Construction of the S Matrix from Its Left-Hand-Cut Discontinuity, When the Latter is Asymptotically Unbounded*

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A new procedure is proposed for constructing the partial-wave S matrix, given its left-hand-cut discontinuity $\lambda(m)$ ($m=-2ik$, where k is the c.m. momentum). The advantage over the N/D method is that this procedure works even when $\lambda(m)$ is unbounded as $m \to \infty$ (provided that it oscillates in a suitable way); in fact, the new technique works even when $\lambda(m)$ is not bounded by any finite power of m, so that not only does the usual N/D decomposition break down, but there exists no partial-wave dispersion relation with a finite number of subtractions. An equivalent potential is introduced that provides considerable insight into the nature of elementary-particle, bound-state, ghost, and Castillejo-Dalitz-Dyson poles. The method is generalized to include inelastic effects.

1. INTRODUCTION

THE reconstruction of the partial-wave scattering amplitude from its discontinuity across the left-hand cut plays a central role in strong-interaction physics. Using the N/D method to perform this operation, first introduced by Chew and Mandelstam,[1] and subsequently discussed by many authors,[2] reduces the problem to the solution of an integral equation. An important drawback of the method is its inability to cope satisfactorily with a discontinuity function (the input of the calculation) that is asymptotically unbounded. In such a case the integral equation is not of Fredholm type, and may not have a solution. Even if it has a solution, in many cases this is not unique,[3] and in almost all cases the solutions contain ghosts, i.e., disallowed poles of the S matrix, so that one has not really found a solution of the problem: To construct a function with specified analyticity properties that is unitary on the right-hand cut and that has the given left-hand-cut discontinuity.

However, there are arguments to suggest that left-hand-cut discontinuities are indeed asymptotically unbounded. In relativistic S-matrix theory, the possibility of exchanging particles of spin greater than or equal to one leads, in Born approximation, to left-hand-cut discontinuities that do not vanish asymptotically, and this is reflected in the uv divergences of perturbative field theory. In nonrelativistic scattering, unbounded left-hand-cut discontinuities are associated with singular potentials; again, as is well known,[4] such potentials constitute a more appropriate model for field theories with uv divergences (be they unrenormalizable or renormalizable) than do regular potentials.

In S-matrix calculations, an asymptotically nonvanishing left-hand-cut discontinuity is usually cut off. It is sometimes stated that this is justified by the fact that the exchanged particle does not have a fixed spin, but should be treated as a Regge pole. However this justification does not seem to stand on very firm ground.[5] In fact, it is quite possible (depending on just what happens when s and t both become large, in the interior of the double spectral-function region) that the left-hand-cut discontinuity in a Regge theory should be an increasing, oscillating function, perhaps of the type discussed in this paper. This idea is even more likely to be relevant if Regge trajectories rise indefinitely,[6] since then presumably the partial-wave amplitudes would not be polynomial-bounded in the energy plane, and in particular along the left-hand cut.[7]

These considerations motivate the search for a method of constructing a partial-wave scattering amplitude from its left-hand-cut discontinuity, which works in cases where the latter is asymptotically unbounded, and even when it is not bounded by any polynomial of the energy. It is the purpose of this paper to provide such a method.

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[1] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).

[2] F. Zachariasen, Phys. Rev. Letters **7**, 112 (1961); **7**, 268 (1961); G. F. Chew and S. C. Frautschi, *ibid.* **124**, 264 (1961); F. Zachariasen and C. Zemach, *ibid.* **128**, 849 (1962); D. Y. Wong, *ibid.* **126**, 1220 (1962).

[3] D. Atkinson and A. P. Contogouris, Nuovo Cimento **39**, 1082 (1965); M. B. Halpern, J. Math. Phys. **7**, 1226 (1966).

[4] L. Bertocchi, S. Fubini, and G. Furlan, Nuovo Cimento **32**, 745 (1964); **35**, 633 (1965); A. Arbuzov, A. T. Filippov, and O. A. Khrustalev, Phys. Letters **8**, 205 (1964); F. Calogero and M. Cassandro, Nuovo Cimento **34**, 1712 (1964). Several other references may be found in F. Calogero, Phys. Rev. **139**, B602 (1965).

[5] E. J. Squires, Nuovo Cimento **34**, 1277 (1964). This reference points out that the conclusion of R. Omnès [Phys. Rev. **133**, B1543 (1964)], namely, that Reggeization eliminates all divergences (and indeed all subtractions), is not generally justified.

[6] S. Mandelstam, Phys. Rev. **166**, 1539 (1968).

[7] Indeed, it appears that the left-hand-cut discontinuity of the partial-wave projection of the Veneziano ansatz [G. Veneziano, Nuovo Cimento **57A**, 190 (1968)] does oscillate asymptotically; and it blows up, surprising as it may appear, if the parent trajectory is *low* enough. In this case the oscillations are so strong that, even if the parent trajectory is low, the dispersion integral of the left-hand-cut discontinuity can be defined (by analytic continuation) and vanishes asymptotically, so that the usual integral equation for the N function is of the Fredholm type. For a more detailed discussion of this point, see a forthcoming paper by D. Atkinson, L. A. P. Balázs, F. Calogero, P. Di Vecchia, A. Grillo, and M. Lusignoli (to be published).

If the left-hand-cut discontinuity is asymptotically unbounded, then one would expect that it oscillates indefinitely, since otherwise either unitarity or analyticity would break down. (The usual way this happens is by the occurrence of poles in unacceptable positions or with residues of the wrong sign, the so-called ghosts.) In fact, Kinoshita's results[8] imply that a left-hand-cut discontinuity which is not bounded by a polynomial must have an infinite number of zeros if the partial-wave scattering amplitude $A_l(s)$ is bounded by $\exp(C|s|^{1-\epsilon})$, for some $\epsilon>0$. On the other hand, one can certainly construct ghost-free unitary amplitudes with unbounded, and even polynomially unbounded, but infinitely oscillating left-hand-cut discontinuities (see below). Arguments will be given later which point to these polynomially unbounded discontinuities as likely candidates for physical relevance (especially in connection with interactions that give rise to singularities in field-theoretic perturbation expansions).

The new method was suggested by analogy with singular potential theory, and it proceeds through the solution of an equivalent potential scattering problem. It is, however, worth emphasizing that this technique is in no sense tied to a nonrelativistic potential model, but that it may be viewed as a general mathematical procedure for constructing a function $S(k)$, "the S matrix," of the complex variable k, which is holomorphic[9] in the upper half-plane, cut from $\frac{1}{2}i\mu$ to $i\infty$ ("the left-hand cut"), is unitary for real k [$S(k)S^{\dagger}(k)=1$], and has an assigned discontinuity across the above-mentioned cut. [In potential scattering, k is proportional to the momentum of the scattered particle, and $S(k)$ is the partial-wave scattering matrix. In the relativistic case, k is the momentum in the c.m. frame; its relation to the square of the c.m. energy s, as well as the connection between $S(k)$ and the physical scattering amplitude, will be discussed in Sec. 4.] An additional requirement for the applicability of the procedure is that the discontinuity should not increase faster than exponentially in $|k|$. The question of the asymptotic behavior, both of the discontinuity and of the function $S(k)$ itself, in all directions in the upper half-plane, and the related problem of uniqueness, will be discussed in detail below.

The introduction of an equivalent potential problem does add considerable physical insight, which remains relevant when the method is employed within the framework of relativistic S-matrix theory. It is known that, in potential theory, one can find a perfectly reasonable (and, in particular, ghost-free) S matrix, starting from a potential that is highly singular as $r\rightarrow 0$, so long as this singularity is repulsive.[4,10] This reflects the obvious physical fact that if an interaction is singular and repulsive at the origin, there is no catastrophe (as there is in the singularly attractive case, due to a collapse into the origin), because the particle is simply constrained not to approach the origin too closely. However, more than this can be said, for since the probability of finding the particle very close to $r=0$ is small, one should expect that the S matrix does not depend very strongly on the precise form of the potential as $r\rightarrow 0$, so long as it is highly singular and repulsive. This is indeed the case; and, although the usual procedure for obtaining the Jost function from the Schrödinger equation, and from this the S matrix, breaks down for highly singular potentials, it is quite feasible to compute the phase-shift itself by solving numerically the radial Schrödinger equation. An alternative procedure, which is based on a convenient expression for the S matrix, has been introduced by one of us.[11] This method automatically minimizes the effect of the region near $r=0$, when the potential has a strongly repulsive singularity for $r\rightarrow 0$, in concordance with the physical picture given above.

Therefore, one should expect that, at least for a class of unbounded left-hand-cut discontinuities, namely those which correspond to equivalent potentials that are repulsively singular for $r\rightarrow 0$, there exists a ghost-free S matrix that is physically entirely acceptable. At the same time, one can understand why the N/D method should not be expected to give this acceptable solution, by analogy with the potential problem. For here, as has been mentioned already, the usual Jost function decomposition of the S matrix does not exist, even though the S matrix itself is preeminently well behaved. The difficulties of the N/D method are associated, not with any misbehavior of the scattering amplitude, but with the simple fact that the usual N and D functions do not exist.

Thus, the purpose of this paper is to show how to construct the S matrix, without having recourse to the N/D representation. This is done by constructing a "radial Schrödinger equation" from the left-hand-cut discontinuity and the requirements of unitarity and analyticity. An equivalent potential will appear in this equation with the property that, if used in this nonrelativistic context, it reproduces the exact S matrix $S(k)$. This task corresponds to solving the so-called inverse scattering problem; a powerful technique for doing this has been provided by Marchenko.[12] An adaptation of this method to the problem at hand has been given in a paper by Cox and one of us,[13] and this provided the first suggestion for the technique developed here. A much simplified derivation is given below, in which the powerful Fredholm theory is used, and in

[8] T. Kinoshita, Phys. Rev. Letters **16**, 869 (1966).

[9] Except possibly for elementary or bound-state poles (see below).

[10] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Pergamon Press, London, 1958), paragraph 35.

[11] F. Calogero and M. B. de Stefano, Phys. Rev. **146**, 1195 (1966); R. G. Newton, *Scattering Theory of Particles and Waves* (McGraw-Hill Book Co., New York, 1966), Sec. 12.4.

[12] Z. S. Agranovich and V. A. Marchenko, *The Inverse Problem of Scattering Theory* (Gordon and Breach, Science Publishers, Inc., New York, 1963).

[13] F. Calogero and J. R. Cox, Nuovo Cimento **55A**, 786 (1968).

which no prior assumption is made about the existence of a Schrödinger equation. In fact, this equation is derived, and an integral representation of the equivalent potential is written down. The class of acceptable unbounded left-hand-cut discontinuities[14] is then the class for which this potential is an acceptable scattering potential (and, in particular, if it is singular at the origin, as is generally the case for unbounded left-hand cuts, it is repulsive there). For this class, the S matrix may be constructed either by integrating numerically the Schrödinger equation, or by employing the technique mentioned above[11] (which is outlined in the present paper, for completeness); the S matrix is then guaranteed to be ghost-free. The method is so powerful that it even works for the class of left-hand-cut discontinuities mentioned above, namely, those not bounded by any power of $-s$, so that, in these cases, not only does the usual N/D decomposition break down, but there is in fact no dispersion relation for the partial-wave amplitude, with a finite number of subtractions.

Finally, it should be remarked that, while this paper focuses attention upon the integral equation for the D function, and its necessary generalization to the case of unbounded left-hand-cut discontinuities, there does exist an alternative formulation of the N/D equations, in which an integral equation is written for N, and in which the input is the dispersion integral over the left-hand-cut discontinuity. If an unbounded, but oscillating, left-hand-cut discontinuity is such that this dispersion integral can be defined, and vanishes asymptotically, so that the integral equation for N is of Fredholm type, then this may provide an alternative route to the S matrix. However, the standard definition of an infinite integral,

$$\int^{\infty} dm\, f(m) \equiv \lim_{B\to\infty} \int^{B} dm\, f(m)\,, \tag{1.1}$$

can be easily shown not to work for many cases in which the procedure given in this paper does work. A regularized definition, that is suggested by the treatment of this paper, namely,

$$\int^{\infty} dm\, f(m) \equiv \lim_{r\to 0+} \int^{\infty} dm\, f(m) e^{-mr}\,, \tag{1.2}$$

might be adequate to guarantee the existence of the input to the N equation. The problem would still remain of showing that this equation is of Fredholm type, or at least that it has a solution, and moreover one that is ghost-free. It should be emphasized that all such difficulties are overcome in the treatment to be found in this paper. The two questions raised above remain, however, quite interesting.

In Sec. 2, the mathematical method is described. In Sec. 3, the domain of validity of the procedure is examined; and a number of related topics are discussed in Sec. 4 [the relation between the mathematical technique and the various kinematics of relativistic scattering; elementary particles, bound states, and ghosts; the Castillejo-Dalitz-Dyson (CDD) ambiguity; inelasticity; higher partial waves]. In Appendix A, certain aspects of singular potential scattering are discussed and, in Appendix B, a mathematical analysis of the basic integral equation is given, including a demonstration of the correspondence between double poles of the equivalent potential $V(r)$ for positive r and disallowed poles of the corresponding S matrix (ghosts).

2. RECONSTRUCTION OF S MATRIX

It is required to find a function $S(k)$ which is holomorphic in the upper half k plane, cut from $k=\frac{1}{2}i\mu$ to $i\infty$, which satisfies the unitarity condition

$$S^*(k)S(k)=1\,, \tag{2.1}$$

for real k, which has the given left-hand-cut discontinuity[15]

$$\lambda(m)=[S(\tfrac{1}{2}im-\epsilon)-S(\tfrac{1}{2}im+\epsilon)]/(2i)\,. \tag{2.2}$$

In this section, it is assumed that the S matrix has neither bound-state nor CDD poles.

In ordinary potential theory, there is a Jost solution $f(k,r)$ of the Schrödinger equation[16]

$$[(\partial/\partial r)^2+k^2]f(k,r)=V(r)f(k,r)\,, \tag{2.3}$$

$$f(k,r) \underset{r\to\infty}{\longrightarrow} e^{-ikr}\,, \tag{2.4}$$

which can be shown[13,15] to be determined also by the following integral equation:

$$f(-\tfrac{1}{2}im,\, r) = e^{-mr/2}+\frac{1}{\pi}\int_{\mu}^{\infty}\frac{dm'}{m'+m}\lambda(m') \times e^{-(m'+m)r/2}f(-\tfrac{1}{2}im',r)\,, \tag{2.5}$$

and by the analytic continuation of its solution

$$f(k,r)=e^{-ikr}\left[1+\frac{1}{\pi}\int_{\mu}^{\infty}\frac{dm'}{m'+2ik}\lambda(m') \times e^{-m'r/2}f(-\tfrac{1}{2}im',\, r)\right]. \tag{2.6}$$

The S matrix is then given by

$$S(k)=\lim_{r\to 0} f(k,r)/f(-k,\, r)\,, \tag{2.7}$$

so that, if $f(k,0)$ exists, as it does for a bounded $\lambda(m)$,

[14] For convenience, throughout this paper, the term "unbounded left-hand-cut discontinuity" means that $\lambda(m)\log m$ is not bounded by any constant asymptotically. For this class, the usual N/D method leads to non-Fredholm equations.

[15] Note that there is a difference of a factor of π between this definition of $\lambda(m)$ and that of Ref. 13.

[16] Throughout this paper, $\rightarrow$ means limiting equality, while $\sim$ means limiting proportionality.

one may set $r=0$ in Eqs. (2.5)–(2.7):

$$f(-\tfrac{1}{2}im,0)=1+\frac{1}{\pi}\int_\mu^\infty \frac{dm'}{m'+m}\lambda(m')f(-\tfrac{1}{2}im',0)\,, \quad (2.8)$$

$$f(k,0)=1+\frac{1}{\pi}\int_\mu^\infty \frac{dm'}{m'+2ik}\lambda(m')f(-\tfrac{1}{2}im',0)\,, \quad (2.9)$$

$$S(k)=f(k,0)/f(-k,0)\,. \quad (2.10)$$

This is the so-called f/f formulation[17] of the N/D method. However, if $\lambda(m)\log m$ is asymptotically unbounded, Eq. (2.8) is not of the Fredholm type, and it might not have solutions. If it does have a solution, this will usually not be unique and, what is worse, it will almost always give an $S(k)$ with ghosts.[3]

When $\lambda(m)$ is unbounded as $m\to\infty$, but diverges less quickly than exponentially in m, then Eq. (2.5) is Fredholm for $r>0$, although it is not Fredholm when $r=0$. This is true even if $\lambda(m)$ increases more quickly than any power of m, e.g.,[16]

$$\lambda(m)\underset{m\to\infty}{\sim}\exp(Bm^{1-\epsilon})\,,\quad \epsilon>0\,,\quad B>0 \quad (2.11)$$

or even

$$\lambda(m)\underset{m\to\infty}{\sim}\exp(Cm\log^{-\epsilon}m)\,,\quad \epsilon>0\,,\quad C>0\,. \quad (2.12)$$

This gives the clue to the new method. Starting with Eq. (2.5), one can solve for $f(\frac{1}{2}im,r)$, $r>0$, by standard techniques and thus obtain $f(k,r)$, $r>0$. In principle, this is enough to determine $S(k)$, from Eq. (2.7), but it would not be a very accurate procedure to evaluate $f(k,r)/f(-k,r)$ for small values of r, and then to extrapolate to $r=0$, since numerator and denominator do not exist separately in the limit $r\to 0$.

Fortunately, there is another way to compute $S(k)$, a procedure, moreover, that minimizes the importance of small values of r. To exploit this method, it is necessary first to have a Schrödinger equation like Eq. (2.3). However, it is not necessary to *assume* that $f(k,r)$ satisfies such an equation, for this fact will now be *proved*. Given $\lambda(m)$, *define* the function $f(-\frac{1}{2}im,r)$ by means of the integral equation (2.5), which is, by assumption, Fredholm for $\mathrm{Re}\,r>0$. After a little algebra, one finds, from Eq. (2.5),

$$\left[\left(\frac{\partial}{\partial r}\right)^2-\tfrac{1}{4}m^2\right]f(-\tfrac{1}{2}im,r) =e^{-mr/2}V(r)+\frac{1}{\pi}\int_\mu^\infty \frac{dm'}{m'+m}e^{-(m'+m)r/2}\lambda(m') \times\left[\left(\frac{\partial}{\partial r}\right)^2-\tfrac{1}{4}m'^2\right]f(-\tfrac{1}{2}im',r)\,, \quad (2.13)$$

[17] A. Martin, Nuovo Cimento **19**, 1257 (1961); V. de Alfaro and T. Regge, *ibid.* **20**, 956 (1961); *Potential Scattering* (North-Holland Publishing Co., Amsterdam, 1965); H. Cornille, J. Math. Phys. **8**, 2268 (1967).

where

$$V(r)=-\frac{1}{\pi}\frac{\partial}{\partial r}\int_\mu^\infty dm\, e^{-mr/2}\lambda(m)f(-\tfrac{1}{2}im,r)\,. \quad (2.14)$$

The quantity $V(r)$, which will be identified as the equivalent potential, is not known in advance, since it is defined in terms of the solution of (2.5). However, the important point is that it depends only on r, and not on m. On comparing Eqs. (2.13) with (2.5), one sees that

$$\left[\left(\frac{\partial}{\partial r}\right)^2-\tfrac{1}{4}m^2\right]f(-\tfrac{1}{2}im,r)$$

satisfies exactly the same Fredholm integral equation as does $V(r)f(-\frac{1}{2}im,r)$, and hence must be exactly the same function. That is,

$$\left[\left(\frac{\partial}{\partial r}\right)^2-\tfrac{1}{4}m^2\right]f(-\tfrac{1}{2}im,r)=V(r)f(-\tfrac{1}{2}im,r)\,. \quad (2.15)$$

This is implied by the uniqueness of the solution of an inhomogeneous Fredholm equation, which follows whenever the homogeneous equation has no nontrivial solution. Hence the validity of Eq. (2.15) is established for all but a discrete set of values of r, at which points $f(-\frac{1}{2}im,r)$ develops poles in the complex r plane. The possible occurrence and location of these poles is connected with the question of ghosts, as will be discussed in detail in Sec. 3.

The differential equation (2.15) can be continued from real m to $m=2ik$, giving

$$\left[\left(\frac{\partial}{\partial r}\right)^2+k^2-V(r)\right]f(k,r)=0\,, \quad (2.16)$$

which is the Schrödinger equation. Moreover, it can be seen, from Eq. (2.6), that[16]

$$f(k,r)\underset{r\to\infty}{\longrightarrow}e^{-ikr}\,, \quad (2.17)$$

so that $f(k,r)$ is identified as the Jost solution of the Schrödinger equation.

Next, define

$$S(k,r)=f(k,r)/f(-k,r)\,. \quad (2.18)$$

From Eq. (2.6), it may be seen that $f(k,r)$ is analytic in k, except for a cut from $k=\frac{1}{2}i\mu$ to $k=i\infty$, across which its discontinuity is given by

$$[f(\tfrac{1}{2}im-\epsilon,r)-f(\tfrac{1}{2}im+\epsilon,r)]/(2i) =\lambda(m)f(-\tfrac{1}{2}im,r)\,. \quad (2.19)$$

Thus the discontinuity of $S(k,r)$ across this cut is

$$[S(\tfrac{1}{2}im-\epsilon,r)-S(\tfrac{1}{2}im+\epsilon,r)]/(2i)=\lambda(m)\,. \quad (2.20)$$

Moreover, again from Eq. (2.6), it can be seen that, for real k and real r,

$$f^*(k,r)=f(-k,r)\,, \quad (2.21)$$

and therefore

$$S^*(k,r)S(k,r)=1\,, \tag{2.22}$$

for all real k and real r. Clearly, Eqs. (2.20) and (2.22) show that $S(k,r)$ is a unitary function that has the given "left-hand-cut" discontinuity $\lambda(m)$.

Thus a whole class of unitary functions $S(k,r)$, which have the given left-hand-cut discontinuity, have been constructed, depending on the continuum non-negative variable r. On the other hand, it follows from Eq. (2.6), and the definition (2.18), that[16]

$$S(k,r) \xrightarrow[k\to\infty]{} e^{-2ikr}\,, \tag{2.23}$$

for $r>0$. Therefore, for all positive nonzero values of r, the functions $S(k,r)$ would not be acceptable if one requires that the S matrix grow less than exponentially in all directions in the complex k plane. Hence, the unique function

$$S(k)\equiv \lim_{r\to 0} S(k,r) \tag{2.24}$$

remains as a possible candidate to represent the S matrix, provided, of course, that this limit exists (see below).

The interpretation of these results in terms of the equivalent potential model is illuminating. In fact it follows from the Schrödinger equation (2.16), and the definition (2.18), that $S(k,R)$ is just the S-wave scattering matrix corresponding to a potential that coincides with $V(r)$, Eq. (2.14), for $r>R$, and which is infinitely repulsive for $0\leqslant r\leqslant R$. This may be considered to provide a justification for the exclusion of $S(k,r)$, $r>0$, as an acceptable S matrix, inasmuch as "infinitely hard" cores are not believed to exist in nature.

The equivalent potential approach is also useful for an understanding of the limit $r\to 0$ [Eq. (2.24)]. Clearly the limit $S(k)$, if it exists, is just the nonrelativistic S-wave scattering matrix produced by the equivalent potential $V(r)$, which is obtained from the left-hand-cut discontinuity $\lambda(m)$ through Eqs. (2.5) and (2.14). Thus $S(k)$ will be satisfactory if this equivalent potential is itself acceptable. What is meant here by an "acceptable potential" is, fortunately, easily defined, thanks to the insight afforded by the detailed understanding of potential scattering theory that is currently available. In particular, the limit $r\to 0$ in Eq. (2.24) should be discussed in the framework of singular potential scattering theory. The fact that $V(r)$ will be singular as $r\to 0$, in the case that $\lambda(m)$ is unbounded[14] as $m\to\infty$, can be seen from Eqs. (2.5) and (2.14). In fact, when $r=0$, Eq. (2.5) is no longer Fredholm and, as a consequence, $f(-\frac{1}{2}im, r)$ will in general become unbounded as $r\to 0$. This implies, through Eq. (2.14), that $V(r)$ is a singular potential. On the other hand, if $\lambda(m)\log^{1+\epsilon}m$ is bounded as $m\to\infty$, Eq. (2.5) remains Fredholm and $V(r)$ is regular. In this case the reconstruction of $S(k)$ can be achieved by the usual procedure, as has already been noted, but it could also be accomplished by the new method. However, for definiteness the discussion will now focus upon the singular case, corresponding to an unbounded $\lambda(m)\log m$.

An acceptable singular equivalent potential $V(r)$ is defined to be one that is repulsive as $r\to 0$ and that is bounded for all real $r>0$.[18] For such potentials, the limit in Eq. (2.24) certainly exists[19,4]; moreover, the S matrix that is constructed from such a potential has no ghosts (this can be proved in the standard way; see Appendix A). Inasmuch as $V(r)$ is defined in terms of $\lambda(m)$, one has given a condition (albeit an implicit one) on the left-hand-cut discontinuity, sufficient to ensure that the corresponding S matrix (uniquely defined by the procedure given above) is acceptable. This class of good left-hand-cut discontinuities will be characterized in greater detail in the following section. Here it will suffice to point out that, if $\lambda(m)\log m$ is unbounded as $m\to\infty$, it must oscillate infinitely.

The new procedure can be conveniently summarized at this point, if one assumes that a good but unbounded $\lambda(m)$ is given: The first step is to solve the Fredholm equation (2.5) for $r>0$. This gives $f(-\frac{1}{2}im, r)$, and so $V(r)$, from Eq. (2.14), for $r>0$. One can then integrate the Schrödinger equation, for the regular solution, from[20] $r\approx 0$ to the asymptotic region, thus obtaining the phase shift $\delta(k)$ and hence the S matrix $S(k)=e^{2i\delta(k)}$. Alternatively, one can use a procedure[11] which requires neither the explicit computation of $V(r)$, nor the integration of a differential equation, and which, moreover, automatically minimizes the contribution from small values of r. This consists in using the integral expression[11]

$$S(k)=1+2ik\int_0^\infty dr\{[f(-k,r)]^{-2}-e^{-2ikr}\}\,, \tag{2.25}$$

where $f(k,r)$ is defined by Eq. (2.6). For a repulsively singular potential, $f(-k,r)$ diverges strongly and monotonically as $r\to 0$, and so the contribution of $[f(-k,r)]^{-2}$ to the integral (2.25) is very small for small r.

3. PERMISSIBLE LEFT-HAND CUTS

In this section, the class of permissible left-hand-cut discontinuities will be discussed in greater detail, and the mechanism by which a "bad" left-hand cut produces pathologies will be elucidated. Throughout this section, only left-hand-cut discontinuities that are asymptotically unbounded, but less than exponentially so

[18] $V(r)$ must vanish asymptotically for large r, but this is automatic if, as is assumed throughout, the "left-hand cut" does not reach the real axis ($\mu>0$). In fact, generally $V(r)\sim\exp(-\mu r)$ for $r\to\infty$.

[19] A. Pais and T. T. Wu, Phys. Rev. **134**, B1303 (1964).

[20] In practice, the integration would start at a very small positive value of r (since the equation is meaningless at $r=0$). This method of computing the scattering phase shift is, in fact, quite convenient for potentials that are singularly repulsive at the origin (Ref. 21).

$[\lambda(m)e^{-mr}\to 0$ as $m\to\infty$, for all $r>0]$, will be considered. There are two distinct problems: first, that of ensuring that the limit of Eq. (2.24) exists, so that an S matrix $S(k)$ can be defined, with an admissible asymptotic behavior (less than exponential behavior in the complex k plane); second, that of making sure that $S(k)$ has the correct analytic properties, in particular, that it has no ghosts (unwanted poles for $\text{Im}k>0$). As will be shown, the first point is connected with the asymptotic behavior of the left-hand-cut discontinuity $\lambda(m)$ as $m\to\infty$ and the associated behavior of the equivalent potential $V(r)$ as $r\to 0$ (attractive or repulsive singularity). The second point, which of course is only relevant if $\lambda(m)$ has an acceptable asymptotic behavior, is connected with the detailed dynamics, and therefore cannot lead to simple conditions on $\lambda(m)$; however, it turns out to be quite illuminating to discuss also this question through the equivalent potential. As will be shown, each ghost of $S(k)$ would correspond to a double pole of $V(r)$ on the positive real axis in the complex r plane.

The most convenient procedure for ascertaining the allowed asymptotic behaviors of the left-hand-cut discontinuity is as follows: First, it is required that the equivalent potential be *repulsively* singular at the origin, i.e., $V(r)>0$ for all $0<r\leqslant r_0$, for some $r_0>0$; and a definite ansatz for this behavior is assumed. Second, the asymptotic behavior of the phase shift $\delta(k)$ as $k\to\infty$ is deduced from the behavior of $V(r)$ as $r\to 0$. Finally, by analytically continuing the asymptotic expression for the S matrix, $S(k)=e^{2i\delta(k)}$, the corresponding expression for the asymptotic behavior of the left-hand-cut discontinuity is obtained. This program is explicitly carried out for three classes of singular potentials, thus yielding three classes of admissible asymptotic behaviors of the left-hand-cut discontinuities, which should cover all cases of interest. Of course, once the method is understood, there is no difficulty in extending it to other categories.

The three classes of potential are as follows[16,21]:

$$V_1(r)\xrightarrow[r\to 0]{} g_1r^{-2}|\log r|^n,\quad n>0,\quad g_1>0 \qquad (3.1\text{a})$$

$$V_2(r)\xrightarrow[r\to 0]{} g_2r^{-p}|\log r|^n,\quad p>2,\quad g_2>0 \qquad (3.1\text{b})$$

$$V_3(r)\xrightarrow[r\to 0]{} g_3\exp(Cr^{-q}),\quad C>0,\quad q>0,\quad g_3>0. \qquad (3.1\text{c})$$

The asymptotic behaviors of the corresponding scattering phase shifts are as follows[22]:

$$\delta_1(k)\xrightarrow[k\to\infty]{} -A_1(\log k)^{n/2}, \qquad (3.2\text{a})$$

$$\delta_2(k)\xrightarrow[k\to\infty]{} -A_2k^{1-2/p}(\log k)^{n/2}, \qquad (3.2\text{b})$$

$$\delta_3(k)\xrightarrow[k\to\infty]{} -A_3k(\log k)^{-q}, \qquad (3.2\text{c})$$

where A_1, A_2, and A_3 are positive constants (they could be explicitly computed using, for example, the techniques of Ref. 21). To compute the asymptotic behavior of the left-hand-cut discontinuity, one must analytically continue $S(k)=e^{2i\delta(k)}$ from real k to $k=\frac{1}{2}im$, $m>0$, and then take the imaginary part. This presents no difficulty, and one obtains

$$\lambda_1(m)\xrightarrow[m\to\infty]{} \exp[f_1(m)(\pi n/4\ln m)]\sin[f_1(m)], \qquad (3.3\text{a})$$

$$\lambda_2(m)\xrightarrow[m\to\infty]{} \exp[f_2(m)\cos(\pi/p)]\times\sin[f_2(m)\sin(\pi/p)], \qquad (3.3\text{b})$$

$$\lambda_3(m)\xrightarrow[m\to\infty]{} \exp[f_3(m)]\sin[f_3(m)(q\pi/2\ln m)], \qquad (3.3\text{c})$$

where

$$f_1(m)=2A_1(\log m)^{n/2},\quad n>0,\quad A_1>0 \qquad (3.4\text{a})$$

$$f_2(m)=2A_2(\tfrac{1}{2}m)^{1-2/p}(\log m)^{n/p},\quad p>2\quad A_2>0 \qquad (3.4\text{b})$$

$$f_3(m)=A_3m(\log m)^{-q},\quad q>0,\quad A_3>0. \qquad (3.4\text{c})$$

Equations (3.3) and (3.4) can be regarded as a survey of the permissible left-hand-cut discontinuities [intermediate possibilities, involving extra factors $(\log\log m)^t$, etc., can easily be supplied by the interested reader]. It should be emphasized that in the above equations only the leading behavior has been kept. The limitations on the various parameters in Eqs. (3.4) ensure that the equivalent potentials of Eqs. (3.1) are strictly singular and repulsive [i.e., $r^2V(r)\to_{r\to 0}+\infty$]. Note, however, that the envelope of the left-hand-cut discontinuity of Eq. (3.3a) tends to unity for $n<2$, to some general constant larger than one for $n=2$; for $n>2$ in Eq. (3.3a), and for all cases in Eqs. (3.3b) and (3.3c), the function $\lambda(m)$ is unbounded. In all cases, the asymptotic behavior is oscillatory: The three cases are distinguished by the relative rate of oscillation and growth (in addition to the different rate of growth).

Thus, it is quite simple to find general classes of acceptable left-hand-cut discontinuities by starting from good potentials, as above. It should also be possible to reverse the procedure by showing that appropriate S matrices can be constructed from these left-hand-cut discontinuities by the technique of Sec. 2 [in particular, the limit (2.24) should exist]. This program is quite difficult, in spite of the fact that, leaning on the results of potential theory, one only has to show that $V(r)$ is positive definite as $r\to 0+$, since clearly the equivalent potential will be singular in this limit. The exact nature of the singularity at $r=0$ is not known in general, since Eq. (2.5) is not Fredholm at this point.

Much more can be said about the singularities of $f(-\frac{1}{2}im,r)$ and $V(r)$ in the right-hand half of the complex plane, $\text{Re}r>0$, since Eq. (2.5) is Fredholm there. In fact, since both the kernel and the inhomogeneous terms are holomorphic functions of r, it follows that, in the right-hand half of the r plane ($\text{Re}r>0$),

[21] F. Calogero, *Variable Phase Approach to Potential Scattering* (Academic Press Inc., New York, 1967), p. 112.
[22] See Ref. 21, p. 220.

$f(-\frac{1}{2}im, r)$, the solution of Eq. (2.5), can only have poles (of course at m-independent positions). It is shown in Appendix B that $V(r)$ can only have double poles in the right-hand half of the complex r plane ($\mathrm{Re}r>0$), and that, at such a double pole, say $r=r_0$, the Laurent expansion of $V(r)$ has the form

$$V(r)=[B_{-2}/(r-r_0)^2]+B_0+B_1(r-r_0)+\cdots, \quad \mathrm{Re}r_0>0 \quad (3.5)$$

so that the single-pole component is missing. Moreover,

$$B_{-2}=l(l+1), \quad (3.6)$$

where l, an integer, is the order of the corresponding pole of $f(-\frac{1}{2}im, r)$ at $r=r_0$. Thus, except for a dynamical accident, $B_{-2}=2$.

The fact that $V(r)$ is analytic for $\mathrm{Re}r>0$, except for the double-pole singularities of the forms (3.5) and (3.6), looks very special from the point of view of potential scattering theory. This is, of course, a consequence of the fact that the analysis is restricted to those potentials that produce an S matrix with the required analyticity in the k plane, and a less than exponential behavior in k, as $k\to\infty$.

These considerations of the analyticity properties for $\mathrm{Re}r>0$ allow an immediate discussion of the second point mentioned at the beginning of this section, namely, the condition for the absence of ghosts, i.e., poles of $S(k)$ for $\mathrm{Im}k\geqslant 0$.[23] Clearly, no such ghosts can occur if $V(r)$ is nonsingular for real $r>0$ [the question of the necessary positive-definiteness of a singular $V(r)$, as $r\to 0+$, has already been discussed]. In other words, if none of the double poles of $V(r)$ lies on the positive real r axis, then there can be no ghosts. This is, of course, a consequence of the nonexistence of ghosts in a scattering theory with a potential bounded for $r>0$ (see Appendix A). On the other hand, it will be shown in Appendix B that when a double pole of $V(r)$ appears on the positive real axis, a ghost materializes.

If $\lambda(m)$ were to increase without oscillating, as $m\to\infty$, then an infinite number of double poles of $V(r)$ would indeed occur on the positive real axis, accumulating at $r=0$ (see Appendix B). Such a behavior is, of course, unacceptable, as has already been emphasized; the present discussion illustrates the mechanism of the pathology. One sees therefore that the two pathological mechanisms discussed in this section are not really completely separable. However, as emphasized above, once the asymptotic behavior of $\lambda(m)$, $m\to\infty$, is acceptable, the discussion of ghosts becomes relevant. The criterion of acceptability then reduces to the problem of excluding these ghosts; and this implies a nontrivial limitation on the strength of the left-hand-cut discontinuity. In terms of a "coupling constant" g multiplying the left-hand-cut discontinuity, the limitation would in general be on the admissible size of g, consistent with the absence of ghosts.[24]

[23] The possibility of bound-state or elementary-particle poles will be considered in Sec. 4.

4. GENERALIZATIONS

In this section, the following subjects are discussed: the connection between the mathematical formalism and the actual physical partial-wave scattering amplitudes for various (two-body) scattering processes; bound-state and elementary-particle poles, and ghosts; the CDD ambiguity; inelasticity; higher partial waves.

The problem to be discussed first is the relation between the square of the total energy in the c.m. system s and the variable k of Secs. 2 and 3, which is in fact the c.m. momentum of either of the scattered particles (for definiteness, in the intial state). This is

$$k^2=\frac{[s-(M_1+M_2)^2][s-(M_1-M_2)^2]}{4s}, \quad (4.1a)$$

$$s=[(k^2+M_1^2)^{1/2}+(k^2+M_2^2)^{1/2}]^2, \quad (4.1b)$$

where M_1 and M_2 are the masses of the initial particles.

The elastic unitarity condition for the partial-wave scattering amplitude $A(s)$ may be written in general as

$$\mathrm{Im}A(s)=\rho(k)|A(s)|^2, \quad (4.2)$$

where $\rho(k)$ is the phase-space factor. In the case that the masses involved in the initial and final states are the same ("elastic" scattering), the phase-space factor is

$$\rho(k)=2k/s^{1/2}, \quad (4.3)$$

for boson-boson or fermion-fermion scattering, whereas for boson-fermion scattering, one simply has[25]

$$\rho(k)=2k/(M_1+M_2). \quad (4.4)$$

In all cases, the amplitude $A(s)$ in Eq. (4.2) is assumed to be the partial-wave projection of a helicity amplitude.[26] The analytic structure of $A(s)$ is supposed to be rather simple in the k variable. There is a cut on the positive imaginary axis from $k=\frac{1}{2}i\mu$ to $i\infty$, where μ is the minimum mass that can be exchanged in the crossed t channel.[27] In addition, there will be a cut on

[24] Considerations of this type have been used to obtain approximate bounds on the pion-nucleon coupling constant. In this case one is dealing with a nonsingular situation. D. Atkinson and F. Calogero, Phys. Rev. **171**, 1767 (1968).

[25] The factor $2/(M_1+M_2)$ has been introduced for dimensional reasons, and so that the nonrelativistic limits of Eqs. (4.3) and (4.4) are the same. Actually, the conventional choice of the constant (and even its dimension) depends on the way that the helicity amplitudes are defined.

[26] M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) **7**, 404 (1959).

[27] This is true, provided that the u-channel contribution to the left hand cut ("exchange force") does not extend below the t-channel contribution ("direct force"). The condition for this is

$$\mu_u^2\equiv\frac{[m_u^2-(M_1-M_2)^2][(M_1+M_2)^2-m_u^2]}{2(M_1^2+M_2^2)-m_u^2}\geqslant m_t^2\equiv\mu_t^2,$$

where m_u and m_t are the smallest masses that can be exchanged in the u and t channels. In any case, it is assumed throughout that $\mu_u^2>0$, so that $\mu=\min(\mu_u,\mu_t)$ is real (and positive).

the real k axis, starting from the inelastic threshold and extending to infinity. Any bound-state poles must lie on the positive imaginary axis, as usual. There are no other singularities in the upper half k plane.[28] The elastic unitarity equation (4.2) holds for real k from $k=0$ to the first inelastic threshold.

If the masses of the final particles are different from those of the initial particles, then the relevant phase space is $[\rho_I(k)\rho_F(k)]^{1/2}$, where I and F refer to initial and final states, where $\rho_{I,F}(k)$ is again given by Eqs. (4.3) or (4.4), whichever applies, and the relation between s and k is given by Eq. (4.2) with the appropriate initial and final masses. The analytic structure in the k plane is then somewhat more complicated,[29] because there is a finite excursion of the "left-hand cut" away from the positive imaginary axis, which, however, is irrelevant to any asymptotic consideration. For simplicity, the discussion will be limited to "elastic" scattering (possibly of particles with different masses). In any case, since the main emphasis of this paper is concerned with the asymptotic behavior of left-hand-cut discontinuities the basic approach and conclusions will apply in the general case.

It should be recalled that the problem which has been solved is to construct a function $S(k)$ which is unitary for real k $[S(k)S^*(k)=1]$, and which has a given discontinuity across a cut that lies along the positive imaginary axis from $\frac{1}{2}i\mu$ to $i\infty$, $\mu>0$. The function $S(k)$ is by assumption analytic in the upper half-plane, except for this cut (and except possibly for bound-state poles on the positive imaginary axis; see below). If one now sets

$$S(k)=1+2i\rho(k)A(s)\,, \tag{4.5}$$

then the unitarity condition (4.2) is automatically satisfied for all real values of k.[30] This is correct in the elastic region. The modifications introduced by inelastic processes will be considered below.

The input information required for the construction of $S(k)$ was its left-hand-cut discontinuity $\lambda(m)$. In terms of the scattering amplitude $A(s)$, this input has the form

$$\lambda(m)=\mathrm{Re}[2\rho(\tfrac{1}{2}im)A(s)]\,, \tag{4.6}$$

where now

$$s=[(M_1^2-\tfrac{1}{4}m^2)^{1/2}+(M_2^2-\tfrac{1}{4}m^2)^{1/2}]^2\,, \tag{4.7}$$

$m\geqslant\mu$. For large positive m, Eq. (4.6) becomes[16]

$$\lambda(m)\xrightarrow[m\to\infty]{}2\,\mathrm{Re}A(-m^2)\,, \tag{4.8}$$

for boson-boson or fermion-fermion scattering,[31] while for boson-fermion scattering, one has

$$\lambda(m)=-[2m/(M_1+M_2)]\mathrm{Im}A(-m^2+i\epsilon)\,,\quad \epsilon>0\,. \tag{4.9}$$

The next problem to be discussed is the possible presence of bound states, and of CDD poles. It is of utmost importance in this connection to define precisely the nature of the problem that is to be solved. Specifically, a possible element of confusion is the fact that a δ-function contribution to the left-hand-cut discontinuity (the input of the calculation) corresponds to a pole of the S matrix on the positive imaginary axis in the k plane, and that the S matrix resulting from a given left-hand cut is allowed to have, on the positive imaginary axis in the k plane, poles other than those arising from the δ functions in the input. The physical interpretation of these poles (elementary particles, bound states, ghosts) will be discussed below.

It is therefore convenient to display separately the input poles (if any) and the continuum left-hand-cut contribution, and to write

$$\lambda(m)=\lambda'(m)\theta(m-\mu)-2\pi\sum_{j=1}^{N}R_j\delta(m-m_j)\,, \tag{4.10}$$

where, by assumption, $\lambda'(m)$ is free of δ-function singularities (and has an acceptable asymptotic behavior; see Sec. 3). N is finite and the m_j are real and positive (but they may be larger or smaller than μ). The output of the calculation is a unitary S matrix $S(k)$ which is meromorphic in the upper half k plane, cut from $k=\frac{1}{2}i\mu$ to $i\infty$. The discontinuity of $S(k)$ across the cut is to be $\lambda'(m)$ (where the δ functions have been removed). The function $S(k)$ must also have poles at $k=\frac{1}{2}im_j$, $j=1, 2, \ldots, N$, with residues

$$\lim_{k\to\frac{1}{2}im_j}[(k-\tfrac{1}{2}im_j)S(k)]=iR_j\,, \tag{4.11}$$

for consistency with Eq. (4.10). In addition, $S(k)$ may have other poles on the upper imaginary axis in the k plane with residues of the form iP_j, $j=1, 2, \ldots, M$. In other words, the S matrix must have the form

$$S(k)=S'(k)+i\sum_{j=1}^{N}\frac{R_j}{k-\frac{1}{2}im_j}+i\sum_{j=1}^{M}\frac{P_j}{k-\frac{1}{2}ib_j}\,, \tag{4.12}$$

where $S'(k)$ has no poles in the upper half-plane and has the left-hand-cut discontinuity $\lambda'(m)$, where m_j, R_j are the positions and residues of the input poles, and b_j, P_j those of the additional poles that result from the calculation.

In a calculation performed within the framework of S-matrix theory, the input poles play the role of

[28] J. Hamilton, in *Strong Interactions and High Energy Physics*, edited by R. G. Moorhouse (Oliver and Boyd, London, 1964), Sec. 3.

[29] J. Kennedy and T. D. Spearman, Phys. Rev. **126**, 1596 (1961); J. L. Petersen, Nordita Report, 1968 (unpublished).

[30] In a situation involving spin, $S(k)$ should be regarded simply as a mathematical construction. It need not be the conventional projection of the scattering matrix on a particular helicity state.

[31] It is usually believed that a better guess can be made for the imaginary than for the real part (mainly because the absorptive part is simpler than the full amplitude in a field-theoretic context). An extension of the method to bypass this difficulty is nontrivial. It appears that, in such an extension, the role of k is taken by s, so that the limiting admissible behavior is exponential in m^2, rather than in m.

elementary particles, the position of the pole giving the mass, and the residue the coupling of the particle in question. In fact, the residue of the pole in the k^2 plane, namely, $-m_jR_j$, is the square of the coupling constant (it measures the strength of the interaction in the crossed channel, due to the exchange of the elementary particle). It is therefore constrained to be positive. The positions and residues of the additional poles of $S(k)$ give the masses and coupling constants of the bound states that result from the calculation. The corresponding residues $-b_jP_j$ must also be positive. If any one should turn out to be negative, this "state" would not be a bound state, but a ghost. Its appearance must be interpreted as an indication that the input left-hand-cut discontinuity is inconsistent with the analyticity and unitary properties that the S matrix must have. It should be reemphasized that the above discussion of elementary-particle and bound-state poles refers only to the upper half of the complex k plane, which corresponds to the physical sheet of the s plane. Poles in the lower half k plane will be discussed below, in connection with the CDD phenomenon.

From the point of view of the equivalent potential approach, a pole introduced as input appears as a pole of $f(k)$, whereas a bound state is a zero of $f(-k)$. Here $f(k)$ is by definition the Jost function. In the regular case

$$f(k)=f(k,0);\tag{4.13}$$

but when the equivalent potential is singular (and repulsive) at the origin, the following definition replaces Eq. (4.13):

$$f(k)=\lim_{r\to 0}\left\{[V(r)]^{1/4}\times\exp\left(\int_r^{\bar r} dr'[V(r')]^{1/2}\right)f(k,r)\right\},\tag{4.14}$$

where $\bar r$ is a constant such that $V(r)>0$ for $0<r\leqslant\bar r$.

In the inverse potential scattering problem, it is well known that,[12,13] to a given partial-wave amplitude, there corresponds an n-parameter family of potentials, all of which reproduce it, n being the number of poles of $S(k)$ on the positive imaginary axis. Suppose, for simplicity, that $S(k)$ has just one pole at $k=\frac{1}{2}i\bar m$ with a residue Q, so that[16]

$$S(k)\xrightarrow[k\to\frac{1}{2}i\bar m]{}\frac{iQ}{k-\frac{1}{2}i\bar m}.\tag{4.15}$$

One possibility is that the Jost function $f(k)$ has a pole at $k=\frac{1}{2}i\bar m$. In this case, there is no bound state. Another possibility is that a bound state exists at $k=\frac{1}{2}i\bar m$, and that the corresponding bound-state wave function satisfies the normalization equation

$$Q=-\left\{\int_0^\infty dr[f(-\tfrac{1}{2}i\bar m,r)]^2\right\}^{-1}.\tag{4.16}$$

In this case, the spectral function (inverse Laplace transform) of the potential, does not contain a δ function at $m=\bar m$. The Jost function $f(k)$ does not have a pole at $k=\frac{1}{2}i\bar m$, but it does have a zero at $k=-\frac{1}{2}i\bar m$. There is, moreover, a whole class of cases which share this last property [i.e., the pole of $S(k)$ comes from a zero of $f(-k)$, and not a pole of $f(k)$, at $k=\frac{1}{2}i\bar m$]. In all these cases, there does exist a bound state at $k=\frac{1}{2}i\bar m$, but now the bound-state normalization constant

$$B=\left\{\int_0^\infty dr[f(-\tfrac{1}{2}i\bar m,r)]^2\right\}^{-1}\tag{4.17}$$

no longer coincides with $-Q$, where Q is the residue of the S matrix, Eq. (4.16). Moreover, in these cases, the spectral function of the potential contains a δ function contribution at $m=\bar m$. All these potentials would be obtained by inserting, in the formalism of Sec. 2, discontinuity functions which differ only in the coefficient of the δ-function contribution at $m=\bar m$, and this turns out to be just $-2(B+Q)$.[13] The two special cases mentioned above correspond to $B=0$ (no bound state) and $B=-Q$ (pure bound state), respectively.

The cases of interest in S-matrix theory are these two extreme ones: Either one has a pure bound state that does not correspond to a δ function in the input (the left-hand-cut discontinuity function or, for that matter, the spectral function of the potential) or an elementary particle. In the latter case, one requires that the pole come from a pole of $f(k)$, rather than a zero of $f(-k)$; then the coefficient of the input δ function in the discontinuity is reproduced unmodified as the residue of the S-matrix pole. This justifies the earlier separation between the (input) elementary-particle δ-function contributions to the discontiniuity functions and the (output) bound-state poles, and the requirement that the poles of the S matrix occurring at the input δ-function positions have unaltered coefficients [see Eqs. (4.10)–(4.12)]. This explains why even when poles are present the equivalent potential that was defined above is unique.

The expert reader will doubtless have wondered what has become of the CDD ambiguity, since the above procedure gives a unique answer. In fact, it is possible to generalize the procedure as follows (for simplicity only one CDD pole will be considered, but there is no difficulty in treating several). The integral equation (2.5) has to be generalized to

$$f(-\tfrac{1}{2}im,r)=e^{-mr/2}-\frac{2C}{m+m_c}f(-\tfrac{1}{2}im_c)e^{-(m+m_c)r/2}+\frac{1}{\pi}\int_\mu^\infty\frac{dm'}{m'+m}\lambda(m')e^{-(m'+m)r/2}f(-\tfrac{1}{2}im',r),\tag{4.18}$$

with

$$m_c<0,\tag{4.19}$$

so that in addition to the cut on the positive imaginary axis, $f(k,r)$ has a pole at $k=\frac{1}{2}im_c$ (which is in the lower half of the k plane). It can then be seen, following the technique of Sec. 2, that $f(k,r)$ still satisfies the radial Schrödinger equation (2.16), with the following (modified) expression for the equivalent potential:

$$V(r)=2C\frac{\partial}{\partial r}[e^{-m_cr/2}f(-\tfrac{1}{2}im_c,\, r)] -\frac{1}{\pi}\frac{\partial}{\partial r}\int_\mu^\infty dm\, e^{-mr/2}\lambda(m)f(-\tfrac{1}{2}im,\, r)\,. \quad (4.20)$$

The asymptotic behavior of $f(k,r)$ as $r\rightarrow\infty$ is now changed. From Eq. (4.18) one finds

$$f(-\tfrac{1}{2}im_c,\, r)\underset{r\rightarrow\infty}{\sim}\frac{m_c}{C}e^{m_cr/2}, \quad (4.21)$$

and, in general, for $k\neq-\frac{1}{2}im_c$,

$$f(k,r)\underset{r\rightarrow\infty}{\sim}\frac{2ik-m_c}{2ik+m_c}e^{-ikr}. \quad (4.22)$$

It is therefore convenient to define a new Jost solution

$$\bar{f}(k,r)=\frac{2ik+m_c}{2ik-m_c}f(k,r)\,, \quad (4.23)$$

which has the usual asymptotic behavior (except at $k=-\frac{1}{2}im_c$), and which is free from the CDD pole (but it has instead a zero in the upper half-plane at $k=-\frac{1}{2}im_c$).

It is now seen that the function

$$S(k,r)=f(k,r)/f(-k,\, r) \quad (4.24)$$

is unitary for real k, and that it has the left-hand-cut discontinuity $\lambda(m)$ for all values of r (as in Sec. 2). The asymptotic behaviors of $f(k,r)$ and $S(k,r)$ as $k\rightarrow\infty$ are just as in Sec. 2, so that, for the same reasons, one believes that $\lim_{r\rightarrow 0}S(k,r)\equiv S(k)$ is the unique candidate for an acceptable S matrix. Also the discussion of the existence of the limit $r\rightarrow 0$, and its connection with the asymptotic behavior of $\lambda(m)$ as $m\rightarrow\infty$, would be identical to that given in Secs. 2 and 3.

In conclusion, an S matrix containing one CDD pole at $k=\frac{1}{2}im_c$, $m_c<0$, can be constructed from a given left-hand-cut discontinuity $\lambda(m)$ according to the following procedure. First, one must solve the integral equation (4.18), which can be written

$$f(-\tfrac{1}{2}im,\, r)=e^{-mr/2}\frac{1+(C/m_c)(m-m_c/m+m_c)e^{-m_cr}}{1+(C/m_c)e^{-m_cr}}+\frac{1}{\pi}\int_\mu^\infty\frac{dm'}{m'+m}e^{-(m'+m)r/2}\lambda(m')f(-\tfrac{1}{2}im',\, r) \times\frac{1+(C/m_c)[(m-m_c)(m'-m_c)/(m+m_c)(m'+m_c)]e^{-m_cr}}{1+(C/m_c)e^{-m_cr}}. \quad (4.25)$$

This equation is Fredholm for all Re$r>0$, and for $C<0$ or $C>-m_c$ (remember that $m_c<0$). Having solved Eq. (4.25), one could obtain the S matrix from Eq. (4.24), in the limit $r\rightarrow 0$. As discussed in Secs. 2 and 3, a more convenient procedure when the equivalent potential (4.20) is repulsively singular, would be to compute the phase shift $\delta(k)$ from the Schrödinger equation with this potential, or to use the integral representation

$$\bar{S}(k)=\exp[2i\bar{\delta}(k)]=\lim_{r\rightarrow 0}\frac{\bar{f}(k,r)}{f(-k,\, r)} =1+2ik\int_0^\infty dr\{[\bar{f}(-k,\, r)]^{-2}-e^{-2ikr}\}\,, \quad (4.26)$$

with $\bar{f}(k,r)$ given in terms of $f(k,r)$ by Eq. (4.23). Incidentally, it is easily seen that the first term on the right-hand side of Eq. (4.20) vanishes as $\exp(-\frac{1}{2}\mu' r)$ as $r\rightarrow\infty$, where $\mu'=\min[\mu,\, |m_c|]$.

Once $\bar{S}(k)$ is obtained, the final S matrix is

$$S(k)=[(2ik-m_c)(2ik+m_c)]^2\bar{S}(k)\,, \quad (4.27)$$

as implied by Eq. (4.23). It is easily seen that $\bar{S}(k)$ has a zero at $k=\frac{1}{2}im_c$ and a pole at $k=-\frac{1}{2}im_c$, so that $S(k)$ has indeed a simple pole at $k=\frac{1}{2}im_c$ and a simple zero at $k=-\frac{1}{2}im_c$.

As implied by the above discussion, there are restrictions on the CDD constant C. Should C lie in the forbidden range $0<C<-m_c$, then $f(k,r)$ would have an accumulation of an infinite number of poles at the point $r=r_0\equiv-(1/m_c)\ln|m_c/C|$. This can be seen from Eq. (4.25), since the factor $[1+(C/m_c)e^{-m_cr}]^{-1}$ that multiplies the kernel can be made arbitrarily large by choosing r sufficiently close to r_0.[32]

It may be remarked that the introduction of a CDD pole is formally identical to the addition of a δ-function contribution, $-2\pi C\delta(m-m_c)$, $m_c<0$, to $\lambda(m)$, the left-hand-cut discontinuity [with, of course, a formal extension of the range of the left-hand-cut integration; see Eq. (4.18)]. Thus a CDD pole is very like an elementary-particle pole [cf. Eq. (4.10)], but in the lower half of the k plane.

[32] Equation (4.26) has the remarkable property that, at the point $r=r_0$, the kernel becomes not only singular, but separable as well. As a consequence, $f(-\frac{1}{2}im,\, r_0)$ is finite, even though r_0 is an accumulation point of poles.

Next, the important matter of inelasticity will be discussed. It turns out that by far the most appropriate approach is that proposed by Froissart.[33] The inelasticity $\eta(k)$, which is assumed to be given, is introduced through the equation

$$A(s)=\frac{\eta(k)\exp[2i\delta(k)]-1}{2i\rho(k)}, \tag{4.28}$$

so that $\eta(k)=1$ in the elastic region, and $0\leqslant\eta(k)\leqslant 1$ in the inelastic region. This may be written, following Froissart, as

$$A(s)=\frac{P(k)\exp[2i\alpha(k)]-1}{2i\rho(k)}, \tag{4.29}$$

where

$$P(k)=\exp\left[-\frac{2ik}{\pi}\int_0^\infty \frac{dk'\ln\eta(k')}{k'^2-k^2}\right] \tag{4.30}$$

and

$$\alpha(k)=\delta(k)+\frac{2k}{\pi}P\int_0^\infty \frac{dk'\ln\eta(k')}{k'^2-k^2}. \tag{4.31}$$

It is then easily seen that the amplitude

$$A'(s)\equiv\frac{A(s)}{P(k)}+\frac{1-P(k)}{2i\rho(k)P(k)} \tag{4.32}$$

can also be written

$$A'(s)=\frac{\exp[2i\alpha(k)]-1}{2i\rho(k)}, \tag{4.33}$$

so that it satisfies the elastic unitary condition (4.1) for *all* real values of k. Thus the approach described in this paper can be used to compute $A'(s)$, the only difference being the fact that the input left-hand-cut discontinuity is now

$$\lambda'(m)=\lambda(m)/P(\tfrac{1}{2}im), \tag{4.34}$$

as implied by Eqs. (4.32) and (4.6). Of course, once $A'(s)$ has been computed, $A(s)$ can be recovered from Eq. (4.32).

So long as $\eta(k)$ does not tend to zero at any real point $k=k_1$ faster than $\exp[-(k-k_1)^{-1+\epsilon}]$, $\epsilon>0$, and does not tend to zero faster than $\exp(-k^{1-\epsilon})$ as $k\to\infty$, then $P(k)$ is well defined by Eq. (4.30) and, in particular, $P(\frac{1}{2}im)$ is real and positive (it cannot vanish for a finite m).

If $\eta(k)$ does not vanish asymptotically ($k\to\infty$), then $P(\frac{1}{2}im)$ tends to a constant as $m\to\infty$. On the other hand, if $\eta(k)$ vanishes asymptotically, then so does $P(\frac{1}{2}im)$ as $m\to+\infty$. For instance, if

$$\eta(k)\underset{k\to\infty}{\sim}k^{-n},$$

then

$$P(\tfrac{1}{2}im)\underset{m\to\infty}{\sim}m^{-n}, \tag{4.35}$$

whereas if

$$\eta(k)\underset{k\to\infty}{\sim}\exp(-k^A),$$

then

$$P(\tfrac{1}{2}im)\underset{m\to\infty}{\sim}\exp[-(\tfrac{1}{2}m)^A/\cos\tfrac{1}{2}\pi A], \tag{4.36}$$

where one must restrict $0<A<1$ in Eq. (4.36). Thus, it can be seen, from Eq. (4.34), that if $\eta(k)$ vanishes asymptotically (i.e., if at asymptotic energies the scattering becomes purely absorptive), then the effective left-hand-cut discontinuity $\lambda'(m)$ is more singular than $\lambda(m)$.

The previous analysis of the connection between the asymptotic behavior of the left-hand-cut discontinuity and the existence of a ghost-free scattering amplitude, as given in Sec. 3, remains valid in the inelastic case, with $\lambda'(m)$ used instead of $\lambda(m)$, since a necessary and sufficient condition for $A(s)$ to be ghost-free is that $A'(s)$ be similarly ghost-free [this is implied by Eq. (4.32), the fact that $P(k)$ has no zeros for $\text{Im}k>0$, and that it is positive definite on the positive imaginary k axis]. It should be noted in passing that the analysis of this paper would be required even for a bounded $\lambda(m)$, if $P(\frac{1}{2}im)\to 0$ as $m\to\infty$ in such a way that $\lambda'(m)$ is unbounded. Incidentally, in such cases $\lambda(m)$ is required to oscillate, even though it is bounded, in order to avoid the appearance of "inelastic ghosts."

It is remarkable that, even in the inelastic case, the procedure for the construction of the scattering matrix from its left-hand-cut discontinuity involves a real (rather than a complex) equivalent potential.

Throughout this paper, it has been tacitly assumed that the partial-wave amplitude under consideration was an S wave. However, it is known that the N/D equations have the same form for all partial waves, the only difference being that certain moment conditions on the left-hand-cut discontinuity are implied by the threshold zeros of the amplitudes. Normally this problem is by-passed by the introduction of subtractions (which are completely equivalent to CDD poles in unsubtracted equations).[34] It is also clear that the generalized equations of this paper apply unmodified to any partial wave (with appropriate left-hand-cut discontinuities, of course). If the input left-hand-cut discontinuity satisfies the moment conditions appropriate to the partial wave in question, then the partial-wave amplitude constructed by the method of this paper will indeed have the correct threshold zero, even though it appears as the S-wave amplitude associated with the equivalent potential. This zero will appear as an apparent dynamical accident in the equivalent potential problem.

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[33] M. Froissart, Nuovo Cimento **22**, 191 (1961).

[34] D. Atkinson and D. Morgan, Nuovo Cimento **41**, 561 (1966)

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APPENDIX A

It will be shown in this appendix that if $V(r)$, the effective potential, is bounded for all real $r>0$, and $r^2V(r)$ diverges monotonically to $+\infty$ as $r\to 0+$, then $S(k)$ has no ghosts [in the absence of bound-state poles, this means that $S(k)$ has no poles for $\mathrm{Im}k>0$]. In the regular case, this is a well-known result; the proof for acceptable singular potentials of the above type is straightforward, but does not seem to have been given elsewhere.

Let $k=\bar{k}$ be the position of a pole of $S(k)$. Then

$$f(-\bar{k},r)\underset{r\to 0}{\sim}[V(r)]^{-1/4}\exp\left\{\int_r^{\bar{r}}dr'[V(r')]^{1/2}\right\}\,. \to 0 \tag{A1}$$

Thus the Wronskian theorem for the functions $f(-\bar{k},r)$ and $f^*(-\bar{k},r)$, taken between 0 and r, is

$$f^*(-\bar{k},r)\frac{\partial}{\partial r}f(-\bar{k},r)-f(-\bar{k},r)\frac{\partial}{\partial r}f^*(-\bar{k},r)=-4i\,\mathrm{Re}\bar{k}\,\mathrm{Im}\bar{k}\int_0^r dr'|f(-\bar{k},r)|^2. \tag{A2}$$

For $\mathrm{Im}\bar{k}>0$, $f(-\bar{k},r)\to\exp(i\bar{k}r)$ vanishes asymptotically, as $r\to\infty$, and so Eq. (A2) becomes

$$\mathrm{Re}\bar{k}=0. \tag{A3}$$

Thus the poles of $S(k)$ in the upper half k plane may occur only on the imaginary axis.

Now it will be shown that if there is a pole on the positive imaginary axis, its residue R is negative (so that the pole corresponds to a bound state, and not to a ghost). The residue of the pole of $S(k)$ is given by

$$iR=\lim_{k\to\bar{k}}\lim_{r\to 0}[(k-\bar{k})S(k,r)]=-\lim_{r\to 0}\left[\frac{f(\bar{k},r)}{\dot{f}(-\bar{k},r)}\right], \tag{A4}$$

where

$$\dot{f}(-\bar{k},r)\equiv\partial/\partial k\,f(k,r)\,|_{k=-\bar{k}}. \tag{A5}$$

The Wronskian theorem for $f(-\bar{k},r)$ and $\dot{f}(-\bar{k},r)$ yields

$$\lim_{r\to 0}\left[f(-\bar{k},r)\frac{\partial}{\partial r}\dot{f}(-\bar{k},r)\right]=\bar{k}\int_0^\infty dr[f(-\bar{k},r)]^2. \tag{A6}$$

To obtain this result, one must use Eq. (A1) and

$$\dot{f}(-\bar{k},r)\underset{r\to 0}{\sim}[V(r)]^{-1/4}\exp\left\{-\int_r^{\bar{r}}dr'[V(r')]^{1/2}\right\}, \tag{A7}$$

which together imply

$$\lim_{r\to 0}\left[f(-\bar{k},r)\frac{\partial}{\partial r}\dot{f}(-\bar{k},r)\right]=-\lim_{r\to 0}\left[\dot{f}(-\bar{k},r)\frac{\partial}{\partial r}f(-k,r)\right]. \tag{A8}$$

Equation (A7) comes from the fact that, as soon as one moves away from $k=-\bar{k}$, $f(k,r)$ develops the divergent behavior as $r\to 0+$, which is consequently present also in $\dot{f}(-\bar{k},r)$.

On combining Eqs. (A4) and (A6), one finds

$$iR=-\left\{\bar{k}\int_0^\infty dr[f(-k,r)]^2\right\}^{-1}\times\lim_{r\to 0}\left[f(\bar{k},r)\frac{\partial}{\partial r}\dot{f}(-\bar{k},r)\right]. \tag{A9}$$

The Wronskian of $f(\bar{k},r)$ and $f(-\bar{k},r)$ is

$$f(\bar{k},r)\frac{\partial}{\partial r}f(-k,r)-f(-\bar{k},r)\frac{\partial}{\partial r}f(\bar{k},r)=2i\bar{k}. \tag{A10}$$

This equation, together with

$$\lim_{r\to 0}\left[f(k,r)\frac{\partial}{\partial r}\dot{f}(-\bar{k},r)\right]=-\lim_{r\to 0}\left[\dot{f}(-\bar{k},r)\frac{\partial}{\partial r}f(\bar{k},r)\right], \tag{A11}$$

which is derived by an argument similar to that used to obtain Eq. (A8), yields[35]

$$R=-\left\{\int_0^\infty dr[f(-\bar{k},r)]^2\right\}^{-1}<0. \tag{A12}$$

Equation (A12) is the standard result, but now proved for repulsively singular potentials; $f(-\bar{k},r)$ is real for $\bar{k}=i|\bar{k}|$ and is, in fact, the bound-state wave function, normalized so that[16] $f(-\bar{k},r)\to\exp(-|\bar{k}|r)$ as $r\to\infty$. Moreover, since $R(\bar{k})$ is necessarily finite [from Eq.

[35] The same result obtains for a regular potential, although in that case a factor of 2 would multiply the right-hand side of Eq. (A6). This factor of 2 is removed in the final result by the fact that, for a regular potential, the second term on the left-hand side of Eq. (A10) would vanish in the limit $r\to 0$, so that also

$$\lim_{r\to 0}\left[f(\bar{k},r)\frac{\partial}{\partial r}f(-\bar{k},r)\right]=2i\bar{k}$$

would be twice the corresponding result for the singular case.

(A12)], and since $f(-\bar{k}, r)$ does not vanish everywhere, it follows that the pole of $f(-\bar{k}, r)$ at $k=\bar{k}$ is simple.

APPENDIX B

In this appendix, certain properties of $f(k,r)$ and $V(r)$ are deduced from the defining Eqs. (2.5), (2.6), and (2.14). Since it is always assumed that

$$\lambda(m)e^{-mr} \xrightarrow[m\to\infty]{} 0 \tag{B1}$$

if $\mathrm{Re}r>0$, it follows that the integral equation (2.5) is Fredholm in the half-plane $\mathrm{Re}r>0$. Moreover, the kernel and the inhomogeneous term are analytic functions of r, so that the solution $f(-\frac{1}{2}im, r)$ is meromorphic in this half-plane. Normally the poles of $f(-\frac{1}{2}im, r)$ in the r plane, the positions of which are independent of m, will be simple, but in principle multiple poles may be possible corresponding to coincident zeros of the Fredholm determinant.

It then follows from the definition, Eq. (2.14), that $V(r)$ can only have double poles for $\mathrm{Re}r>0$, and that the Laurent expansion about such a pole contains no simple-pole contribution. These conclusions hold independently of the order of the pole of $f(-\frac{1}{2}im, r)$, as is easily seen from the Schrödinger equation (2.15). Moreover, this equation implies that the coefficient of the double pole of $V(r)$ is just $l(l+1)$, where l is the order of the corresponding pole of $f(-\frac{1}{2}im, r)$.

Incidentally, although the possibility of multiple poles of $f(-\frac{1}{2}im, r)$ has not been ruled out, their occurrence would appear to constitute a dynamical accident. For example, suppose that $f(-\frac{1}{2}im, r)$ has a double pole at $r=r_0$,

$$f\left(-\frac{im}{i}, r\right)=\frac{A(m)}{(r-r_0)^2}+O\left(\frac{1}{r-r_0}\right); \tag{B2}$$

then $A(m)$ must satisfy the following two homogeneous Fredholm equations:

$$A(m)=\frac{1}{\pi}\int_\mu^\infty \frac{dm'}{m'+m}e^{-(m'+m)r/2}\lambda(m')A(m'), \tag{B3}$$

$$mA(m)=-\frac{1}{\pi}\int_\mu^\infty \frac{dm'}{m'+m}e^{-(m'+m)r/2}\lambda(m')m'A(m'). \tag{B4}$$

Equation (B3) follows from Eqs. (2.5) and (B2), whereas (B4) follows from (B3) and from the integral condition

$$\int_\mu^\infty dm\, e^{-mr_0/2}\lambda(m)A(m)=0, \tag{B5}$$

which in turn follows from the fact that $V(r)$, Eq. (2.14), cannot have a triple pole. Equations (B3) and (B4) show that the Fredholm kernel

$$\frac{1}{\pi}\frac{1}{m'+m}e^{-(m'+m)r_0/2}\lambda(m')$$

has both eigenvalue $+1$ and -1. This presumably is very exceptional.

It is convenient now to discuss the positions of the poles of $V(r)$ in the right-hand half of the r plane. Consider first the (unacceptable) case of an asymptotically unbounded but nonoscillating $\lambda(m)$ [and for simplicity assume that $\lambda(m)$ is positive definite]. Then, by introducing the quantity $\lambda^{1/2}(m)f(-\frac{1}{2}im, r)$, the kernel can be cast into a real symmetric form. Hence all its eigenvalues are real, and they accumulate at $+\infty$. This implies that the poles of $f(-\frac{1}{2}im, r)$ are on the real r axis ($r>0$) accumulating at $r=0$. This follows because the norm of the kernel diverges monotonically as $r\to 0+$. It is easily seen that this accumulation occurs for any asymptotically unbounded and nonoscillating $\lambda(m)$.

Consider now the more interesting case of an unbounded oscillating $\lambda(m)$. If the nature of the oscillations is acceptable, the poles of $V(r)$ will be moved away from the positive axis in the r plane, for in this case it is not possible to produce a real symmetric kernel for real r values. Examples of left-hand-cut discontinuities of this kind, associated with acceptable potentials, have been given in Sec. 3.

Finally, it will be shown that a pole of $V(r)$ occurring on the positive real axis gives rise to a pole of $S(k)$ on the positive imaginary axis in the k plane with the wrong sign of the residue for a bound state, so that it is a ghost. First, a general argument will be given to establish this result, and then the mechanism whereby a ghost appears at $k=i\infty$, as a double pole of $V(r)$ passes from real negative to real positive r, will be displayed. It may be recalled (from the inverse potential-scattering problem[12,13]) that, given an acceptable S matrix with one pole on the positive imaginary axis, there is a unique potential that reproduces this S matrix via the Schrödinger equation for which the residue of the pole coincides with the corresponding bound-state wave-function normalization (of course, there are other potentials for which the state would not be a pure bound state; see Sec. 4). This potential is indeed the potential that is produced by the method of this paper, if no δ-function contribution is included in the input left-hand-cut discontinuity. Therefore, one can conclude quite generally that if the sign of the residue of the pole is the correct one for a bound state, then the potential must be free of double poles on the positive real r axis (since such singularity-free potentials are necessarily produced by solving the inverse potential-scattering problem). Therefore, if the potential has instead a double pole on the positive real r axis, the corresponding S-matrix pole must have a residue of the wrong sign.

Admittedly, this argument ignores the fact that the inverse potential-scattering problem has so far been fully analyzed only for potentials $V(r)$ that are regular as $r \to 0$. It seems clear, however, that the same results must hold for the class of acceptable singular potentials that have been discussed in this paper.

It will now be shown in detail how a ghost appears at $k=i\infty$, as some parameter (e.g., one that multiplies the left-hand-cut discontinuity function) is varied, so that a double pole of $V(r)$ just moves from negative to positive real r. Consider first the particular case of a potential

$$V(r)=2/(r-r_0)^2, \tag{B6}$$

i.e., a double pole and nothing else. It is easy to construct the corresponding Jost solution of the Schrödinger equation, which is

$$g(k,r)=e^{-ikr}\left[1-\frac{i}{k(r-r_0)}\right]. \tag{B7}$$

The S matrix is

$$S(k)=(kr_0+i)/(kr_0-i), \tag{B8}$$

and this has a pole at $k_0\equiv i/r_0$. As long as r_0 is negative, and therefore $V(r)$ is nonsingular, this pole is in the lower half k plane, but as soon as r_0 passes through zero and changes sign, the pole changes from $k=-i\infty$ to $k=i\infty$, and then moves down the positive imaginary axis. It is then, of course, not a bound state, but a ghost (in fact, the residue of the S matrix in the k^2 variable is negative).

The fact that as soon as a double pole of $V(r)$ enters the real r axis, a ghost appears at $k=i\infty$ remains true for a general potential of the acceptable class. Consider a potential of the form

$$V(r)=[2/(r-r_0)^2]+v(r), \tag{B9}$$

where $v(r)$ is bounded for all $\mathrm{Re}\,r>0$, and is either regular or is singular and positive, as $r\to 0+$. [Note that the analyticity of $v(r)$ at $r=r_0$ is implied by Eq. (3.5).] Then the Schrödinger equation can be cast into the integral form

$$f(k,r)=g(k,r)+\frac{1}{2ik}\int_r^\infty dr'[g(k,r)g(-k,r') -g(k,r')g(-k,r)]v(r')f(k,r'), \tag{B10}$$

where $g(k,r)$ was defined in Eq. (B7) and is the Jost solution in the case with $v(r)=0$.

The position k of any pole of the S matrix, and in particular a ghost, is defined by the equation

$$\lim_{r\to 0} f(-\bar{k},r)=0. \tag{B11}$$

Note that, in the singular case, $\lim_{r\to 0} f(-k,r)$ is well defined only at these values of k, because in general

$$f(k,r)\underset{r\to 0}{\longrightarrow}[V(r)]^{-1/4}\left\{E_1(k)\exp\left[+\int_r^{\bar{r}} dr'[V(r')]^{1/2}\right] +E_2(k)\exp\left[-\int_r^{\bar{r}} dr'[V(r')]^{1/2}\right]\right\}. \tag{B12}$$

Thus a pole position $k=-\bar{k}$ may equivalently be defined by $E_2(-\bar{k})=0$.

It will now be shown that when r_0 is small and positive, then there is a $\bar{k}$ close to $k_0=i/r_0$. Note that throughout Eqs. (B11), (2.18), and (2.24), $k=\bar{k}$ is by definition a pole of $S(k)$ and will be a ghost since r_0 is positive. This proof will be given explicitly in the regular case [in which case $\lim_{r\to 0} f(-k,r)$ exists for all values of k].

Consider Eq. (B10) written for $r=0$ and $k=-\bar{k}$:

$$0=1-\frac{i}{\bar{k}r_0}+\frac{1}{\bar{k}}\int_0^\infty dr'\left[\left(1-\frac{1}{\bar{k}^2r_0(r-r_0)}\right)\sin\bar{k}r' +\frac{r'}{\bar{k}r_0(r'-r_0)}\cos\bar{k}r'\right]v(r')f(-\bar{k},r'). \tag{B13}$$

It will now be shown that if

$$\bar{k}=k_0(1+\epsilon), \tag{B14}$$

then

$$\epsilon\underset{r_0\to 0}{\longrightarrow}0. \tag{B15}$$

In fact, from Eqs. (B13) and (B14), one has, to first order in ϵ,

$$\epsilon=r_0\int_0^\infty dr'\left[i\left(1+\frac{r_0}{r'-r_0}\right)\sin k_0r'+\frac{r'}{r'-r_0}\cos k_0r'\right] \times v(r')f(-k_0,r'). \tag{B16}$$

To demonstrate Eq. (B15) one must bound this integral. This looks divergent because of the double pole of the integrand at $r=r_0$ [recall that $f(-k_0,r')$ has a simple pole at $r'=r_0$], but it is in fact finite, as can be shown by deforming the contour of integration in the r' plane. The line $[0\leqslant r'<\infty]$ will be replaced by the contour $[r'=r_0(1+e^{i\theta}),\ \pi\geqslant\theta\geqslant 0]$, together with the line $[2r_0\leqslant r'<\infty]$. Let ϵ_1 and ϵ_2 be the contributions from the semicircle and the line, respectively, so that $\epsilon=\epsilon_1+\epsilon_2$. To evaluate ϵ_1, one can substitute in place of $f(-k_0,r')$ just the pole contribution $A(r_0)/(r-r_0)$. One obtains, to the leading order in r_0,

$$\epsilon_1=-2r_0v(r_0)A(r_0)/e. \tag{B17}$$

It is shown below that for small r_0,

$$A(r_0)\approx r_0/e, \tag{B18}$$

and this implies

$$\epsilon_1 \xrightarrow[r_0\to 0]{} 0, \tag{B19}$$

since the assumption that the potential is regular is tantamount to the requirement

$$r_0^2 v(r_0) \xrightarrow[r_0\to 0]{} 0. \tag{B20}$$

As for ϵ_2, the corresponding integrand (B16) can be majorized as follows:

$$|f(-k_0, r')| \leqslant B\left[1+\frac{A(r_0)e}{r'-r_0}\right]e^{-r'/r_0}; \tag{B21}$$

$$\left|i\left(1+\frac{r_0}{r'-r_0}\right)\sin k_0 r' + \frac{r'}{r'-r_0}\cos k_0 r'\right| \leqslant 3e^{r'/r_0}; \tag{B22}$$

$$|v(r)| \leqslant g r^{-2+\eta}, \tag{B23}$$

for a sufficiently small $\eta>0$. Here B is a numerical constant, and Eq. (B21) follows from the fact that $f(-k_0, r')$ has the asymptotic behavior $e^{ik_0 r'}$ when either k_0 or r' becomes very large, and it has a pole at $r'=r_0$ of residue $A(r_0)$. Since the range of the integral defining ϵ_2 only starts at $r'=2r_0$, the pole denominator $r'-r_0$ can be replaced by r_0. In the derivation of (B22), use was made of the same replacement. Equation (B23) follows from the assumed regularity of $V(r)$ as $r\to 0$ and the fact that $v(r)$ vanishes asymptotically as $-2r^{-2}$ [see Eq. (B9)]. [For simplicity marginally regular cases, e.g., $V(r) \sim_{r\to 0} (r \log r)^{-2}$, are ignored, although the proof could easily be extended to cover them.] Using these majorizations, one easily obtains

$$\epsilon_2 \leqslant gB[1+A(r_0)e/r_0][(2r_0)^\eta/(1-\eta)]. \tag{B24}$$

This equation together with (B18) implies that also

$$\epsilon_2 \xrightarrow[r_0\to 0]{} 0, \tag{B25}$$

so that Eq. (B15) has been established.

One should also check that the sign of the residue of the S matrix pole is that of a ghost, and not a bound state. In fact, it can be shown explicitly that the residue of the pole, for small r_0, tends to that of the case when the potential has the simple form (B6). This argument, which is quite similar to that given above for the position of the pole, will not be given here, because in any case one can rely on the general argument given at the beginning of this discussion.

To complete the proof, it is necessary to investigate the behavior of $A(r_0)$ as $r_0\to 0$, which can be done by using again the integral equation (B10), this time at $r=r_1$, with $k=-k_0=-i/r_0$:

$$A(r_0) = \frac{r_0}{e} - r_0^2\int_{r_0}^{\infty} dr'\left[\frac{\sin k_0(r'-r_0)}{k_0(r'-r_0)} - \cos k_0(r'-r_0)\right] \times v(r') f(-k_0, r'). \tag{B26}$$

Note that the integrand is nonsingular at $r'=r_0$, since the term in square brackets has a zero that just cancels the pole of $f(-k_0, r')$ at $r'=r_0$. In fact, the integrand can be majorized by the following procedure:

$$|f(-k_0, r')| \leqslant C[A(r_0)+r'-r_0] \Big/ \left\{r'\left[\exp\left(\frac{r'+r_0}{r_0}\right)-1\right]\right\}, \tag{B27}$$

$$\left|\frac{\sin k_0(r'-r_0)}{k_0(r'-r_0)} - \cos k_0(r'-r_0)\right| \leqslant C'\left[\exp\left(\frac{r'-r_0}{r_0}\right)-1\right]. \tag{B28}$$

Here C and C' are absolute constants (independent of r_0). Equation (B27) follows from the fact that $f(-k_0, r')$ has a pole at $r'=r_0$, with residue $A(r_0)$, has the asymptotic behavior e^{-r'/r_0} as $r'\to\infty$, or as $r_0\to 0$, and is bounded for r' between these two limits. Equation (B28) can be trivially checked.

The majorizations (B27) and (B28) are used together with (B23) in Eq. (B26) to show that

$$|A(r_0)-r_0/e| \leqslant CC'g[A(r_0)/r_0+1/(1-\eta)] \times [r_0^{1+\eta}/(2-\eta)], \tag{B29}$$

and this implies Eq. (B18).

It is rather more difficult to carry out the proof when $V(r)$ is singular as $r\to 0$, even though all the integrals in the above proof [Eqs. (B13), (B16), and (B26)] remain well defined [as implied by the remark after Eq. (B11)]. However, it is easy to see that there is a class of singular potentials, namely, those for which $r_0^2 v(r_0) \ll 1$, for some small (but not too small) value of r_0, such that the above proof is essentially unchanged. This indicates that the same association of double poles of $V(r)$ on the positive real axis with poles of $S(k)$ having $\mathrm{Im}\,k>0$, and a negative sign for the residue if $\mathrm{Re}\,k=0$ (ghosts), obtains for singular as for regular potentials.