Spherically Symmetric Static Solutions of Einstein's Equations

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A formal general solution of Einstein's equations in the static case containing an arbitrary function of r is obtained. A necessary and sufficient condition that the arbitrary function must satisfy in order that the solution be physically meaningful in the neighborhood of the center is established. A mapping from Newtonian solutions is indicated. The case of infinite pressure at the center is considered. New solutions are given as examples.

INTRODUCTION

MOST of the known exact solutions of Einstein's equations in the spherically symmetric static case have been found with *ad hoc* methods which may be described as follows: By manipulating Einstein's equations, a complicated differential equation is obtained connecting two unknown functions; the differential equation becomes simple, however, for particular forms of one of the functions.[1-5]

Although this method has no physical basis and may give unphysical results, it remains a valuable one in view of the scarcity of known exact solutions. It is felt, however, that there is a need for a more physical approach.

The present work is concerned with (a) properties of Einstein's equations and their solutions in the case of perfect fluids with spherical symmetry and (b) new exact solutions and their physical properties.

FORMAL GENERAL SOLUTION

We will use freedom in the choice of coordinates to take the following line element:

$$ds^2=-A(r)dr^2-r^2d\Omega^2+C(r)dt^2 \tag{1}$$

or

$$ds^2=-e^{\omega}dr^2-r^2d\Omega^2+e^{\sigma}dt^2, \tag{2}$$

with

$$d\Omega^2=d\theta^2+\sin^2 d\theta\varphi^2.$$

The field equations for the symmetric energy-momentum tensor are in this case[6]

$$8\pi p=-8\pi T_1{}^1=e^{-\omega}(1/r^2+\sigma'/r)-1/r^2, \tag{3}$$

$$8\pi p=-8\pi T_2{}^2=e^{-\omega}[\tfrac{1}{2}\sigma''+\tfrac{1}{4}\sigma'^2+(\sigma'-\omega')/2r-\tfrac{1}{4}\sigma'\omega'], \tag{4}$$

$$8\pi\rho=8\pi T_4{}^4=e^{-\omega}(\omega'/r-1/r^2)+1/r^2. \tag{5}$$

From $T_1{}^i{}_{;1}=0$ we obtain

$$-\tfrac{1}{2}\sigma'=p'/(p+\rho). \tag{6}$$

We have to solve the equation $T_1{}^1=T_2{}^2$, which is a differential equation relating ω to σ. From the mathematical point of view as well as from the physical point of view, it is more convenient to consider σ as an arbitrary function of r and express ω in terms of σ; reverting to A and C, the equation $T_1{}^1=T_2{}^2$ reads

$$-\frac{A'}{A^2}\left(\frac{C'}{4C}+\frac{1}{2r}\right)+\frac{1}{A}\left(\frac{C''}{2C}-\frac{C'}{2rC}-\frac{C'^2}{4C}-\frac{1}{r^2}\right)+\frac{1}{r^2}=0, \tag{7}$$

which is a linear differential equation in A^{-1}, the solution of which is

$$A=e^{\omega}=r^{-2}e^{\sigma}(2+\sigma'r)^2\exp\left(-4\int\frac{\sigma'dr}{2+r\sigma'}\right)\Big/ -4\int\left[r^{-3}e^{\sigma}(2+r\sigma')\exp\left(-4\int\frac{\sigma'dr}{2+r\sigma'}\right)\right]dr+\text{const}. \tag{8}$$

By giving to σ arbitrary expressions in function of r, we obtain different solutions. We may then look at the properties of the solutions and see if they have any physical meaning. This is an *ad hoc* method similar to that used by Tolman.[1] From a practical point of view, we should choose σ so that the indicated integrations may be performed. A simple way is to take for σ a rational function of r with arbitrary coefficients:

$$\sigma=\frac{a_0+a_1r+a_2r^2+\cdots}{b_0+b_1r+b_2r^2+\cdots}, \tag{9}$$

then $\sigma'/(2+r\sigma')$ will also be a rational fraction; if the coefficients are so chosen that there are no double roots in the denominator of (9), we shall have

$$\sigma'/(2+r\sigma')=\sum[c_i/(r-d_i)], \tag{10}$$

so that $\exp\{-4\int[\sigma'dr/(2+r\sigma')]\}$ will give

$$\exp\left(-4\int\sum\frac{c_idr}{r-d_i}\right)=\prod(r-d_i)^{-4c_i}. \tag{11}$$

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[2] H. A. Buchdahl, Phys. Rev. **116**, 1027 (1959).
[3] H. A. Buchdahl, Astrophys. J. **140**, 1512 (1964).
[4] H. A. Buchdahl, Astrophys. J. **147**, 310 (1967).
[5] B. Kuchowicz, report, 1966 (unpublished).
[6] R. C. Tolman, *Relativity, Thermodynamics, and Cosmology* (Oxford University Press, London, 1934), p. 242.

If we choose the a's and the b's so that every $4c_i$ is a whole number, all the integrations can be performed, and we obtain solutions expressed in terms of known functions.

However, we are interested in finding conditions for the choice of σ so that the resulting solutions will be physically meaningful. We shall do this in three different ways and shall establish conditions on σ so as to obtain (a) physically meaningful solutions with finite pressure at the center, (b) a mapping of Newtonian solutions, and (c) solutions with infinite pressure at the center that may be used in the problem of collapse.

Conditions for Physical Solutions with Finite Pressure at Center

From the three equations (3), (5), and (6) we can eliminate A and C to obtain

$$\begin{aligned}&8\pi r^3(-2p'^2\rho+4pp'^2+2pp'\rho'-2p\rho p''-2p^2p'')\\&\quad+3(8\pi)r^2p'(\rho+p)^2+r[6p'^2-2p''(\rho+p)+2p'\rho'\\&\qquad-8\pi(\rho+p)^2(\rho+3p)]-4p'(\rho+p)=0.\end{aligned}\quad(12)$$

It is known[7] that in the static case and for the metric (1) the pressure is a decreasing function of r; therefore, the maximum of the pressure is at the origin; if an infinity of the pressure is to be avoided it therefore is enough to have finite pressure at the origin. We shall now prove the following results:

Result I. If the origin is a regular point for ρ and p, then at the origin we must have $p_0'=0$.

If the origin is a regular point for ρ and p, they can be expanded in a Taylor series. Introducing these series into Eq. (12) and setting $r=0$, we obtain

$$-4p_0'(\rho_0+p_0)=0, \quad (13)$$

so that we must have $p_0'=0$.

Result II. If the origin is a regular point for ρ and p as functions of r, then ρ and p must be even functions of r.

The proof of this result is given in the Appendix.

Result III. If the origin is a regular point for $\rho(r)$ and if A^{-1} behaves there as ar^n with $n\neq-1$, then we must have at the origin $A=e^{\omega}=1$.

From (5) we may write

$$8\pi r^2\rho+(A^{-1}-1)-A'r/A^2=0. \quad (14)$$

Assuming that A^{-1} behaves like ar^n for small r, we have at the origin for finite ρ

$$(n+1)ar^n-1=0,$$

so that $n=0$ and $a=1$, that is to say, $A(0)=1$.

Result IV. If the origin is a regular point for ρ, then A is an even function of r.

The result is evident upon inspecting Eq. (5) and taking result II into account.

Result V. If the origin is a regular point for ρ and p, we must have

$$p_0''=-(\tfrac{4}{3}\pi)(\rho_0+p_0)(\rho_0+3p_0). \quad (15)$$

Let us introduce in (12) the Taylor expansion of ρ and p in terms of even powers of r (result II), and let us set equal to zero the coefficient of r in the resulting equation; we obtain

$$-2p_0''-(\rho_0+p_0)8\pi(\rho_0+3p_0)-4p_0''=0,$$

which leads to the result.

What is remarkable about this result is that it is independent of the equation of state.

Result VI. If ρ and p are regular at the origin and if A^{-1} does not behave there as r^{-1}, then σ and therefore $C=e^{\sigma}$ admit a Taylor expansion around the origin in even powers of r.

We have from (3)

$$\sigma'=A(8\pi pr+1/r)-1/r. \quad (16)$$

Introducing the Taylor expansion of p and A in even powers of r (results II and V) and taking III into account, we obtain for σ' a Taylor expansion in odd powers of r, the integral of which is a Taylor series in even powers of r.

Now, let us write the equation of state in the form

$$\rho=\rho_0+a(p-p_0)+b(p-p_0)^2+\cdots, \quad (17)$$

and expand p, ρ, A, and σ around the origin:

$$p=p_0-\tfrac{1}{12}(\rho_0+p_0)8\pi(\rho_0+3p_0)r^2+dr^4+\cdots, \quad (18)$$

$$A=1+ur^2+vr^4+wr^6+\cdots, \quad (19)$$

$$\rho=\rho_0+fr^2+gr^4+hr^6+\cdots, \quad (20)$$

$$\sigma=\sigma_0+jr^2+kr^4+\cdots. \quad (21)$$

Introducing these expressions into (6), (3), and (5), we obtain

$$j=\tfrac{4}{3}\pi(\rho_0+3p_0), \quad (22)$$

$$u=8\pi\rho_0/3, \quad (23)$$

$$a=\frac{5}{3}\frac{\rho_0-3p_0}{\rho_0+p_0}-\frac{240k}{(8\pi)^2(\rho_0+p_0)(\rho_0+3p_0)}. \quad (24)$$

We may therefore write for σ

$$\sigma=\sigma_0+\tfrac{4}{3}\pi(\rho_0+3p_0)r^2+(8\pi)^2\times\frac{(\rho_0+p_0)(\rho_0+3p_0)}{240}\left(\frac{5}{3}\frac{\rho_0-3p_0}{\rho_0+p_0}-a\right)r^4+O(r^6). \quad (25)$$

We therefore have the following indications on σ:

(1) σ is an even function of r.

[7] B. Harrison, K. Thorne, M. Wakano, and J. Wheeler, *Gravitational Theory and Gravitational Collapse* (The University of Chicago Press, Chicago, 1965), p. 21.

(2) The constant σ_0 is of course arbitrary and is changed by a rescaling of time.

(3) The coefficient of r^2 in (21) must be positive ($j>0$).

(4) The choice of the coefficient j of r^2 determines the value of ρ_0+3p_0. The knowledge of σ cannot determine ρ_0 and p_0 independently. Concerning this last point, we have from (8) [and taking (19) and (23) into consideration] that

$$A=\frac{r^2X(r)}{r^2Y(r)+\text{const}\times r^2}=1+\frac{8\pi\rho_0}{3}r^2+\cdots, \qquad (26)$$

where $r^2X(r)$ and $r^2Y(r)$ tend to a constant different from 0 for $r=0$. The constant in A is therefore given by

$$-(8\pi\rho_0/3)[r^2X(r)]_{r=0}. \qquad (27)$$

It is therefore clear that the value of the constant in A affects the value of ρ_0. In short, the knowledge of σ determines the solution up to an arbitrary constant in A; this allows for an arbitrary determination of ρ_0; the solution is therefore completely determined by giving σ and ρ_0.

However, once σ (and therefore j) is given, ρ_0 must be restricted in its values for physical reasons; we have from (22)

$$\rho_0=(3j/4\pi)-3p_0. \qquad (28)$$

We will therefore impose on ρ_0 the inequalities

$$3j/16\pi\leqslant\rho_0\leqslant 3j/4\pi, \qquad (29)$$

which correspond to the physical inequalities

$$\rho_0\geqslant p_0\geqslant 0. \qquad (30)$$

We shall impose also the condition $(d\rho/dp)_0\geqslant 1$, i.e. [from (17) and (24)],

$$\frac{5}{3}\left(\frac{\rho_0-3p_0}{\rho_0+p_0}\right)-\frac{240k}{(8\pi)^2(\rho_0+p_0)(\rho_0+3p_0)}\geqslant 1. \qquad (31)$$

Using (22), this may be expressed in terms of j and k so that we have

$$5k/(\pi j)\leqslant(8/3)\rho_0-(3j/2\pi). \qquad (32)$$

For a given j this inequality is the easier to satisfy when ρ_0 takes its greatest value, which according to (29) is $3j/4\pi$. In this case we still have to satisfy

$$k\leqslant j^2/10. \qquad (33)$$

Conversely, it is easy to show that if j and k are given so that (33) is satisfied, it is possible to find a value of ρ_0 satisfying (29) and such that (31) also is satisfied. We therefore reach the following result:

Result VII. The necessary and sufficient condition for a function σ to correspond to a solution such that the pressure is finite at the center, and such that around the origin we have $d\rho/dp\geqslant 1$ and $\rho_0\geqslant p_0\geqslant 0$, is that σ admit a Taylor expansion

$$\sigma=\sigma_0+jr^2+kr^4+\cdots$$

in even powers of r, and that j and k satisfy the inequalities

$$j>0 \quad\text{and}\quad k\leqslant\tfrac{1}{10}j^2 \qquad (34)$$

—or equivalently, that C admit a Taylor expansion

$$C=m(1+nr^2+qr^4+\cdots) \qquad (35)$$

in even powers of r with

$$n>0 \quad\text{and}\quad q\leqslant\tfrac{3}{5}n^2. \qquad (36)$$

Let us apply result VII on the different exact solutions mentioned in Tolman's article.[1]

Solution I is that of Einstein's universe. We have in this case $e^\sigma=\text{const}$, so that $n=q=0$ [Eq. (35)]. This solution does not, therefore, comply with the conditions (34) or (36) of result VII (hereafter referred to simply as "our conditions"). It is known that for this solution $\rho+3p=0$, unless we use the cosmological constant.

Solution II is that of Schwarzschild and de Sitter; we have in this case

$$e^\sigma=\text{const}(1-2M/r-r^2/R^2). \qquad (37)$$

If $M\neq 0$, the solution does not respect our conditions which state that e^σ is to have a Taylor expansion around the origin. In fact, in such a case A^{-1} behaves like r^{-1} at the origin. We have excluded this case in establishing result III. (We have therefore excluded the case of a point mass at the origin.)

The case $M=0$ gives $n=-1/R^2$ and $q=0$, which again contradict our conditions. In fact, we have in this case de Sitter's universe, for which $\rho+p=0$.

Solution III is the Schwarzschild interior solution. We have in this case

$$e^\sigma=[A-B(1-r^2/R^2)^{1/2}]^2, \qquad (38)$$

from which we write

$$\sigma=2\ln(A-B)+\frac{Br^2}{R^2(A-B)}+\frac{AB-2B^2}{4(A-B)^2}\frac{r^2}{R^4}+O(r^4). \qquad (39)$$

The condition $j>0$ here reads $A/B>1$, and the condition $k<j^2/10$ here reads $A/B<12/5$.

The two conditions are compatible, and therefore it should be possible to find a solution satisfying our criteria of physicality. The Schwarzschild solution does not satisfy our criteria since it has $d\rho/dp=0$ everywhere.

However, the expression for e^ω in this solution is not the most general one corresponding to the given e^σ, since it does not contain an arbitrary constant in addition to those that occur in e^σ.

The most general expression for e^ω may be calculated from (8), and is found to be

$$e^{\omega}=\frac{[x(A-Bx)+B(1-x^2)]^2[(x-x_1)^{x_1}/(x-x_2)^{x_2}]^{2/(x_2-x_1)}}{[x(A-Bx)+B(1-x^2)]^2[(x-x_1)^{x_1}/(x-x_2)^{x_2}]^{2/(x_2-x_1)}x^2+ar^2}, \tag{40}$$

in which

$$x=\left(1-\frac{r^2}{R^2}\right)^{1/2},\quad x_1=\frac{A-(A^2+8B^2)^{1/2}}{4B},\quad x_2=\frac{A+(A^2+8B^2)^{1/2}}{4B}, \tag{41}$$

while a is the additional arbitrary constant. The Schwarzschild interior solution corresponds to the particular value $a=0$.

If we give to A/B the value 2, for instance, we shall have satisfied the imposed inequalities. We can then choose the arbitrary constant a so as to have $\rho_0>p_0>0$.

Solution IV is the most interesting of Tolman's solutions. We have in this case

$$e^{\omega}=\frac{1+(2r^2/A^2)}{(1-r^2/R^2)(1+r^2/A^2)}, \tag{42}$$

$$e^{\sigma}=B^2(1+r^2/A^2), \tag{43}$$

$$8\pi\rho=A^{-2}\frac{1+(3A^2/R^2)+3r^2/R^2}{1+(2r^2/A^2)}+\frac{2}{A^2}\frac{1-(r^2/R^2)}{(1+2r^2/A^2)^2}, \tag{44}$$

$$8\pi p=A^{-2}\frac{1-(A^2/R^2)-3r^2/R^2}{1+(2r^2/A^2)}. \tag{45}$$

The equation of state is

$$\rho=\rho_0+5(p-p_0)+8(p-p_0)^2/(\rho_0+p_0), \tag{46}$$

with

$$8\pi\rho_0=3/A^2+3/R^2$$

and

$$8\pi p_0=1/A^2-1/R^2.$$

The condition $j>0$ here gives $A^2>0$; the second condition is identically satisfied here. Granted the positiveness of A^2 we find, in agreement with our results, that it is always possible to determine a convenient value for ρ_0 such that we have at the origin

$$\rho_0>p_0>0 \text{ and } (d\rho/dp)_0>1.$$

In our case $(d\rho/dp)_0=5$, and we should require

$$\frac{3}{A^2}+\frac{3}{R^2}>\frac{1}{A^2}-\frac{1}{R^2}>0 \quad\text{or}\quad \frac{-1}{2A^2}<\frac{1}{R^2}<\frac{1}{A^2}. \tag{47}$$

This can be satisfied either with $R^2>A^2>0$ or with $-R^2>2A^2>0$. This last case was not considered by Tolman, who imposed the inequality $\rho_0/p_0\geqslant 3$ for which we must have $R^2>0$.

Solution V: We have here $e^\sigma=r^{2n}$; σ cannot be expanded into a Taylor series around the origin so that it does not respect our conditions. In fact, the solution has infinite pressure and density at the origin.

Solution VI:

$$e^{\sigma}=(Ar^{1-n}-Br^{1+n})^2,\quad e^{\omega}=2-n^2, \tag{48}$$

$$8\pi\rho=(1-n^2)/r^2(2-n^2), \tag{49}$$

$$8\pi p=[(1-n)^2A-(1+n)^2Br^{2n}]/r^2(2-n^2)(A-Br^{2n}). \tag{50}$$

As given by Tolman, this solution is singular except for $n^2=1$, for which we have $\rho=0$ and $p\neq 0$, an unacceptable situation.

However, from our point of view Tolman's solution is not general enough, since the expression for e^ω does not contain an arbitrary constant in addition to those occurring in e^σ. Now it is possible with the help of Eq. (8) to find a more general expression. The calculations for the case $n=1$ give

$$e^{\sigma}=(A-Br^2)^2=A^2[1-(2B/A)r^2+(B^2/A^2)r^4], \tag{51}$$

$$e^{\omega}=\frac{(A-3Br^2)^{2/3}}{(A-3Br^2)^{2/3}+dr^2}\quad (d=\text{arbitrary const}), \tag{52}$$

$$8\pi p=\frac{4}{(-A/B)+r^2}, \tag{53}$$

$$8\pi\rho=\frac{-3dA+5dBr^2}{(A-3Br^2)^{5/3}}=\frac{2dAp^{5/3}+20dBp^{2/3}}{(-12B-2Ap)^{5/3}}. \tag{54}$$

Tolman's solution corresponds to the particular value $d=0$ of the arbitrary constant.

Our conditions here read $B/A<0$ and $B^2/A^2<12B^2/5A^2$; the second condition is identically satisfied. The solution should therefore be physical around the origin. In fact, there we have

$$d\rho/dp=5dA^{1/3}/2B. \tag{55}$$

It is clear that for $A>0$, $B<0$, and $d<0$, ρ_0 and p_0 are both positive; it is also clear that ρ_0/p_0 and $(d\rho/dp)_0$ can be as great as desired.

Solution VII has such a complicated expression that Tolman has not given any of its properties.

Solution VIII is given by an expression for σ that cannot be expanded in a Taylor series around the origin, so that this solution is unphysical according to our criterion. In fact, the expression for $\rho+p$ as given by Tolman tends to infinity at the center.

Of course, we are not restricted to the known solutions; taking for instance

$$C=e^{\sigma}=\frac{1+ar^2}{1+br^2}=1+(a-b)r^2+(b^2-ab)r^4+\cdots, \quad (56)$$

we may calculate from (8) the corresponding expression for $A=e^{\omega}$, which is given by

$$\frac{4(abr^4+2ar^2+1)^2}{r^2(1+br^2)^3(1+ar^2)}\left(\frac{br^2+(1-b/a)^{1/2}}{br^2+(1+b/a)^{1/2}}\right)^{-(1-b/a)^{1/2}} \bigg/ -4\int\frac{abr^4+2ar^2+1}{r^3(1+br^2)^2}\left(\frac{br^2+(1-b/a)^{1/2}}{br^2+(1+b/a)^{1/2}}\right)^{-(1-b/a)^{1/2}} \times dr+\text{const}. \quad (57)$$

Our conditions are in this case $a-b>0$, $b^2-ab<\frac{3}{5}(a-b)^2$. The two conditions lead to a being greater than the positive one of the two quantities b or $-\frac{2}{3}b$. Let us take for instance $b=\frac{3}{4}a>0$ and $r^2=z$; we obtain for the integral occurring in the denominator

$$\int\frac{abz^2+2az+1}{z^2(1+bz)^2}\left(\frac{2bz+1}{2bz+3}\right)^{-1/2}dz, \quad (58)$$

which can be expressed in terms of known functions. We need not calculate the complicated expression for $A=e^{\omega}$ in order to assert that the solution behaves physically around the origin.

MAPPING OF NEWTONIAN SOLUTIONS

It is clear from (6) that for $p\ll\rho$, $-\frac{1}{2}\sigma'$ corresponds to the Newtonian field intensity; therefore, if in $\sigma=2V$ we introduce for V the solution of a Newtonian problem, we shall have a corresponding general relativistic exact solution. For all the values of the parameters corresponding to weak fields, the solution will be similar to that of Einstein's linear approximation. However, the range of the parameters is not to be fixed by the strength of the field but by requiring the solution to behave physically. We will see that exact solutions so obtained may describe indeed strong gravitational fields.

Let us consider, for instance, a Newtonian incompressible fluid; the Newtonian potential for a sphere of finite radius is

$$V=ur^2+v \quad (u \text{ and } v \text{ are constants}). \quad (59)$$

We may therefore take for our mapping

$$\sigma=ar^2+b \quad (a \text{ and } b \text{ are constants}). \quad (60)$$

We thus rediscover Tolman's solution IV.[1] It may be checked readily that the solution has physical meaning even in the case of strong fields.

We could have taken for σ the expression

$$\sigma=\ln[1+(ar^2+b)]. \quad (61)$$

In the case of weak fields ($ar^2+b\ll1$), this last expression is, up to terms of smaller order, equivalent to the preceding one. However, we obtain in this case a new solution the line element of which is

$$ds^2=-\frac{dr^2}{1-ar^2e^{-1-ar^2}\,\text{Ei}(1+ar^2)+\text{const}} -r^2d\Omega^2+e^{ar^2+b}dt^2, \quad (62)$$

in which Ei is defined by

$$\text{Ei}(x)=\int_{-\infty}^{x}\frac{e^t dt}{t}. \quad (63)$$

SOLUTIONS WITH INFINITE PRESSURE AT CENTER

These solutions have a physical meaning if they can be considered as limiting cases for very high pressures at the center.

Let us suppose that $\rho=kp$ (k is a constant); integrating (6) we find

$$e^{\sigma}=g_{44}=\text{const}\times p^{-2/(1+k)};$$

it is therefore seen that as p tends to infinity g_{44} tends to zero, and reciprocally.

Let us now inquire about the behavior of ρ and p near the origin for infinite values of p_0; we shall suppose that p may be expanded in a Laurent series around the origin and that it has a dominant term $p\approx ar^n$ for small r (with $n<0$); we shall then have $\rho\approx kar^n$, so that Eq. (12) becomes

$$a^3r^{3n+1}[n^2(2-2k)+n(3k^2+8k+5)-k^3-5k^2-7k-3] +a^2r^{2n-1}(4n^2-2n-2kn)\approx0. \quad (64)$$

It can be readily checked that the only possibility compatible with $k>1$ and $n<0$ is to take $n=-2$. We have in this case

$$a=(20+4k)/(k^3+11k^2+31k+5). \quad (65)$$

We may therefore state that when ρ and p are infinite at the center while their ratio ρ/p tends to a constant greater than 1, and if the Laurent series for p has a limited number of terms in negative powers of r, then the pressure and the density behave near the center as r^{-2}.

Introducing $p\approx ar^{-2}$, $\rho\approx kar^{-2}$ into (6), we find

$$\sigma=4\ln r^{1/(1+k)}+\text{const}, \quad\text{or}\quad C=e^{\sigma}=\text{const}\times r^{4/(1+k)}, \quad (66)$$

giving thus the dominant term of C near the origin.

Taking $C=r^{4/(1+k)}$, we find from (8)

$$A=\frac{(k^2+6k+1)/(k+1)^2}{1-ur^{(2k^2+12k+2)/(k^2+4k+3)}}, \quad (67)$$

$$8\pi p=-\frac{5+k}{1+k}\frac{(k+1)^2u}{k^2+6k+1}r^{(4k-4)/(k^2+4k+3)}+\frac{4}{k^2+6k+1}r^{-2}, \quad (68)$$

$$8\pi\rho=\frac{3k^2+16k+5}{k^2+4k+3}\frac{(k+1)^2u}{k^2+6k+1}r^{(4k-4)/(k^2+4k+3)}+\frac{4k}{k^2+6k+1}r^{-2}. \quad (69)$$

We therefore rediscover Tolman's solution V in a different notation.

It is to be remarked that for $k=1$ we obtain the particularly simple solution

$$ds^2=-\frac{2}{1-ur^2}dr^2-r^2(d\theta^2+\sin^2\theta d\varphi^2)+br^2dt^2, \quad (70)$$

$$8\pi p=-\tfrac{3}{2}u+\tfrac{1}{2}r^{-2}, \quad (71)$$

$$8\pi\rho=+\tfrac{3}{2}u+\tfrac{1}{2}r^{-2}, \quad (72)$$

$$\rho=p+3u/8\pi. \quad (73)$$

Taking $b=u$, the line element can be smoothly joined to Schwarzschild's exterior solution

$$ds^2=-(1-2m/r)^{-1}dr^2+r^2(d\theta^2+\sin^2\theta d\varphi^2)+(1-2m/r)dt^2. \quad (74)$$

It can be checked easily that we obtain a spherical body which is surrounded by vacuum and has a coordinate radius

$$R=(3u)^{-1/2}, \quad (75)$$

while the total mass is given by

$$m=\tfrac{1}{3}(3u)^{-1/2}. \quad (76)$$

It is clear that with u small enough, the total mass can be as great as desired. (The relevance of this solution to the problem of gravitational collapse will be the subject of another paper.)

DISCUSSION

We have adopted as the criterion for physicality the condition that pressure and density be finite at the center and that they satisfy the inequalities $\rho>p>0$, $d\rho/dp>1$. Although we believe that there is no naturally occurring infinite pressure, we studied the case of infinite pressure at the center as a limiting case for very high pressures, thus giving a physical meaning to a seemingly unphysical solution (according to our criterion). It may therefore be asked if there are no other cases in which solutions disqualified by our criterion could be so reinterpreted. Such would be the case, for instance, with a solution having a finite pressure at the center with an infinite density; it could be considered as representing a limiting case for the threshold of condensation at the center. It could be asked if we have exhausted the case for which both pressure and density tend to infinity at the center, while their ratio tends to a finite number different from zero. In fact, we have restricted the study to the case in which there is a dominating term in the expression of p in the function of r. We have therefore excluded, for instance, a logarithmic divergence of the pressure. Such a logarithmic behavior is obtained with the following solution:

$$e^{\sigma}=(1-ar)^{-1}, \quad (77)$$

from which we obtain through (8)

$$e^{\omega}=\frac{(2-ar)^6}{r^2(1-ar)^3\{4a^3r+64/r^2+32a^2\ln[(1-ar)/r^2]-4a^2/(1-ar)+\text{const}\}}. \quad (78)$$

The metric is regular at the origin. Nevertheless, the pressure and the density diverge there logarithmically according to

$$p\approx-a^2\ln r \quad \text{and} \quad \rho\approx-a^2(1+\ln r).$$

CONCLUSION

The method developed here is useful for finding new regular solutions either by *ad hoc* considerations or by a mapping from Newtonian solutions. It allows us also to find irregular solutions in which pressure and density diverge as r^{-n} (the only possible value for n being $n=2$ if we impose the condition $\rho/p>1$) while the limit of their ratio at the center can be chosen at will. It allows us also to find out if a solution is irregular by a look at the metric tensor without calculating explicitly the pressure and the density.

APPENDIX

Theorem: If

$$|f(y,-y',y'',-x)|=|f(y,y',y'',x)| \quad (A1)$$

identically in (y,y',y'',x), and if

$$y'(0)=0 \quad (A2)$$

for a solution of

$$f(y,y',y'',x)=0, \quad (A3)$$

then for this solution we have

$$y(-x)=y(x). \quad (A4)$$

The proof consists in the remark that $x \to y(-x)$ also satisfies (A3) and has vanishing derivative at $x=0$, hence $y(-x)=y(x)$ because of uniqueness.

From this theorem we deduce the truth of result II. A look at Eq. (12) supplemented by $\rho=f(p)$ shows that the left side of the Eq. satisfies (A1); now, result I states that $p'(0)=0$ and therefore the solution $p(r)$ of (12) is an even function of r.

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Classical Relativistic Rotator as a Basis for the Elementary Particles*

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A classical Lorentz-covariant generalization of the nonrelativistic theory of a free, stationary, symmetric top is developed. The resulting relativistic theory predicts a physical mass which is a monotonically increasing function of spin asymptotically approaching a linear relation in the limit of large spin. The theory is free of spacelike solutions.

I. INTRODUCTION

THE notion that the elementary-particle resonances may be excited rotational states is not new. It has led to the investigation of the rotational levels of composite systems and to the study of relativistic wave equations based on various rotator models.[1] Perhaps the most detailed study of the applicability of rotational states to the elementary particles is due to Corben.[2] His analysis is based on the model of a symmetric top. It is in the spirit of Corben's approach, that a properly formulated quantum theory of a relativistic rotator is founded on a properly formulated classical theory of that same rotator, that we undertake the present analysis. Some of the introductory material has appeared in the literature; it is reiterated, in Secs. I and II, for the sake of coherence. The rest of the analysis and the emerging rotator theory differ in content from previous formulations.

We develop a classical Lorentz covariant generalization of the free, nonrelativistic, symmetric top and discuss those features of the relativistic theory which indicate its relevance to the elementary particles. We focus especially on the two important features: (a) the predicted relation between the physical mass and spin of the rotator, and (b) the question of spacelike solutions.[3] These two crucial aspects prove to be directly related in our formulation, for the condition which ensures that the physical mass increase monotonically with spin also rules out the possibility of spacelike four-momenta.

Our present purpose is only to indicate the relevance of the model of the symmetric top to a discussion of the elementary particles. Therefore, in the nonrelativistic theory, we make the relatively simple choice of collinear spin angular momentum $\mathbf{S}$ and angular velocity $\boldsymbol{\omega}$. Thus the rotational kinetic energy in the nonrelativistic theory is

$$T=\tfrac{1}{2}\mathbf{S}\cdot\boldsymbol{\omega} = S^2/2I, \quad (1)$$

where I is the moment of inertia about the axis of rotation.[4] The energy-spin relation (1) forms the basis for a highly successful quantum theory of the rotational levels of symmetric molecules and heavy symmetric nuclei.[5] This quantum theory follows almost trivially by merely replacing the classical spin variable $\mathbf{S}$ by $\hbar\mathbf{J}$, where $\mathbf{J}$ is the spin operator in units of $\hbar$. Equation (1) then gives the energy eigenvalues for states of well-defined angular momentum with the continuous classical variable S^2 replaced by the discrete values $\hbar^2 j(j+1)$.

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