for a solution of

$$f(y,y',y'',x)=0\,, \tag{A3}$$

then for this solution we have

$$y(-x)=y(x)\,. \tag{A4}$$

The proof consists in the remark that $x \to y(-x)$ also satisfies (A3) and has vanishing derivative at $x=0$, hence $y(-x)=y(x)$ because of uniqueness.

From this theorem we deduce the truth of result II. A look at Eq. (12) supplemented by $\rho=f(p)$ shows that the left side of the Eq. satisfies (A1); now, result I states that $p'(0)=0$ and therefore the solution $p(r)$ of (12) is an even function of r.

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Classical Relativistic Rotator as a Basis for the Elementary Particles*

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A classical Lorentz-covariant generalization of the nonrelativistic theory of a free, stationary, symmetric top is developed. The resulting relativistic theory predicts a physical mass which is a monotonically increasing function of spin asymptotically approaching a linear relation in the limit of large spin. The theory is free of spacelike solutions.

I. INTRODUCTION

THE notion that the elementary-particle resonances may be excited rotational states is not new. It has led to the investigation of the rotational levels of composite systems and to the study of relativistic wave equations based on various rotator models.[1] Perhaps the most detailed study of the applicability of rotational states to the elementary particles is due to Corben.[2] His analysis is based on the model of a symmetric top. It is in the spirit of Corben's approach, that a properly formulated quantum theory of a relativistic rotator is founded on a properly formulated classical theory of that same rotator, that we undertake the present analysis. Some of the introductory material has appeared in the literature; it is reiterated, in Secs. I and II, for the sake of coherence. The rest of the analysis and the emerging rotator theory differ in content from previous formulations.

We develop a classical Lorentz covariant generalization of the free, nonrelativistic, symmetric top and discuss those features of the relativistic theory which indicate its relevance to the elementary particles. We focus especially on the two important features: (a) the predicted relation between the physical mass and spin of the rotator, and (b) the question of spacelike solutions.[3] These two crucial aspects prove to be directly related in our formulation, for the condition which ensures that the physical mass increase monotonically with spin also rules out the possibility of spacelike four-momenta.

Our present purpose is only to indicate the relevance of the model of the symmetric top to a discussion of the elementary particles. Therefore, in the nonrelativistic theory, we make the relatively simple choice of collinear spin angular momentum $\mathbf{S}$ and angular velocity $\boldsymbol{\omega}$. Thus the rotational kinetic energy in the nonrelativistic theory is

$$T=\tfrac{1}{2}\mathbf{S}\cdot\boldsymbol{\omega} \;=\; S^2/2I\,, \tag{1}$$

where I is the moment of inertia about the axis of rotation.[4] The energy-spin relation (1) forms the basis for a highly successful quantum theory of the rotational levels of symmetric molecules and heavy symmetric nuclei.[5] This quantum theory follows almost trivially by merely replacing the classical spin variable $\mathbf{S}$ by $\hbar\mathbf{J}$, where $\mathbf{J}$ is the spin operator in units of $\hbar$. Equation (1) then gives the energy eigenvalues for states of well-defined angular momentum with the continuous classical variable S^2 replaced by the discrete values $\hbar^2 j(j+1)$.

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[1] D. Bohm, P. Hillion, T. Takabayasi, and J. P. Vigier, Progr. Theoret. Phys. (Kyoto) **23**, 496 (1960); T. Takabayasi, *ibid.* **23**, 915 (1960); **36**, 660 (1966); E. J. Sternglass, Phys. Rev. **123**, 391 (1961); Nuovo Cimento **35**, 227 (1965); H. C. Corben, Proc. Natl. Acad. Sci. (U.S.) **48**, 1559 (1962); **48**, 1746 (1962); J. Math. Phys. **5**, 1664 (1964); Nuovo Cimento **47**, 486 (1967); M. Gell-Mann, Phys. Today **17**, 23 (1964); A. O. Barut, *High Energy Physics and Elementary Particles* (International Atomic Energy Agency, Vienna, 1965), p. 679; K. Rafanelli, Phys. Rev. **175**, 1947 (1968).

[2] H. C. Corben, *Classical and Quantum Theories of Spinning Particles* (Holden-Day Inc., San Francisco, 1968).

[3] E. Abers, I. T. Grodsky, and R. Norton, Phys. Rev. **159**, 1222 (1967); I. T. Grodsky and R. F. Streater, Phys. Rev. Letters **20**, 695 (1968); S. J. Chang and L. O'Raifeartaigh, Phys. Rev. **170**, 1316 (1968).

[4] H. Goldstein, *Classical Mechanics* (Addison-Wesley Publishing Co., Reading, Mass., 1959), Chap. 5.

[5] G. Herzberg, *Infrared and Raman Spectra of Polyatomic Molecules* (D. Van Nostrand, Inc., Princeton, N. J., 1945); K. Kotajima and D. Vinciguerra, Phys. Rev. Letters **8**, 68 (1964); L. D. Landau and E. M. Lifshitz, *Quantum Mechanics, Non-Relativistic Theory* (Addison-Wesley Publishing Co., Reading, Mass., 1958), p. 373.

It has been pointed out that the nonrelativistic theory of a symmetric top cannot apply to the elementary particles because (a) the energy (1) rises too rapidly with spin to describe the Regge recurrences, and (b) nonrelativistic theory simply does not apply to a rotator whose spin is finite but whose dimensions are on the order of its Compton wave length.[2] Hence, if we want to consider the applicability of the model of a symmetric top to the elementary particles, we must discuss the model within the framework of a relativistic theory. Fortunately, however, the above condition (b) also rules out questions of rigidity in the relativistic theory.[6] The relativistic generalization of (1) that we consider here emerges from an extension of Frenkel's theory of a spinning point particle.[7] Therefore, we briefly review the salient features of the Frenkel formulation and indicate its inherent limitations. We then develop an extension of Frenkel's theory which has (1) as the nonrelativistic limit.

Before we begin a detailed discussion, it is well to state the degree to which any classical theory of spinning particles can be expected to reflect properties of the elementary particles. Clearly, precise quantitative predictions are not going to bear fruit. For example, if we consider the energy levels of the nonrelativistic symmetric top, the difference between S^2 and $S(S+1)$ vanishes in the limit of large spin, but cannot be ignored for spin values in the range of well established experimental findings. Hence one should expect no more of the classical theory than a semiquantitative reflection of quantal behavior, with asymptotic coincidence.

II. FRENKEL THEORY

In 1926 Frenkel proposed a Lorentz-covariant description of a charged point particle whose magnetic properties are described by the antisymmetric spin tensor, $S_{\mu\nu}=-S_{\nu\mu}$. The magnetic and electric dipole moments are proportional to the space, and mixed space-time components of $S_{\mu\nu}$, respectively:

$$\mathbf{S}=(S_{23},S_{31},S_{12}), \tag{2a}$$

$$\boldsymbol{\tau}=i(S_{41},S_{42},S_{43}). \tag{2b}$$

There is a constraint in Frenkel's theory,

$$S_{\mu\nu}v_\nu=0, \tag{3}$$

which asserts that the electric dipole moment arises only from the motion of the particle,[8] vanishing with the instantaneous particle velocity $\mathbf{V}$, i.e.,

$$\boldsymbol{\tau}=(1/c)(\mathbf{V}\times\mathbf{S}). \tag{4}$$

Henceforth we shall refer to the frame in which the particle is instantaneously at rest as the intrinsic rest frame (IRF). In mathematical language, the constraint (3) states the condition that the internal homogeneous Lorentz group of the relativistic motion maps into the homogeneous rotation group of three dimensions in the limit of small velocities.[9]

Frenkel's classical equations of motion follow from a variational principle and include the effects of external electromagnetic fields which may vary spatially and temporally. In the absence of external fields, these equations are a statement of the homogeneity and isotropy of space-time:

$$\dot{P}_\mu=0, \tag{5}$$

stating the conservation of the momentum-energy four-vector $P_\mu=(\mathbf{P},iE/c)$, and

$$\dot{S}_{\mu\nu}=P_\mu v_\nu-P_\nu v_\mu, \tag{6}$$

stating the conservation of the total angular momentum tensor,

$$M_{\mu\nu}=x_\mu P_\nu-x_\nu P_\mu+S_{\mu\nu}, \tag{7}$$

without demanding the separate conservation of both orbital and spin contributions. The intrinsic rest mass, i.e., the mass in the IRF, is a parameter m which is covariantly defined by the scalar relation

$$v_\mu P_\mu+mc=0 \tag{8}$$

and is a constant of the motion in virtue of the constraint (3).

Let us now examine the relation between the physical mass and spin predicted by the Frenkel theory. By physical mass and physical spin we mean the quantities $E/c^2=(-i/c)P_4$ and S, respectively, evaluated in the Lorentz frame $P_\mu=(0,iE/c)$. This frame is hereafter referred to as the momentum rest frame (MRF). In the MRF the Frenkel equations become

$$d\mathbf{S}/dt=0, \tag{9a}$$

$$d\mathbf{V}/dt=-(\mathbf{S}\times\mathbf{V})E/S^2, \tag{9b}$$

$$\mathbf{V}\cdot d\mathbf{V}/dt=\mathbf{V}\cdot\mathbf{S}=\mathbf{S}\cdot d\mathbf{V}/dt=0, \tag{9c}$$

which show that the instantaneous particle position executes uniform circular motion of radius

$$|\mathbf{r}|=|-(\mathbf{V}\times\mathbf{S})/E|=VS/E=c\tau/E \tag{10}$$

in the plane orthogonal to $\mathbf{S}$. This oscillatory motion is the classical analog of Zitterbewegung. The defining relation (8) then becomes

$$E=mc^2/\gamma=mc^2S_0/S, \tag{11}$$

where S_0 is the intrinsic rest spin, i.e., the spin in the IRF, and is related to the physical spin S, since

$$(S_0)^2=S_{\mu\nu}S_{\mu\nu}/2=S^2-\tau^2=S^2/\gamma^2 \tag{12}$$

as a consequence of the equations of motion (9).[2]

[6] R. Hart, Am. J. Phys. **33**, 1006 (1965).

[7] J. Frenkel, Z. Physik **37**, 243 (1926).

[8] In our notation, $x_\mu=(\mathbf{x},ict)$; s is the proper time of the particle; hence $d/ds=(\gamma/c)d/dt$ gives $v_\mu v_\mu=-1$, where $\dot{x}_\mu=dx_\mu/ds=v_\mu=(\gamma/c)(\mathbf{V},ic)$, $\mathbf{V}=d\mathbf{x}/dt$, and $\gamma=[1-(V/c)^2]^{-1/2}$; Greek indices run from 1 to 4, and Latin indices run from 1 to 3.

[9] K. Rafanelli, Phys. Rev. **175**, 1761 (1968); E. Abers, R. White, and R. Norton, TRW Report, 1968 (unpublished).

If S_0 is a fixed parameter, (11) predicts that the physical mass of the Frenkel point particle decreases with increasing spin, a behavior not indicative of the observed elementary particles.[10]

III. INERTIAL FRAMES OF REFERENCE

It is essential that we understand the nature of the relativistic generalization upon which the Frenkel theory is founded in order that it may be extended and Eq. (11) replaced by a relation more indicative of the elementary particles. The point particle in one whose energy is

$$E_0=mc^2 \tag{13}$$

as viewed by an observer at rest with respect to the particle, i.e., an observer in the IRF. Invariance of the classical theory of the point particle with respect to the inhomogeneous Lorentz transformations generated by P_μ and $M_{\mu\nu}$ gives rise to the equations of motion, (5) and (6), if the point particle has an intrinsic spin. Hence the translationally equivalent inertial frames of special relativity may be labelled by the continuum of different values for the three-momentum $\mathbf{P}$. Therefore, the MRF is inertial. Equations (9) indicate that in this inertial frame the instantaneous particle motion is uniform circular, which is accelerated motion. Therefore, the IRF is not inertial, except for the special degenerate case of separately conserved spin and orbital angular momenta.

The following argument, based on the Thomas precession,[11] shows that the factor γ in the relation (11), between the physical mass and the intrinsic rest mass, is the observable kinematic effect of the noninertial, rotating IRF. The general expression for the rate with which axes fixed in an accelerated particle precess with respect to an inertial set of axes is[12]

$$\boldsymbol{\omega}_T=\frac{(\gamma-1)}{V^2}(\mathbf{V}\times d\mathbf{V}/dt). \tag{14}$$

Thus for the spinning point particle, Eqs. (9) indicate a Thomas precessional frequency, as viewed from the MRF, of

$$\boldsymbol{\omega}_T=(1-\gamma)E\mathbf{S}/S^2. \tag{15}$$

The kinematic effect of this precession on the energy is then given by

$$E_0=E-\mathbf{S}\cdot\boldsymbol{\omega}_T=\gamma E=mc^2, \tag{16}$$

in agreement with (13). Therefore, for the Frenkel point particle, the observable effect of the classical Zitterbewegung is expressed in the relation

$$SE=S_0E_0, \tag{17}$$

where the quantities on the left-hand side are the physical spin and energy, while their intrinsic counterparts appear on the right-hand side.

On the basis of the circular motion admitted by Eqs. (9) we conclude that the MRF and IRF are inertially inequivalent frames of reference, the latter being accelerated with respect to the former. It is always possible to satisfy Eqs. (9) by setting $\mathbf{V}$ identically equal to zero. Then the IRF is at rest with respect to inertial space and coincides with the MRF. Only then is the IRF one of the inertial frames of special relativity. As long as $\mathbf{V}\neq 0$ with respect to inertial space, the Zitterbewegung accompanying the relativistic motion of the Frenkel particle causes the IRF to be accelerated, hence noninertial. This poses a very important, but generally overlooked, interpretational difficulty. Namely, if the particle considered is a point, then indeed (13) is the intrinsic rest energy. In particular, it is the rest energy as viewed by an observer in the inertial IRF. Since all the inertial frames of special relativity are reached via Lorentz transformations, one should expect to generalize (13) by transforming to another, arbitrary, inertial frame, i.e., $\mathbf{P}\neq 0$. However, if the particle has intrinsic spin, then coordinate axes fixed in the particle form an inertial coordinate system only in the inertial IRF, in any other inertial frame axes fixed in the particle undergo a Thomas precession. How then do we specify the continuum of coordinate axes, connected by Lorentz transformations to the inertial IRF, for surely these axes are not fixed in the particle.

Specification of such coordinate axes requires the notion of the c.m. of a spinning particle, a nontrivial notion, as attested to by the quantity of literature on the subject.[13] Let us just mention its significance in the Frenkel theory. The center of the circular motion admitted by Eqs. (9) is given by

$$\mathbf{X}=\mathbf{x}-\mathbf{r}, \tag{18}$$

where

$$\mathbf{r}=-(\mathbf{V}\times\mathbf{S})/E=-c\boldsymbol{\tau}/E \tag{19}$$

from which it follows that

$$d\mathbf{X}/dt=0. \tag{20}$$

In the MRF, then, the position $\mathbf{X}$ is the time-averaged position of the particle, and is the c.m. of the Frenkel particle.[2] This notion of the mass center is Lorentz-covariant, for we note that the four-vector

$$X_\mu=x_\mu+S_{\mu\nu}P_\nu/P_\sigma P_\sigma \tag{21}$$

reduces to (18) if $\mathbf{P}=0$. It also follows from (5) and (6) that

$$\dot{X}_\mu=\kappa P_\mu, \tag{22}$$

with $\kappa=(v_\mu P_\mu)/(P_\sigma P_\sigma)$, so that the velocity of the mass center is constant and collinear with the momentum. Hence a nonrotating coordinate system, fixed to the

[10] K. Rafanelli, J. Math. Phys. **8**, 1440 (1967).
[11] L. H. Thomas, Phil. Mag. 3, 1 (1927); Nature **117**, 514 (1926).
[12] J. D. Jackson, *Classical Electrodynamics* (John Wiley & Sons, Inc., New York, 1962), Sec. 11.5.
[13] M. H. L. Pryce, Proc. Roy. Soc. (London) **A195**, 62 (1948); K. Rafanelli, Phys. Rev. **155**, 1420 (1967). See also the reference list of Ref. 2, Chap. 2.

c.m., is inertial, and in fact furnishes a coordinate system for the MRF; i.e., the Lorentz frame in which the center of mass is at rest is the MRF. Therefore, we may refer, without ambiguity, to either the MRF or the c.m. frame. Thus the continuum of co-moving c.m. coordinate systems provide the needed specification of inertial coordinate axes. Clearly, the distinction between the mass center and the instantaneous position of a spinning particle is a relativistic phenomenon, for the two coincide in the inertial IRF.

In light of the above analysis we see that if (13) defines the energy of a particle at rest with respect to inertial space, then its Lorentz scalar generalization is

$$\dot{X}_\mu P_\mu + mc = 0\,, \tag{23}$$

achieved via a boost to moving axes which stay fixed to the c.m. However, we see from (22) that

$$\dot{X}_\mu P_\mu = v_\mu P_\mu\,, \tag{24}$$

stating that since the component of the instantaneous velocity in the direction of the momentum is unaccelerated, no distinction need be made between (8) and (23).

IV. RELATIVISTIC SYMMETRIC TOP

It is evident from the analysis of Secs. II and III that the Frenkel theory is a relativistic generalization appropriate to a point particle, possessing an intrinsic spin, whose intrinsic rest mass is a parameter whose value is the same to all the inertial observers of special relativity. The Zitterbewegung accompanying the relativistic motions of such a particle gives rise to the relation (17), which predicts a behavior not observed in the elementary particles. The nature of the relativistic generalization on which the Frenkel theory is founded leads us to assert that if the particle considered is not a point, so that the energy in the inertial IRF can be in part rotational, i.e., due to some rotation about an axis through the c.m., then all inertial observers must agree that the energy they observe is in part due to rotation of the body about the c.m. they observe. As long as the Frenkel equations of motion, (5) and (6), remain valid, along with the constraint (3), then the Zitterbewegung accompanying the relativistic motion of the body will decrease the dependence of the physical mass on spin by the multiplicative factor S_0/S.

We are therefore encouraged to seek a Frenkel type of relativistic generalization for the stationary symmetric top whose nonrelativistic kinetic energy is given by (1). In other words, we would like to construct a classical Lorentz-covariant theory consistent with Eqs. (3), (5), and (6), but with (8) replaced by

$$v_\mu P_\mu + Mc = 0\,, \tag{25}$$

where the new mass term, M, (a) is a Lorentz scalar, i.e., has the same value in all the inertial frames of special relativity, (b) is a constant of the motion, and (c) has (1) as its nonrelativistic limit. The last requirement means that we should consider

$$E_0 = (S_0)^2/2I_0 + mc^2 \tag{26}$$

to be the total energy of the rotator in the inertial IRF. The parameter m now denotes the nonrotational part of the rest energy. As before, the subscript zero refers to intrinsic quantities.

We now know that the inertial frames of reference associated with the relativistic motions of a spinning particle are specified by the coordinates of the c.m. (21). Therefore, in order to ascertain the effect of rotation about the c.m., we decompose the total angular momentum into what are now separately conserved orbital and spin contributions:

$$M_{\mu\nu} = X_\mu P_\nu - X_\nu P_\mu + \Sigma_{\mu\nu}\,. \tag{27}$$

Using (21), comparison with (7) gives

$$\Sigma_{\mu\nu} = S_{\mu\nu} - \frac{S_{\mu\sigma} P_\sigma P_\nu}{(P_\rho P_\rho)} + \frac{S_{\nu\sigma} P_\sigma P_\mu}{(P_\rho P_\rho)}\,. \tag{28}$$

We call $\Sigma_{\mu\nu}$ the spin tensor with respect to the c.m. coordinates since it is the contribution to the total angular momentum over and above the orbital motion of the mass center. We may directly verify that $\dot{\Sigma}_{\mu\nu} = 0$, $\Sigma_{\mu\nu} P_\nu = 0$, and hence $\Sigma_{ij} = S_{ij}$ in the MRF. The Lorentz scalar

$$\Sigma_{\mu\nu}\Sigma_{\mu\nu} = S_{\mu\nu} S_{\mu\sigma} [\delta_{\nu\sigma} - 2P_\nu P_\sigma/(P_\rho P_\rho)] \tag{29}$$

has the invariant value

$$\Sigma_{\mu\nu}\Sigma_{\mu\nu} = 2S^2\,, \tag{30}$$

where S is the physical spin. In the inertial IRF, of course, $S = S_0$.

Since the kinetic-energy term in (26) is due to rotation about an axis through the c.m. of the stationary rotator, and since $\Sigma_{\mu\nu}$ is the net rotation about the c.m. of the moving rotator, we are led to the mass term

$$Mc = (1/4Ic)\Sigma_{\mu\nu}\Sigma_{\mu\nu} + mc\,, \tag{31}$$

where I is the moment of inertia about the c.m. of the moving rotator. In the MRF, then, (25) becomes

$$SE/S_0 = S^2/2I + mc^2\,, \tag{32}$$

which clearly shows that if $S = S_0$ we recover (26) with $I = I_0$. It is essential that we understand the moment-of-inertia parameter I, appearing in (32). Because the particle motion in the MRF is uniform circular in the plane orthogonal to the fixed orientation of $\mathbf{S}$, the parameters I and I_0 are related via the parallel-axis theorem, i.e.,

$$I = I_0 + mr^2\,. \tag{33}$$

With r given by (10), and $V^2/c^2 = 1 - 1/\gamma^2 = 1 - (S_0/S)^2$, then

$$I = I_0 + (mc^2 S^2/E^2)[1 - (S_0/S)^2]\,. \tag{34}$$

The functional dependence of the physical mass on spin is then given by the roots of the cubic equation

$$E^3+pE^2+qE+b=0\,, \tag{35}$$

where the coefficients are[14]

$$p=-(SS_0/2I_0)(1+2mc^2I_0/S^2)\,, \tag{36a}$$

$$q=(mc^2S^2/I_0)[1-(S_0/S)^2]\,, \tag{36b}$$

$$b=-(m^2c^4SS_0/I_0)[1-(S_0/S)^2]. \tag{36c}$$

As a check, we note that if $S=S_0$ the three roots of the cubic equation are real, and given by

$$E_1=S_0^2/2I_0+mc^2\,,\quad E_2=E_3=0. \tag{37}$$

Thus, as we must, we recover the stationary top as one of the roots. The other two roots are trivial for the case $S=S_0$; however, their behavior for relativistic motions is important.

In accord with what we can expect to learn from a classical theory, let us examine the asymptotic behavior, i.e., $S/S_0\rightarrow\infty$. In this limit, all three roots are real only if

$$S_0^2/2I_0\geq 8mc^2. \tag{38}$$

If the equality holds, the roots are

$$E_1=E_3=4mc^2(S/S_0)\,,\quad E_2=0. \tag{39}$$

If the inequality holds, then the roots are distinct. For example, if

$$S_0^2/2I_0=9mc^2\,, \tag{40}$$

we have

$$E_1=6mc^2S/S_0\,,\quad E_2=0\,,\quad E_3=3mc^2S/S_0. \tag{41}$$

More generally, if

$$S_0^2/2I_0=gmc^2\,, \tag{42}$$

then for any $g\geq 8$ the roots of (35) are real, and are given by

$$E_k=(mc^2S/3S_0)\{g+2[g(g-6)]^{1/2}\times\cos[\tfrac{1}{3}\varphi+(k-1)(120°)]\}\,, \tag{43}$$

where $k=1$, 2, 3, and

$$\cos\varphi=\pm[g(g-9)^2/(g-6)^3]^{1/2}\,, \tag{44}$$

where the plus sign is for $g>9$, and the minus sign is for $g<9$. As $g\rightarrow\infty$ we have

$$E_1=gmc^2S/S_0\,,\quad E_2=E_3=0. \tag{45}$$

The root E_1 characterizes the behavior of the symmetric top. For the range of the parameter, $g\geq 8$, the energy of the top starts at $(g+1)mc^2$ in the inertial IRF. Then, as viewed from axes which stay fixed to the c.m., i.e., the MRF, the energy increases monotonically with increasing spin, eventually becoming a linear function of the spin in the limit of large spin. It is interesting to note that for all $g\geq 8$, extrapolation of the constant slope of E_1 versus S/S_0, at large spin, back to $S/S_0=1$, gives an intercept less than $(g+1)mc^2$. Hence, semiquantitatively at least, the trajectory of the root E_1, on an energy-spin plot, lies on a curve whose functional dependence is like $(S/S_0)^n$, where $0<n<1$ for finite spin with n approaching 1 in the limit of large spin. There is no reason for preferring any value for g as long as we realize that it is not meaningful to try to fit the elementary-particle data with a classical theory. Such a determination must await the quantal treatment of this top.

There is probably no profound significance to the roots E_2 and E_3. Their reality, for $g\geq 8$, however, is quite significant. For a particle which obeys the Frenkel equations of motion, it follows that

$$P_\mu P_\mu=-E^2/c^2=-(Mc^2/\gamma)^2/c^2 \tag{46}$$

regardless of the structure of M. However, if and only if the possibility of imaginary values for E are excluded, are we assured of having only timelike four-momenta. Such is the case for the model of a symmetric top, provided the condition (38) is satisfied.

The implications of the foregoing analysis shed some light on the problems plaguing present theories based on Poincaré invariance. There is one condition which ensures both a relation between physical mass and spin in semiquantitative agreement with the trajectories of observed elementary-particle states, and the existence of single-particle states which are timelike only. The condition is that the relativistic rotator considered have as its nonrelativistic limit the model of a particle whose spatial extent is not zero, i.e., not a point, so that a minimum contribution to the intrinsic rest energy may be made by rotation about an axis fixed in the body, enough to counterbalance the relativistic effect of the Zitterbewegung, which is to drag down the physical energy with increasing spin. We see then the shortcoming of Frenkel's original formulation, for the point particle emerges as the limiting case of the top, where $S_0^2/2I_0\ll mc^2$. Our results, however, show that this is not a limit realized in nature, since it is beset with the coupled difficulties of a descending mass spectrum and single-particle spacelike states.

V. SIMPLE TWO-PARTICLE ANALOG

The point of view we have adopted here is relevent to the argument presented by Chang and O'Raifeartaigh[3] on the spacelike states of a two-particle system, as shown by the following argument. In an arbitrary Lorentz frame the motion of particle, obeying Frenkel's equations of motion, is helical. If the superimposed oscillations are rapid,[15] the space-time picture of the motion is that of a particle which exists somewhere on the surface of a world tube, concentric with the straight world line of the mass center, and whose spatial extent is determined by the ratio S/S_0. After a moment's

[14] R. S. Burington, *Handbook of Mathematical Tables and Formulas* (Handbook Publishers, Sandusky, Ohio, 1956), p. 7.

[15] K. Rafanelli, Nuovo Cimento **52**, 342 (1967).

thought we see that the relatively simple system of two spinless particles which, to an observer fixed to the c.m., remain a fixed distance apart, presents the same space-time picture as that of the single spinning particle. It is not surprising then that conclusions akin to those reached for the single spinning particle apply to this simple composite system.

Consider the two particles, each of rest mass m, located at the space-time points $x_\mu^{(1)}$ and $x_\mu^{(2)}$. The total four-momentum of the system is

$$P_\mu = p_\mu^{(1)} + p_\mu^{(2)}, \tag{47}$$

and the relative four-momentum is

$$p_\mu = p_\mu^{(1)} - p_\mu^{(2)} = m\dot{\rho}_\mu, \tag{48}$$

where

$$\rho_\mu = x_\mu^{(1)} - x_\mu^{(2)}. \tag{49}$$

With $c=1$, we have

$$P_\mu P_\mu = -4m^2 - p_\mu p_\mu, \tag{50}$$

and with p_μ given by

$$p_\mu = (\mathbf{p}, i\epsilon), \tag{51}$$

where $\mathbf{p} = \mathbf{p}^{(1)} - \mathbf{p}^{(2)}$, and $\epsilon = E^{(1)} - E^{(2)}$, we have

$$P_\mu P_\mu = -4m^2 - \mathbf{p}^2 + \epsilon^2. \tag{52}$$

If the two particles do not interact, then $\mathbf{p}=0$. Such a system is timelike for $E^{(1)} = E^{(2)}$, since then $\epsilon = 0$. On the other hand, if $E^{(1)} = -E^{(2)}$, then $\epsilon^2 = 4(E^{(1)})^2$, and if each of the constituent particles has only timelike four-momenta, then (52) becomes

$$P_\mu P_\mu = 4(\mathbf{p}^{(1)})^2 > 0, \tag{53}$$

and the system is spacelike.

If the system of two particles, on the other hand, is to possess "elementary" characteristics, then some interaction must bind the constituents. Then, regardless of the details of the interaction, $\mathbf{p}$ is not required to be zero. If the particles are separated by a fixed distance, as viewed in the c.m. frame, or equivalently the MRF, we have

$$\mathrm{KE}_{\mathrm{rot}} = \mathbf{p}^2/4m \tag{54}$$

as the rotational kinetic energy of the relative motion. Therefore, even the composite particle-antiparticle system considered above is prevented from being spacelike as long as the rotational kinetic energy of the relative motion is greater than some minimum portion of the total kinetic energy of the system; the condition here is

$$\mathrm{KE}_{\mathrm{rot}} \geq (\mathbf{p}^{(1)})^2/m \tag{55}$$

VI. CONCLUSION

We have developed a classical relativistic theory of the symmetric top, and derived the criteria under which (a) the physical mass increases with spin, and (b) there are no spacelike solutions.

The analysis presented was strictly classical. However, it has been argued that the quantum counterpart of the Frenkel point-particle theory is the Majorana wave equation $(i\Gamma_\mu P_\mu + \kappa)\psi = 0$.[10,16] Such a wave equation obtains directly from (8) if $i\Gamma_\mu$ is the velocity operator and the parameter mc is replaced by the parameter κ. It is not surprising, then, that the difficulties inherent in the classical Frenkel formulation are reflected in the Majorana equation,[3] for they are both descriptions of a point particle with intrinsic spin. If we adopt the attitude that within the classical theory lie the guidelines for construction of the corresponding quantum theory, then the semiquantitative features of the classical theory of the relativistic symmetric top should encourage us to study the quantum theory appropriate to this model. In particular if we want to study linear relativistic wave equations of the Majorana type, i.e., which contain all spin states, which display the same kind of relevance to the elementary particles contained in this classical theory, we should consider wave equations of the form $(i\Gamma_\mu P_\mu + Mc)\psi = 0$, where Mc is an operator. The form of the mass operator, in order to properly account for the details of the Zitterbewegung, is dictated by its classical counterpart (31). It is reasonable to assert, in fact, that we need only replace the classical variables in (31) by their operator counterparts, i.e., $P \to -i\hbar\partial_\mu$, and $S_{\mu\nu} \to -i\hbar\Gamma_{\mu\nu}$, where $-i\Gamma_{\mu\nu}$ is the spin operator in units of $\hbar$. The validity of this assertion is currently being tested.

The operator counterpart of the c.m. coordinate (21) has been mentioned in the literature.[13] It is a nonlocal position operator. Thus it would seem that the price of properly accounting for the Zitterbewegung of a spinning particle, in a manifestly Lorentz covariant way, is nonlocal field theory. The dynamical origin of the nonlocality is the Zitterbewegung, whose spatial extent is on the order of the Compton wavelength.

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[16] E. Majorana, Nuovo Cimento **9**, 335 (1932).