 (in the c.m. system) may be computed by changing the integration over t_0 into integration over η_0. However, for the computation of angular correlations among final-state particles in the c.m. system, only Eqs. (23) and (26) are appropriate.

B. Decay Processes

The phase-space integral to be evaluated for the decay process

$$A(Q)\rightarrow\sum_{i=1}^{n} a_i(p_i) \tag{39}$$

is of the form

$$\mathfrak{D}_n=\prod_{i=1}^{n}[d^4p_i\delta(p_i^2+m_i^2)]\times\delta^4(Q-\sum_{i=1}^{n}p_i)F(Q;p_i),\quad n\geqslant 3. \tag{40}$$

The phase-space factor in this case is the same as in the case of collision processes with s replaced by M^2, where $M^2=-Q^2$. However, in general, the transformation in this case would be somewhat different. In particular, it is obvious that the set of variables t_r will have to be defined in a different manner. The $(3n-7)$-independent Mandelstam-like variables which uniquely define a point in the phase space may be defined as follows:

$$s_r=-(Q-\sum_{i=1}^{r}p_i)^2,\quad u_r=-(Q-p_{r+1})^2,\qquad 1\leqslant r\leqslant n-2 \tag{41}$$
$$t_r=-(Q-\sum_{i=2}^{r+1}p_i)^2,\qquad 2\leqslant r\leqslant n-2.$$

The meaning of symbols s_0, s_{n-1}, u_0, and t_1 which are used in the expressions below is quite obvious ($s_0\equiv M^2$, $s_{n-1}\equiv m_n^2$, $u_0\equiv s_1$, $t_1\equiv u_1$). The phase-space integral $\mathfrak{D}_n$ is then transformed into the following definite integral of rank $(3n-7)$:

$$\mathfrak{D}_n=\frac{\pi^2}{4M^2}M^{2(n-3)}\prod_{r=1}^{n-2}\left[\int_{s_{r-}}^{s_{r+}}ds_r\right]\int_{u_{1-}}^{u_{1+}}du_1\prod_{r=2}^{n-2}\left[\int_{u_{r-}}^{u_{r+}}\frac{du_r}{[\lambda(M^2,s_r,s_r')\lambda(M^2,m_{r+1}^2,u_r)]^{1/2}}\right]$$
$$\times\prod_{r=2}^{n-2}\left[\int_{t_{r-}}^{t_{r+}}\frac{dt_r}{[\lambda(M^2,t_{r-1},t_{r-1}')]^{1/2}[(1-\xi_r^2)(1-\eta_r^2)(1-\zeta_r^2)]^{1/2}}\right]F(s_r;u_r;t_r), \tag{42}$$

where s_r' is given by (6) (with s replaced by M^2) and

$$t_r'=-(\sum_{i=2}^{r+1}p_i)^2=t_r+(r-1)M^2+\sum_{i=2}^{r+1}m_i^2-\sum_{j=1}^{r}u_j, \tag{43}$$

$$\xi_r=\frac{\lambda(M^2,s_r,s_r')+\lambda(M^2,t_{r-1},t_{r-1}')-\lambda(M^2,m_1^2,s_1)}{2[\lambda(M^2,s_r,s_r')\lambda(M^2,t_{r-1},t_{r-1}')]^{1/2}}=\frac{(M^2+s_r'-s_r)(M^2+t_{r-1}'-t_{r-1})-2M^2(s_r'+t_{r-1}'-m_1^2)}{[\lambda(M^2,s_r,s_r')\lambda(M^2,t_{r-1},t_{r-1}')]^{1/2}}, \tag{44}$$

$$\eta_r=\frac{\lambda(M^2,s_{r+1},s_{r+1}')-\lambda(M^2,s_r,s_r')-\lambda(M^2,m_{r+1}^2,u_r)}{2[\lambda(M^2,s_r,s_r')\lambda(M^2,m_{r+1}^2,u_r)]^{1/2}}=\frac{2M^2(s_r+m_{r+1}^2-s_{r+1})-(M^2+m_{r+1}^2-u_r)(M^2+s_r-s_r')}{[\lambda(M^2,s_r,s_r')\lambda(M^2,m_{r+1}^2,u_r)]^{1/2}}, \tag{45}$$

$$\zeta_r=(\omega_r-\xi_r\eta_r)[(1-\xi_r^2)(1-\eta_r^2)]^{-1/2}, \tag{46}$$

$$\omega_r=\frac{\lambda(M^2,t_r,t_r')-\lambda(M^2,t_{r-1},t_{r-1}')-\lambda(M^2,m_{r+1}^2,u_r)}{2[\lambda(M^2,t_{r-1},t_{r-1}')\lambda(M^2,m_{r+1}^2,u_r)]^{1/2}}=\frac{2M^2(t_{r-1}+m_{r+1}^2-t_r)-(M^2+m_{r+1}^2-u_r)(M^2+t_{r-1}-t_{r-1}')}{[\lambda(M^2,t_{r-1},t_{r-1}')\lambda(M^2,m_{r+1}^2,u_r)]^{1/2}}. \tag{47}$$

The limits of integrations of variables s_r and u_r are given by (11) and (12) and the limits of integration of variables t_r are

$$t_{r\pm}=t_{r-1}+m_{r+1}^2-\frac{(M^2+m_{r+1}^2-u_r)(M^2+t_{r-1}-t_{r-1}')}{2M^2}+\frac{[\lambda(M^2,m_{r+1}^2,u_r)\lambda(M^2,t_{r-1},t_{r-1}')]^{1/2}}{2M^2}\times\{-\xi_r\eta_r\pm[(1-\xi_r^2)(1-\eta_r^2)]^{1/2}\}. \tag{48}$$

ξ_r, η_r, and ζ_r defined by Eqs. (44)–(46) have the following geometrical meaning in the rest frame of decaying particle:

$$\xi_r=(\sum_{i=1}^{r}\mathbf{p}_i)\cdot(\sum_{i=2}^{r}\mathbf{p}_i)/|\sum_{i=1}^{r}\mathbf{p}_i||\sum_{i=2}^{r}\mathbf{p}_i|,$$
$$\eta_r=\mathbf{p}_{r+1}\cdot(\sum_{i=1}^{r}\mathbf{p}_i)/|\mathbf{p}_{r+1}||\sum_{i=1}^{r}\mathbf{p}_i|, \tag{49}$$
$$\zeta_r=\cos\phi_{r+1},$$

where ϕ_{r+1} is the angle between the planes defined by the momentum vectors $\mathbf{p}_1$ (or $\sum_{i=2}^{r}\mathbf{p}_i$) and $\sum_{i=1}^{r}\mathbf{p}_i$ and the vectors $\mathbf{p}_{r+1}$ and $\sum_{i=1}^{r}\mathbf{p}_i$ (or $\sum_{i=r+2}^{n}\mathbf{p}_i$).

The invariant mass spectra and energy spectra of final-state particles in the rest frame of decaying particle are obtained from (42) in the same way as discussed in the case of collision processes. Angular correlation between the particles a_1 and a_2 is obtained by changing the u_1 integration into integration over the variable η_1 which gives the cosine of angle between the vectors $\mathbf{p}_1$ and $\mathbf{p}_2$ in the rest frame of A. We have[10]

$$\mathfrak{D}_n=\frac{\pi^2}{4M^2}M^{2(n-3)}\int_{-1}^{+1}d\eta_1\int_{s_{1-}}^{s_{1+}}\int_{s_{2-}}^{s_{2+}}\frac{ds_1ds_2[\lambda(M^2,m_1^2,s_1)\lambda(M^2,m_2^2,u_1)]^{1/2}\theta(u_{1+}-u_1)\theta(u_1-u_{1-})}{\{(M^2+s_1-m_1^2)+\eta_1(M^2+m_2^2-u_1)[\lambda(M^2,m_1^2,s_1)/\lambda(M^2,m_2^2,u_1)]^{1/2}\}}$$
$$\times\prod_{r=3}^{n-2}\left[\int_{s_{r-}}^{s_{r+}}ds_r\right]\prod_{r=2}^{n-2}\left[\int_{u_{r-}}^{u_{r+}}\frac{du_r}{[\lambda(M^2,s_r,s_r')\lambda(M^2,m_{r+1}^2,u_r)]^{1/2}}\right]$$
$$\times\prod_{r=2}^{n-2}\left[\int_{t_{r-}}^{t_{r+}}\frac{dt_r}{[\lambda(M^2,t_{r-1},t_{r-1}')]^{1/2}[(1-\xi_r^2)(1-\eta_r^2)(1-\zeta_r^2)]^{1/2}}\right]F(s_r;u_r;t_r),\quad(50)$$

where u_1 and $u_{1\pm}$ are given by (24) and (25) with s replaced by M^2. Using the relation between ζ_r and t_r given by Eqs. (46) and (47), integration over t_r can be changed into integration over ζ_r and distributions in ζ_r can be obtained as in the case of collision processes. In the particular cases when $F(s_r;u_r;t_r)$ does not depend on some of the variables, lower-rank integrals may be obtained for $\mathfrak{D}_n$.

In transforming the phase-space integrals into definite integrals over Mandelstam-like variables, we have throughout defined the variables s_r and u_r as given in Eq. (3). Though this choice is convenient, it is apparant that it is not unique and in some particular cases, other definitions of these variables may be useful. As an example we describe an alternative transformation of the phase-space integral $\mathfrak{D}_n$ in terms of the following independent variables:

$$s_r=-\Big(\epsilon_r\sum_{i=r-1}^{n}p_i-\sum_{i=r-\epsilon_r}^{r-\epsilon_r+1}p_i\Big)^2,$$
$$t_r=-\Big(Q-\sum_{i=2}^{r+1}p_i\Big)^2,\qquad 1\leqslant r\leqslant n-2\qquad(51)$$
$$u_r=-[Q-(1-\epsilon_r)p_r-p_{r+1}]^2,\quad 2\leqslant r\leqslant n-2$$

where

$$\epsilon_r=1,\quad\text{for } r \text{ an even integer}$$
$$=0,\quad\text{for } r \text{ an odd integer}.$$

$\mathfrak{D}_n$ is then transformed into the following definite integral:

$$\mathfrak{D}_n=(\pi^2/4M^2)M^{2(n-3)}$$
$$\times\prod_{r=1}^{n-2}\left[\int_{s_{r-}}^{s_{r+}}ds_r\right]\prod_{r=2}^{n-2}\left[\int_{u_{r-}}^{u_{r+}}\frac{du_r}{[\lambda(s_{r+1},u_{r+1},M^2)\lambda(u_r,m_{r+1}^2,M^2)]^{\epsilon_r/2}[\lambda(M^2,s_{r-1},s_{r-1}')\lambda(M^2,s_r,u_r)]^{(1-\epsilon_r)/2}}\right]$$
$$\times\int_{t_{1-}}^{t_{1+}}dt_1\prod_{r=2}^{n-2}\left[\int_{t_{r-}}^{t_{r+}}\frac{dt_r}{[\lambda(M^2,t_{r-1},t_{r-1}')]^{\epsilon_r/2}[\lambda(M^2,t_{r-2},t_{r-2}')]^{(1-\epsilon_r)/2}[(1-\xi_r^2)(1-\eta_r^2)(1-\zeta_r^2)]^{1/2}}\right]$$
$$\times F(s_r;u_r;t_r),\quad(52)$$

where

$$s_{2r}'=s_{2r}+(r-1)M^2+\sum_{j=1}^{r}(s_{2j-1}-u_{2j-1}),\quad(53)$$

$$t_{2r-1}'=t_{2r-1}-t_1+(r-1)M^2+m_2^2+\sum_{j=2}^{r}(s_{2j-1}-u_{2j-1}),\quad(54)$$

$$\xi_{2r}=[2M^2(s_{2r+1}+t_{2r-1}-t_{2r+1})-(M^2+s_{2r+1}-u_{2r+1})(M^2+t_{2r-1}-t_{2r-1}')]\times[\lambda(M^2,s_{2r+1},u_{2r+1})\lambda(M^2,t_{2r-1},t_{2r-1}')]^{-1/2},\quad(55)$$

$$\xi_{2r+1}=[(M^2+s_{2r}-s_{2r}')(M^2+t_{2r-1}-t_{2r-1}')-2M^2(s_{2r}+t_{2r-1}-m_1^2)]\times[\lambda(M^2,s_{2r},s_{2r}')\lambda(M^2,t_{2r-1},t_{2r-1}')]^{-1/2},\quad(56)$$

$$\eta_{2r}=[(M^2+s_{2r+1}-u_{2r+1})(M^2+m_{2r+1}^2-u_{2r})-2M^2(s_{2r+1}+m_{2r+1}^2-m_{2r+2}^2)]\times[\lambda(M^2,s_{2r+1},u_{2r+1})\lambda(M^2,m_{2r+1}^2,u_{2r})]^{-1/2},\quad(57)$$

[10] A formula for the angular correlations in three-body decays is given by M. M. Nieto, Rev. Mod. Phys. **40**, 140 (1968).

$$\eta_{2r+1}=[2M^2(s_{2r}+s_{2r+1}-s_{2r+2})-(M^2+s_{2r+1}-u_{2r+1})(M^2+s_{2r}-s_{2r}')]\times[\lambda(M^2,s_{2r+1},u_{2r+1})\lambda(M^2,s_{2r},s_{2r}')]^{-1/2}, \quad (58)$$

$$\zeta_r=(\omega_r-\xi_r\eta_r)[(1-\xi_r^2)(1-\eta_r^2)]^{-1/2}, \quad (59)$$

$$\omega_{2r}=[2M^2(t_{2r-1}-t_{2r}+m_{2r+1}^2)-(M^2+m_{2r+1}^2-u_{2r})(M^2+t_{2r-1}-t_{2r-1}')]\times[\lambda(M^2,m_{2r+1}^2,u_{2r})\lambda(M^2,t_{2r-1},t_{2r-1}')]^{-1/2}, \quad (60)$$

$$\omega_{2r+1}=[2M^2(t_{2r-1}-t_{2r+1}+s_{2r+1})-(M^2+s_{2r+1}-u_{2r+1})(M^2+t_{2r-1}-t_{2r-1}')]\times[\lambda(M^2,s_{2r+1},u_{2r+1})\lambda(M^2,t_{2r-1},t_{2r-1}')]^{-1/2}. \quad (61)$$

The limits of integration are[11]

$$[s_{2r-1}]_-=(m_{2r-1}+m_{2r})^2,\quad [s_{2r-1}]_+=(\sqrt{s_{2r-2}}-\sum_{i=2r+1}^{n} m_i)^2, \quad (62)$$

$$[s_{2r}]_-=(\sum_{i=2r+1}^{n} m_i)^2,\quad [s_{2r}]_+=(\sqrt{s_{2r-2}}-\sqrt{s_{2r-1}})^2, \quad (63)$$

$$[u_{2r+1}]_\pm=M^2+s_{2r+1}-\frac{(M^2+s_{2r}-s_{2r}')(s_{2r}+s_{2r+1}-s_{2r+2})}{2s_{2r}}\pm\frac{[\lambda(M^2,s_{2r},s_{2r}')\lambda(s_{2r},s_{2r+1},s_{2r+2})]^{1/2}}{2s_{2r}}, \quad (64)$$

$$[u_{2r}]_\pm=M^2+m_{2r+1}^2-\frac{(M^2+s_{2r+1}-u_{2r+1})(s_{2r+1}+m_{2r+1}^2-m_{2r+2}^2)}{2s_{2r+1}}\pm\frac{[\lambda(M^2,s_{2r+1},u_{2r+1})\lambda(s_{2r+1},m_{2r+1}^2,m_{2r+2}^2)]^{1/2}}{2s_{2r+1}}, \quad (65)$$

$$t_{1\pm}=M^2+m_2^2-\frac{(s_1+m_2^2-m_1^2)(M^2+s_1-s_2)}{2s_1}\pm\frac{[\lambda(s_1,m_1^2,m_2^2)\lambda(M^2,s_1,s_2)]^{1/2}}{2s_1}, \quad (66)$$

$$[t_{2r}]_\pm=t_{2r-1}+m_{2r+1}^2-\frac{(M^2+m_{2r+1}^2-u_{2r})(M^2+t_{2r-1}-t_{2r-1}')}{2M^2}+\frac{[\lambda(M^2,m_{2r+1}^2,u_{2r})\lambda(M^2,t_{2r-1},t_{2r-1}')]^{1/2}}{2M^2}\times[X_{2r}]_\pm, \quad (67)$$

$$[t_{2r+1}]_\pm=t_{2r-1}+s_{2r+1}-\frac{(M^2+s_{2r+1}-u_{2r+1})(M^2+t_{2r-1}-t_{2r-1}')}{2M^2}+\frac{[\lambda(M^2,s_{2r+1},u_{2r+1})\lambda(M^2,t_{2r-1},t_{2r-1}')]^{1/2}}{2M^2}\times[X_{2r+1}]_\pm, \quad (68)$$

$$[X_r]_\pm=-\xi_r\eta_r\pm[(1-\xi_r^2)(1-\eta_r^2)]^{1/2}. \quad (69)$$

The meaning of the symbols s_{n-1}, u_{n-1}, t_{n-1}, s_0, and u_1 is clear from the definitions given in (51) ($s_{n-1}\equiv s_{n-2}$, $u_{n-1}\equiv s_{n-2}'$ for n even; $s_{n-1}\equiv m_n^2$ for n odd; $s_0\equiv M^2$, $u_1\equiv s_2$ and $t_{n-1}\equiv m_1^2$).

In the particular case when the function F is independent of all u_r and t_r, we have

$$\mathfrak{D}_n=\frac{1}{M^2}(\tfrac{1}{2}\pi)^{n-1}\prod_{r=1}^{n-2}\left[\int_{s_{r-}}^{s_{r+}}\frac{ds_r}{s_r}\right]\times\prod_{r=1}^{n-1}\{[\lambda(s_r,m_r^2,m_{r+1}^2)]^{(1-\epsilon_r)/2}\times[\lambda(s_{r-2},s_{r-1},s_r]^{\epsilon_r/2}\}F(s_r). \quad (70)$$

We conclude this section with the following remarks:

(a) The range of integration over the variables s_r, u_r, and t_r becomes narrower with increasing r. It is particularly so for the variables u_r and t_r. It is because the transformed phase-space integrals $\mathcal{P}_n$ and $\mathfrak{D}_n$ are either of the form $\prod[\int ds_r]\prod[\int du_r]\prod[\int dt_r]$ or of the form $\prod[\int ds_r]\prod[\int dt_r]\prod[\int du_r]$. It is therefore obvious that in the former case, for example, the range of integration of variable t_r for fixed values of variables s_i, u_i, and t_j ($j<r$) (on which depend the limits of integration of t_r) will be very much restricted. Hence, in any practical calculation, the values of the variables u_r or t_r or both (for $r>1$, say) may be restricted to such a narrow range that the function $F(s_r;u_r;t_r)$ remains practically constant within the range of integration, in which case the integration becomes trivial.

(b) It has been stated that all of $\frac{1}{2}N(N-1)$ scalar products of the type $P_j\cdot P_k$ ($j\neq k$) can be expressed in terms of ($3N-10$) such independent scalar products. For $N\leqslant 5$, all such scalar products can be expressed as

[11] It should be noticed that the integration over u_{2r+1} precedes the integration over u_{2r} [unless $2r=n-2$, in which case $u_{2r+1}=u_{n-1}$, which is not one of the ($3n-7$) independent variables].

a *linear* combination of $(3N-10)$ independent variables (x_i):

$$P_j\cdot P_k=\sum_i \alpha_i x_i,$$

where the coefficients α_i are independent of x_i. In such a case the coefficients α_i can be obtained simply by using energy-momentum conservation. However, for $N\geqslant 6$ some of the scalar products $P_j\cdot P_k$ (whose number goes on increasing with increasing N) will always be such that the coefficients α_i themselves depend on variables x_i. It is not possible, in such a case, to obtain α_i simply by using energy-momentum conservation. The coefficients α_i may, however, be calculated by making use of symmetry arguments which do not depend on geometry. This is illustrated in the Appendix.

Some of the results of this work were applied to study some production processes.[12–14] After the manuscript was submitted for publication, we learnt that a scheme similar to ours had been used for the construction of a Monte Carlo program for generating n-particle production amplitudes at CERN.[15]

ACKNOWLEDGMENTS

The author is indebted to Professor S. N. Biswas for reading the manuscript and for useful comments and suggestions. He also wishes to thank Dr. S. H. Patil and Dr. K. Datta for their comments on the manuscript. He is grateful to Professor R. C. Majumdar and Professor F. C. Auluck for their kind interest in this work.

APPENDIX

Some illustrations of the use of the results stated in the text are given below.

A. Three-Body Decays

We consider the decay process

$$A(Q)\to a_1(p_1)+a_2(p_2)+a_3(p_3),$$

which is described by the T-matrix element defined by

$$\langle f|S|i\rangle=i(2\pi)^4\delta^4(Q-\sum_{j=1}^{3} p_j)\times(2V\mathcal{E})^{-1/2}\prod_{j=1}^{3}(2VE_j)^{-1/2}\langle f|T|i\rangle. \quad \text{(A1)}$$

Now, $|\langle f|T|i\rangle|^2$ summed up over the final spin states and averaged over the initial spin states is a Lorentz-invariant quantity which we denote by $F(Q,p_i)$, and can be expressed in terms of two independent Mandelstam-like variables which may be taken to be

$$s_1=-(Q-p_1)^2,\quad u_1=-(Q-p_2)^2. \quad \text{(A2)}$$

The transition probability per second is then given by

$$R=\frac{1}{\tau}=\frac{1}{256\pi^3M^3}\int_{(m_2+m_3)^2}^{(M-m_1)^2}ds_1\int_{u_{1-}}^{u_{1+}}du_1\,F(s_1,u_1), \quad \text{(A3)}$$

where the limits of integration of variable u_1 are

$$u_{1\pm}=M^2+m_2^2-\frac{(s_1+m_2^2-m_3^2)(M^2+s_1-m_1^2)}{2s_1}\pm\frac{[\lambda(s_1,m_2^2,m_3^2)\lambda(M^2,s_1,m_1^2)]^{1/2}}{2s_1}. \quad \text{(A4)}$$

If $F(s_1,u_1)$ is simple enough, the integration over u_1 may be done analytically; in particular, if $F(Q;\,p_i)=F(s_1)$, we have

$$\int_{u_{1-}}^{u_{1+}}du_1\,F(s_1)=[F(s_1)/s_1][\lambda(s_1,m_2^2,m_3^2)\lambda(M^2,s_1,m_1^2)]^{1/2}. \quad \text{(A5)}$$

The energy and momentum spectra of particle a_1, for example, in the rest frame of particle A are given by the parametric equations

$$\frac{dR}{dE_1}=\frac{1}{128\pi^3M^2}\int_{u_{1-}}^{u_{1+}}du_1\,F(s_1,u_1),\qquad E_1=(M^2+m_1^2-s_1)/2M, \quad \text{(A6)}$$

and

$$\frac{dR}{d|\mathbf{p}_1|}=(128\pi^3M^2)^{-1}(M^2+m_1^2-s_1)^{-1}\times[\lambda(M^2,m_1^2,s_1)]^{1/2}\int_{u_{1-}}^{u_{1+}}du_1\,F(s_1,u_1),\qquad |\mathbf{p}_1|=[\lambda(M^2,m_1^2,s_1)]^{1/2}/2M, \quad \text{(A7)}$$

where the parameter s_1 takes values in the range $(m_2+m_3)^2$ to $(M-m_1)^2$. The invariant mass spectrum of particles a_2+a_3 is given by

$$\frac{dR}{ds_1}=\frac{1}{256\pi^3M^3}\int_{u_{1-}}^{u_{1+}}du_1\,F(s_1,u_1),\qquad s_1=M_{23}^2. \quad \text{(A8)}$$

The angular correlation between particles a_1 and a_2 in

[12] S. C. Bhargava, Phys. Rev. **174**, 1969 (1968).
[13] S. C. Bhargava, Nuovo Cimento **58A**, 815 (1968).
[14] R. Dutt and P. Nanda, Nuovo Cimento **60A**, 706 (1969).
[15] Program FOWL, CERN Library W-505 (unpublished). We also received a report (unpublished) by E. Byckling and K. Kajantie, in which a Monte Carlo method for phase-space integration is given.

the rest frame of particle A is obtained as

$$\frac{dR}{d\eta_1}=\frac{1}{256\pi^3M^3}\int_{(m_2+m_3)^2}^{(M-m_1)^2}\frac{ds_1[\lambda(M^2,m_1^2,s_1)\lambda(M^2,m_2^2,u_1)]^{1/2}\theta(u_{1+}-u_1)\theta(u_1-u_{1-})F(s_1,u_1)}{\{(M^2+s_1-m_1^2)+\eta_1(M^2+m_2^2-u_1)[\lambda(M^2,m_1^2,s_1)/\lambda(M^2,m_2^2,u_1)]^{1/2}\}}, \quad (A9)$$

where $\eta_1=\mathbf{p}_1\cdot\mathbf{p}_2/|\mathbf{p}_1||\mathbf{p}_2|$ and u_1 stands for the following:

$$u_1=M^2+m_2^2-\tfrac{1}{2}\{(M^2+s_1-m_1^2)(s_1+m_2^2-m_3^2)-\eta_1[\lambda(M^2,s_1,m_1^2)]^{1/2}$$
$$\times[\lambda(s_1,m_2^2,m_3^2)-(m_2^2/M^2)\lambda(M^2,m_1^2,s_1)(1-\eta_1^2)]^{1/2}\}[s_1+\lambda(M^2,m_1^2,s_1)(1-\eta_1^2)/4M^2]^{-1}. \quad (A10)$$

B. Four-Body Decays

The five independent Mandelstam-like variables describing the kinematics of four body-decay processes may be defined as [cf. Eq. (41)]

$$s_1=-(Q-p_1)^2,\qquad u_1=-(Q-p_2)^2,$$
$$s_2=-(Q-p_1-p_2)^2,\qquad u_2=-(Q-p_3)^2,\qquad t_2=-(Q-p_2-p_3)^2. \quad (A11)$$

The phase-space integral $\mathfrak{D}_4$ is then transformed into the following:

$$\mathfrak{D}_4=\frac{\pi^2}{4}\int_{(m_2+m_3+m_4)^2}^{(M-m_1)^2}ds_1\int_{(m_3+m_4)^2}^{(\sqrt{s_1}-m_2)^2}ds_2\int_{u_{1-}}^{u_{1+}}\frac{du_1}{[\lambda(M^2,s_2,s_2')\lambda(M^2,m_2^2,u_1)]^{1/2}(1-\xi_2^2)^{1/2}}$$
$$\times\int_{u_{2-}}^{u_{2+}}\frac{du_2}{[\lambda(M^2,m_3^2,u_2)]^{1/2}(1-\eta_2^2)^{1/2}}\int_{t_{2-}}^{t_{2+}}\frac{dt_2}{(1-\zeta_2^2)^{1/2}}F(s_1,s_2;\,u_1,u_2;\,t_2). \quad (A12)$$

Changing the t_2 integration into integration over ζ_2, we obtain

$$\mathfrak{D}_4=\frac{\pi^2}{8M^2}\int_{s_{1-}}^{s_{1+}}ds_1\int_{s_{2-}}^{s_{2+}}ds_2\int_{u_{1-}}^{u_{1+}}\frac{du_1}{[\lambda(M^2,s_2,s_2')]^{1/2}}\int_{u_{2-}}^{u_{2+}}du_2\int_{-1}^{+1}\frac{d\zeta_2}{(1-\zeta_2^2)^{1/2}}F(s_1,s_2;\,u_1,u_2;\,t_2(\zeta_2)), \quad (A13)$$

where t_2 is obtained in terms of ζ_2 from Eqs. (46) and (47); the variables s_1, u_1, and u_2 measure the c.m. energies E_1, E_2, and E_3, respectively; s_2 is the two-particle (a_3+a_4) invariant mass squared; and ζ_2 is the cosine of the angle (ϕ) between the planes defined by the momentum vectors $(\mathbf{p}_1,\mathbf{p}_2)$ and by $(\mathbf{p}_3,\mathbf{p}_4)$. The matrix element squared, $F(s_1,s_2;\,u_1u_2;\,t_2(\zeta_2))$, is proportional to the differential transition probability multiplied by the momentum $|\mathbf{p}_1+\mathbf{p}_2|$;

$$F(s_1,s_2;\,u_1,u_2;\,t_2(\phi))=\left(\frac{\pi^2}{16M^3}\right)^{-1}|\mathbf{p}_1+\mathbf{p}_2|\frac{\partial^5\mathfrak{D}_4}{\partial s_1\partial s_2\partial u_1\partial u_2\partial\phi}. \quad (A14)$$

The distribution in angle ϕ is obtained as[16]

$$\frac{d\mathfrak{D}_4}{d\phi}=\frac{\pi^2}{8M^2}\int_{s_{1-}}^{s_{1+}}ds_1\int_{s_{2-}}^{s_{2+}}ds_2\int_{u_{1-}}^{u_{1+}}\frac{du_1}{[\lambda(M^2,s_2,s_2')]^{1/2}}\times\int_{u_{2-}}^{u_{2+}}du_2\,F(s_1,s_2;\,u_1,u_2;\,t_2(\phi)). \quad (A15)$$

[16] A study of such angular correlations has been suggested by Nelson for the determination of the spin and parity of isosinglet boson resonances decaying into four pions [T. J. Nelson, Phys. Rev. **172**, 1701 (1968)].

A somewhat different choice of variables is obtained from (51). We have $s_1=-(p_1+p_2)^2$, and t_1, s_2, u_2, t_2 are defined in the same way as u_1, s_2, u_2, and t_2 in Eq. (A11). The transformed phase-space integral in this case is

$$\mathfrak{D}_4=\frac{\pi^2}{8M^2}\int_{(m_1+m_2)^2}^{(M-m_3-m_4)^2}ds_1\int_{(m_3+m_4)^2}^{(M-\sqrt{s_1})^2}\frac{ds_2}{[\lambda(M^2,s_1,s_2)]^{1/2}}\times\int_{t_{1-}}^{t_{1+}}dt_1\int_{u_{2-}}^{u_{2+}}du_2\int_{-1}^{+1}\frac{d\zeta_2}{(1-\zeta_2^2)^{1/2}}F(s_1,s_2;\,t_1,t_2;\,u_2),$$

where ζ_2 has the same geometrical meaning as before and is given by Eqs. (59) and (60). It may be seen that with the set of independent variables defined in (51), single-particle energy spectra cannot be obtained. However, this set of variables may be useful in some particular cases. For example, if p_3 and p_4 denote the electron and neutrino four-momenta in a four-body leptonic decay of a hadron, it may be possible to integrate over u_2 and t_2 (or ζ_2) analytically, because of the simplicity of lepton current and because $m_\nu=0$, $m_e\approx0$. In that case, the invariant mass spectrum of the final-state hadrons (with four-momenta p_1 and p_2) is easily obtained in terms of a lower-rank integral. This invariant mass spectrum could also be obtained with the choice of variables defined by (A11) by identifying the electron and neutrino momenta with p_1 and p_2, but this identification would make the u_2 and t_2 (or ζ_2) integrations more difficult.

C. Production Processes with Three Particles in the Final State

In terms of the variables defined by (3), viz.,

$$s\equiv s_0=-Q^2,\quad t_0=-(q_1-p_1)^2,$$
$$s_1=-(Q-p_1)^2,\; u_1=-(Q-p_2)^2,\; t_1=-(q_1-p_2)^2, \tag{A16}$$

the phase-space integral $\mathcal{P}_3$ is transformed into the following:

$$\mathcal{P}_3=\frac{\pi s}{2\lambda(s,M_1^2,M_2^2)}\int_{(m_2+m_3)^2}^{(\sqrt{s}-m_1)^2}\frac{ds_1}{[\lambda(s,s_1,m_1^2)]^{1/2}}\times\int_{t_{0-}}^{t_{0+}}\frac{dt_0}{(1-\xi_1^2)^{1/2}}\int_{u_{1-}}^{u_{1+}}\frac{du_1}{[\lambda(s,m_2^2,u_1)]^{1/2}(1-\eta_1^2)^{1/2}}\times\int_{t_{1-}}^{t_{1+}}\frac{dt_1}{(1-\zeta_1^2)^{1/2}}F(s,s_1;\,t_0,t_1;\,u_1). \tag{A17}$$

The angular correlation between the initial-state particle $A_1(q_1)$ and the final-state particle $a_1(p_1)$ (in the c.m. system) is given by

$$\frac{d\sigma}{d\zeta_0}\propto\frac{d\mathcal{P}_3}{d\zeta_0}=\frac{\pi}{4[\lambda(s,M_1^2,M_2^2)]^{1/2}}\int_{s_{1-}}^{s_{1+}}\frac{ds_1}{(1-\xi_1^2)^{1/2}}\times\int_{u_{1-}}^{u_{1+}}\frac{du_1}{[\lambda(s,m_2^2,u_1)]^{1/2}(1-\eta_1^2)^{1/2}}\int_{t_{1-}}^{t_{1+}}\frac{dt_1}{(1-\zeta_1^2)^{1/2}}\times F(s,s_1;\,t_0(\zeta_0),t_1;\,u_1), \tag{A18}$$

where $\zeta_0=\mathbf{p}_1\cdot\mathbf{q}_1/|\mathbf{p}_1||\mathbf{q}_1|$. The distributions in the angle contained between the production plane of $a_1(p_1)$ and the plane defined by the momentum vectors of particles $a_2(p_2)$ and $a_3(p_3)$ are obtained as

$$\frac{d\sigma}{d\phi}\propto\frac{d\mathcal{P}_3}{d\phi}=\tfrac{1}{4}\pi[\lambda(s,M_1^2,M_2^2)]^{-1/2}\int_{s_{1-}}^{s_{1+}}\frac{ds_1}{[\lambda(s,s_1,m_1^2)]^{1/2}}\times\int_{t_{0-}}^{t_{0+}}dt_0\int_{u_{1-}}^{u_{1+}}du_1\,F(s,s_1;\,t_0,t_1(\phi);\,u_1), \tag{A19}$$

where t_1 as a function of $\cos\phi(=\zeta_1)$ is given by Eqs. (9) and (10).

Production cross section and energy spectra (or equivalently, two-particle invariant mass spectra) of final-state particles in the processes double pion photoproduction,[12] radiative pion nucleon scattering,[13] and pion production[14] in pion-nucleon collisions, have been obtained using the results stated in the text.

D. Production Processes with Four Particles in the Final State

The eight independent Lorentz-invariant variables which uniquely specify a point in the phase space may be defined as [cf. Eq. (3)]

$$\begin{aligned}&s\equiv s_0=-Q^2, && t_0=-(q_1-p_1)^2,\\ &s_1=-(Q-p_1)^2, && u_1=-(Q-p_2)^2,\\ & && t_1=-(q_1-p_2)^2,\\ &s_2=-(Q-p_1-p_2)^2, && u_2=-(Q-p_3)^2,\\ & && t_2=-(q_1-p_3)^2.\end{aligned} \tag{A20}$$

It may be seen that the scalar products $p_1\cdot p_3$, $p_2\cdot p_3$, $p_1\cdot p_4$, and $p_2\cdot p_4$ cannot be expressed as a linear combination of these eight variables. The combinations $(p_1+p_2)\cdot p_3$, $(p_1+p_2)\cdot p_4$, $(p_3+p_4)\cdot p_1$, and $(p_3+p_4)\cdot p_2$ can, however, be expressed as a linear combination of the variables defined by (A20). If t_2 is defined to be equal to $-(p_1+p_3)^2$ rather than as given in (A20), the scalar products $q_1\cdot p_3$, $q_2\cdot p_3$, $q_1\cdot p_4$, and $q_2\cdot p_4$ cannot be expressed as a linear combination of the eight independent variables. In either case, if only one of the four scalar products is known, others can be expressed as a linear combination of these nine scalar products. With the eight independent variables defined by (A20), $p_1\cdot p_3$ can be obtained in the following way.

By symmetry considerations, we must have

$$\int d^4p_3\,\delta(p_3^2+m_3^2)\delta((Q-p_1-p_2-p_3)^2+m_4^2)\times\delta((Q-p_3)^2+u_2)\delta((q_1-p_3)^2+t_2)(p_3)_\mu=[\alpha(Q-p_1-p_2)_\mu+\beta Q_\mu+\gamma(q_1)_\mu]I, \tag{A21}$$

where

$$I=\int d^4p_3\,\delta(p_3^2+m_3^2)\delta((Q-p_1-p_2-p_3)^2+m_4^2)\times\delta((Q-p_3)^2+u_2)\delta((q_1-p_3)^2+t_2). \tag{A22}$$

The quantities α,β,γ are obtained, as usual, by contracting with $(Q-p_1-p_2)_\mu$, Q_μ, and $(q_1)_\mu$ and solving the three simultaneous equations in α, β, and γ. Not only do we have

$$\int d^4p_3\,\delta(p_3^2+m_3^2)\delta((Q-p_1-p_2-p_3)^2+m_4^2)\times\delta((Q-p_3)^2+u_2)\delta((q_1-p_3)^2+t_2)(p_1\cdot p_3)=[\alpha(Q-p_1-p_2)\cdot p_1+\beta Q\cdot p_1+\gamma q_1\cdot p_1]I,$$

but $p_1\cdot p_3$ *itself must be equal to* $[\alpha(Q-p_1-p_2)\cdot p_1+\beta Q\cdot p_1+\gamma q_1\cdot p_1]$.

In this way, all $\frac{1}{2}N(N-1)$ scalar products of the type $P_i\cdot P_j$ $(i\neq j)$ can be expressed in terms of $(3N-10)$-independent variables algebraically, without the aid of complicated geometry.