Constant Acceleration in Curved Space-Time

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Based on an analysis of various gedankenexperiments, criteria are given that determine when a particle moves with constant acceleration in curved space. It is shown that for a certain restricted case, the criteria are equivalent to those for constant acceleration proposed previously by Rindler and by Marder.

IN flat space, a particle following a hyperbolic trajectory defined by

$$(x'-x_i')^2-(x^4-x_i^4)^2=1/a^2 \tag{1}$$

can be regarded as possessing constant acceleration in the sense that its acceleration is found to be a constant, a, when measured by any free observer who is instantaneously at rest with respect to the particle. Rindler[1] and Marder[2] have proposed extensions of the definition of constant acceleration to include also motions in curved space-time. Rindler has shown, however, that his definition is equivalent to one of the definitions proposed by Marder, so that for our purposes it will be sufficient to comment only on Rindler's work. In brief, Rindler's criteria for constant acceleration in a general space-time are as follows. Let a particle move along a world line $x^\mu(\tau)$ whose tangent vector is $F^\mu=dx^\mu/d\tau$, and let the absolute derivative of F^μ be designated by[3]

$$A^\mu=\frac{\delta F^\mu}{\delta\tau}=\frac{dF^\mu}{d\tau}+\Gamma_{\alpha\beta}{}^\mu F^\alpha F^\beta.$$

According to Rindler, the particle will move with constant acceleration if its world line has the property that

$$A_\mu A^\mu=\alpha^2=\text{const}, \tag{2a}$$

$$\delta A^\mu/\delta\tau=\alpha^2F^\mu. \tag{2b}$$

In the preceding paper,[4] concerned with the principle of equivalence, we have obtained results which lead in a natural manner to criteria that determine when a particle undergoes motion with constant acceleration in curved space. Our criteria are more general than those proposed by Rindler, and they include Rindler's definition as a special case.

What we have done in Ref. 4 is to analyze various gedankenexperiments that are performed equivalently in flat space and in the presence of a gravitational field. For example, we have considered the situation where an observer measures the local velocity and acceleration of a particle by means of the Fock radar method coupled with a proper time clock. With the Fock radar method, an observer reflects a light signal from the particle and defines the distance to the particle and the time of the measurement as

$$s=\tfrac{1}{2}(t_2-t_1),\quad t=\tfrac{1}{2}(t_2+t_1),$$

where t_1 and t_2 are the times on the observers' clock when the signal was sent and received.

If the observer follows a geodesic trajectory in a curved space and the particle moves along an arbitrary timelike world line with tangent vector F^μ, it is found that the measured local velocity and acceleration are related by

$$d^2s/dt^2=U_\mu A^\mu[1-(ds/dt)^2]^{3/2},$$

where U^μ is a unit vector orthogonal to F^μ, i.e.,

$$U_\mu U^\mu=1,\quad U_\mu F^\mu=0.$$

An equivalently performed experiment in flat space consists of a free observer measuring, by means of the radar method, the local velocity and acceleration of a particle following a hyperbolic trajectory defined by (1). For such an experiment it is found that

$$d^2s/dt^2=a[1-(ds/dt)^2]^{3/2}.$$

For this and the other equivalent experiments discussed in Ref. 4, one always finds a correspondence between the acceleration constant a and the quantity $U_\mu A^\mu$, which suggests that $U_\mu A^\mu$ should be interpreted as the gravitational acceleration.

Now the value of the quantity $U_\mu A^\mu$ may change along the particle's world line because of the nature of the trajectory or because of the change of the direction of U^μ. It seems quite natural, therefore, to regard the particle as possessing constant acceleration if $U_\mu A^\mu$ remains constant along its world line. It is necessary, however, to specify a law of transport for the orthogonal unit vector U^μ, for otherwise the direction of U^μ would be free to change in an arbitrary manner. The condition that we shall place upon U^μ is that it be transported in such a manner that it does not rotate. As is well known, this condition is realized if U^μ undergoes Fermi transport[5] along the world line F^μ, i.e., if U^μ satisfies

$$\delta U^\mu/\delta\tau=U_\rho A^\rho F^\mu.$$

Thus a particle will be uniformly accelerated in the di-

[1] W. Rindler, Phys. Rev. **119**, 2082 (1960).
[2] L. Marder, Proc. Cambridge Phil. Soc. **53**, 194 (1957).
[3] Greek indices range from 1 to 4. Latin indices range from 1 to 3. The signature is $(+++-)$.
[4] J. L. Anderson and R. Gautreau, preceding paper, Phys. Rev. **185**, 1656 (1969).
[5] J. L. Synge, *Relativity: The General Theory* (North-Holland Publishing Co., Amsterdam, 1960), p. 15.

rection U^μ if

$$U_\mu A^\mu = \text{const}$$

when U^μ is Fermi-transported along the world line F^μ of the particle.

It is entirely possible that a particle may possess a constant acceleration in one direction and have a variable acceleration in a different direction. To express the directional relationship of the acceleration, consider an orthonormal triad

$$U_{(i)\mu}U^{(j)\mu} = \delta_{(i)}{}^{(j)},$$

where raising and lowering of the bracketed indices labeling each member of the triad is accomplished with the diagonal invariant matrix

$$\eta_{(\alpha\beta)} = \eta^{(\alpha\beta)} = \mathrm{diag}(1, 1, 1, -1),$$

and let each member be orthogonal to the world line F^μ;

$$F_\mu U_{(i)}{}^\mu = 0. \tag{3}$$

If we now let each member of the orthonormal triad propagate with Fermi transport, so that

$$\delta U_{(i)}{}^\mu/\delta\tau = U_{(i)\rho}A^\rho F^\mu, \tag{4}$$

we can define the component of the acceleration in each of the three directions as

$$a_{(i)} = A_\mu U_{(i)}{}^\mu.$$

Thus, for the acceleration to be a constant in a particular direction specified by $U_{(i)}{}^\mu$, we must have $a_{(i)} = \text{const}$.

If the three $a_{(i)}$'s remain constant so that the acceleration of the particle is a constant in three mutually orthogonal spacelike directions, we can regard the particle as moving with absolutely constant acceleration. What we shall now show is that our condition for absolutely constant acceleration is equivalent to Rindler's criteria for constant acceleration. To this end, we introduce a fourth timelike unit vector

$$U_{(4)}{}^\mu = F^\mu,$$

which from Eq. (3) is orthogonal to the three orthonormal vectors $U_{(i)}{}^\mu$. We thus have an orthonormal tetrad $U_{(\alpha)}{}^\mu$ with the property

$$U_\mu{}^{(\alpha)}U_{(\beta)}{}^\mu = \delta_{(\beta)}{}^{(\alpha)}, \quad U_\mu{}^{(\alpha)}U_{(\alpha)}{}^\nu = \delta_\mu{}^\nu, \tag{5}$$

along which any vector may be resolved. In particular, we have

$$A_\mu U_{(i)}{}^\mu = a_{(i)}, \quad A_\mu U_{(4)}{}^\mu = a_{(4)} = 0. \tag{6}$$

Using (5), we have

$$a^{(\alpha)}a_{(\alpha)} = A_\mu A^\mu,$$

so that, since $a_{(4)} = a^{(4)} = 0$,

$$a^{(i)}a_{(i)} = A_\mu A^\mu. \tag{7}$$

Thus, if each $a_{(i)}$ is a constant, we see that

$$A_\mu A^\mu = \alpha^2 = \text{const},$$

which is one of the conditions [Eq. (2a)] that Rindler imposes for constant acceleration.

To show that Rindler's other condition also follows when all the $a_{(i)}$'s are constant, we note that by multiplying $a_{(\alpha)}$ with $U_{(\beta)}{}^\mu$ and contracting with respect to the bracketed indices, we obtain from (5)

$$a^{(\alpha)}U_{(\alpha)}{}^\mu = A^\mu$$

and, since $a_{(4)} = a^{(4)} = 0$, we find

$$A^\mu = a^{(i)}U_{(i)}{}^\mu. \tag{8}$$

Taking the absolute derivative of both sides of (8), we have, if $a_{(i)}$ is a constant,

$$\delta A^\mu/\delta\tau = a^{(i)}\delta U_{(i)}{}^\mu/\delta\tau.$$

If we now use (4) and (7), we arrive at the result that

$$\delta A^\mu/\delta\tau = \alpha^2 F^\mu,$$

which is the other condition [Eq. (2b)] imposed by Rindler for constant acceleration.

Thus Rindler's criteria for constant acceleration are met only when a particle possesses what we have termed absolutely constant acceleration. Our criteria, on the other hand, serve also to define constant acceleration in the less restricted case where one or two of the acceleration components may not be constant.

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