In this paper we have used Noether's theorem to derive the c.m. theorem from the invariance of the Lagrangian under a 3-parameter set of infinitesimal transformations which will not, in general, generate a finite group of transformations in configuration space. We could use instead the Hamiltonian formalism and consider canonical transformations in phase space. Then it proves possible to define a 3-parameter Abelian group of canonical transformations in phase space (existing independently of any particular Hamiltonian, of course) whose generators, for Hamiltonians conserving total energy and momentum, reduce to the generators of symmetry transformations corresponding to the ones we have used in the Lagrangian approach. These transformations, as well as the canonical symmetry transformations generated by the c.m. constants of the motion considered here will be discussed in a separate paper.

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Shear-Free Gravitational Radiation

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It is shown that, with the exception of the Schwarzschild metric, gravitational systems described by suitably regular vacuum solutions of Einstein's equations admitting a shear- and twist-free congruence of diverging null rays must be radiative. Robinson and Trautman have demonstrated the existence of type-II solutions of this kind, which describe radiating gravitational systems with bounded sources. However, attempts to display an explicit radiative solution through specilaization to conformally spherically symmetric, Kerr-Schild, conformally Kerr-Schild, and type-*D* space times lead to singular metrics. Finally, important physical properties of these systems, including energy, angular momentum, radiation flux, and trapped surfaces, are discussed.

I. INTRODUCTION

MANY of the known solutions to Einstein's equations are shear free.[1] The plane-wave type, with divergence-free ray vectors, clearly represent an excessively unphysical extrapolation of gravitational systems with sources confined to a bounded region. Robinson and Trautman[2,3] were the first to investigate systematically shear-free vacuum metrics with diverging ray vectors. This paper is concerned with the question: Which types of gravitational systems with bounded sources can be described by the Robinson-Trautman metrics? The Schwarzschild metric provides an important example and suggests the possible existence of other cases of physical interest. Robinson and Trautman confined their original analysis to hypersurface-orthogonal shear-free metrics. Although they later generalized their approach to include twisting solutions,[4] such as Kerr's, we will restrict our attention here to the twist-free case.

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[1] J. Ehlers and W. Kundt, in *Gravitation: An Introduction to Current Research*, edited by L. Witten (John Wiley & Sons, Inc., New York, 1962), Chap. 2.

[2] I. Robinson and A. Trautman, Phys. Rev. Letters 4, 431 (1960).

[3] I. Robinson and A. Trautman, Proc. Roy. Soc. (London) **A265**, 463 (1962).

[4] I. Robinson and A. Trautman, *Proceedings on Theory of Gravitation* (PWN-Polish Scientific Publishers, Warsaw, 1964), pp. 107–114.

Since we are concerned with vacuum solutions exterior to some bounded region containing sources, the Goldberg-Sachs[5] theorem is applicable: Shear-free vacuum metrics are algebraically special. This means that the Weyl tensor for such solutions has particularly simple algebraic properties. Correspondingly, there exists a coordinate system in which the metric is also algebraically simpler than usual. For this purpose, Robinson and Trautman used null coordinates based upon the shear-free family of diverging null hypersurfaces. Their work showed that in such a coordinate system, the analytical properties of the metric also simplify considerably. This feature can best be appreciated in terms of some work by Newman and Unti[6] concerning the Lienard-Wiechert potentials of an accelerating charged particle in the context of special relativity. In terms of a null coordinate system based upon the shear-free family of null cones, $u=\text{const}$, emanating from the world line of the accelerating particle, the description of the electromagnetic field becomes especially simple. A gauge can be found in which the vector potential satisfies

$$A_\alpha = A u_{,\alpha}. \tag{1.1}$$

This algebraic statement is unusual from the customary point of view of describing the radiation field as trans-

[5] J. N. Goldberg and R. K. Sachs, Acta Phys. Polon. **22**, 13 (1962).

[6] E. T. Newman and T. W. J. Unti, J. Math. Phys. 4, 1467 (1963).

verse. The vector potential given in Eq. (1.1) is purely longitudinal, i.e., in terms of the retarded time coordinate, $x^0=u$; the only nonvanishing component of A_α is A_0. Only the field strengths $F_{\mu\nu}$ are transverse in the radiation zone. Furthermore, Newman and Unti found that the proportionality function A has the analytically simple feature of possessing only two nonzero terms in a $1/r$ expansion along the null cones. These simplifications in the description of the electromagnetic field were accompanied, of course, by making the metrical description of Minkowski geometry more complicated in terms of coordinates tied to a noninertial origin. Consequently, the resulting metric does not even asymptotically approach a metric with time-independent components. This is why the usual description of the radiation zone is no longer appropriate in these coordinates.

Robinson and Trautman's description of shear-free gravitational systems is analogous to Newman and Unti's description of the Lienard-Wiechert potentials. The metric geometry is both algebrically and analytically simple in terms of a null coordinate system, so that Einstein's vacuum equations take on an almost tractable form. Unfortunately, however, techniques for constructing the general solution are still not available. Robinson and Trautman gave various specialized solutions and indicated how to obtain the most general type N solution in their initial paper. Later, Foster and Newman[7] showed how some of these solutions could be used to generate further solutions by means of a conformal mapping. In this way, they are able to find an infinite family of type-III solutions. As indicated by Robinson and Trautman, these algebraic types are too specialized to represent physical systems with bounded sources.[3] For this purpose, only the most general type of Robinson-Trautman metrics, type II, can be expected to be suitable. We give a proof in Sec. II. Hence, even with their infinite class of newly found solutions, Foster and Newman still had to resort to an approximation procedure in order to discuss systems with localized sources.

On the other hand, Robinson and Trautman[4] have presented a proof of the existence of type-II solutions with suitable regularity properties to describe bounded systems. We discuss these regularity properties in Sec. II and show that, except in the Schwarzschild case, *these metrics must be radiative*. In addition, we rule out the existence of suitably regular solutions of various types, including conformally spherically symmetric metrics, twist-free Kerr-Schild[8] metrics, and type-D metrics, the Schwarzschild metric again being the only exception.

Notwithstanding the nonexistence of explicit solutions, type-II Robinson-Trautman metrics still provide the simplest exact model of the exterior field of a bounded source emitting gravitational waves. Some of the physical properties of these systems, such as energy, angular momentum, radiation flux, and trapped surfaces, are discussed in Sec. III.

The formalism we use is very similar to those of Robinson and Trautman[4] and of Foster and Newman.[7] We adopt the same retarded time coordinate $x^0=u$, where the shear-free null hypersurfaces are given by $u=\text{const}$. For the coordinates x^A labeling the null rays on these hypersurfaces, we use polar coordinates $x^2=\theta$, $x^3=\varphi$. For a radial coordinate along the null hypersurfaces we use a luminosity distance $x^1=r$. Robinson and Trautman and Foster and Newman use stereographic coordinates as ray labels and an affine parameter as radial coordinate. The transformations connecting our formalism to that used by Foster and Newman are given in Appendix A. We arrive at the metric

$$ds^2=(K+2\psi_2{}^0/rW)du^2+2W du\, dr+2rW_{,A}du\, dx^A -r^2q_{AB}dx^A dx^B\,, \quad (1.2)$$

where $W=W(u,x^A)$, $\psi_2{}^0=-1$ or 0, q_{AB} is the metric of the unit sphere, and

$$K=W^2[1-L^2(\ln W)]. \quad (1.3)$$

Here L^2 is the total angular momentum operator and K is just the Gaussian curvature of a two-space with metric $W^{-2}q_{AB}$. For Eq. (1.2) to be a vacuum metric, one of Einstein's equations remains to be satisfied,

$$6\psi_2{}^0(\ln W^2)_{,0}+W^2L^2K=0. \quad (1.4)$$

For details see Appendix A.

Equations (1.2)–(1.4) display the simplicity of shear-free systems. Knowledge of the function W completely determines the metric. Because W has no radial dependence, this implies that an asymptotic solution is now equivalent to a global solution. Even though these systems are in general radiative, the transverse metric is completely time-independent. As in the Lienard-Wiechert case, the coordinates leading to this simplification give an unusual metric description of the radiation zone. In the customary null-coordinate treatments of the radiation zone, such as given by Bondi *et al.*,[9] by Sachs,[10] and by Newman and Penrose,[11] in the limit of large r the metric approaches a null polar coordinate version of the Minkowski metric. This is not true of the Robinson-Trautman metrics. In general, the shear-free family of null hypersurfaces does not define a frame at null infinity which in any asymptotic sense can be considered to be inertial.

To be explicit, we word this paper to treat a family of shear-free null hypersurfaces diverging out to future null infinity, but all statements can be time reversed to treat a family converging in from past null infinity.

[7] J. Foster and E. T. Newman, J. Math. Phys. **8**, 189 (1967).
[8] R. P. Kerr and A. Schild, American Mathematical Society Symposium, New York, 1964 (unpublished).
[9] H. Bondi, M. G. J. van der Burg, and A. W. K. Metzner, Proc. Roy. Soc. (London) **A269**, 21 (1962).
[10] R. K. Sachs, Proc. Roy. Soc. (London) **A270**, 103 (1962).
[11] E. Newman and R. Penrose, J. Math. Phys. **3**, 566 (1962).

II. REGULARITY

For the regularity conditions appropriate to the exterior shear-free region of a bounded source system, we impose the usual asymptotic flatness conditions plus the following additional restriction. We require that the shear-free null hypersurfaces $u=\text{const}$ form a regular diverging family of hypersurfaces with topology $S^2\times E^1$ such that consecutive members do not intersect in the neighborhood of future null infinity. In flat space time, this would rule out families of null cones whose vertices lie on spacelike or null lines but would include the more physically interesting case of a family emanating from a timelike line discussed in the preceding description of the Lienard-Wiechert potentials. This additional restriction is probably not necessary but it conveniently rules out many nonphysical situations which otherwise would require more complicated global considerations to eliminate. It does not rule out any situations to which the Robinson-Trautman existence theorem applies, but we stress that by ***regularity*** in this paper we imply a stronger condition than asymptotic flatness alone.

Penrose[12,13] has given a mathematically rigorous coordinate-independent statement of the asymptotic flatness conditions. Using this approach, he has established that the radial behavior of the Robinson-Trautman line element is completely consistent with asymptotic flatness.[13] In the present terminology, this implies that the Riemann tensor displays the asymptotic "peeling" behavior[6,10]

$$R_{\mu\nu\rho\sigma}=r^{-1}N_{\mu\nu\rho\sigma}+r^{-2}\mathrm{III}_{\mu\nu\rho\sigma}+r^{-3}\mathrm{II}_{\mu\nu\rho\sigma}. \qquad (2.1)$$

Because of the algebraic special nature of shear-free systems, the series terminates at type II. In Appendix B, we give the key components of the Riemann tensor. Penrose's treatment of asymptotic flatness can be combined with our regularity requirement on the $u=\text{const}$ hypersurfaces to show that the coordinates $x^\alpha=(u,r,x^A)$ give a nonsingular description of the metric components in Eq. (1.2). Therefore, $W(u,x^A)$ must be a smooth function on the sphere with $0<W<\infty$ for the correct metric signature. Then W^{-1} and $\ln W$ are also smooth functions on the sphere.

The search for physically interesting Robinson-Trautman metrics now reduces to finding solutions of Eqs. (1.3) and (1.4) consistent with these smoothness conditions on W. There are two distinct classes of solutions corresponding to $\psi_2{}^0=0$ or -1. We examine these two possibilities separately.

A. $\psi_2{}^0=0$

In this case the algebraic type is either N or III. All scalars of the Riemann tensor vanish. The field equation, Eq. (1.4), reduces to

$$L^2K=0, \qquad (2.2)$$

so that smoothness on the sphere demands that

$$K=K(u). \qquad (2.3)$$

Inserting Eq. (1.3), the expression for K in terms of W, into Eq. (2.2) and rewriting the result using the $\eth$ calculus[14] discussed in Appendix A, we find

$$L^2K=(\eth^2W)(\eth^{*2}W)-W\eth^2\eth^{*2}W.$$

Here $\eth$ denotes the spin-weight raising operator and $\eth^*$ is the spin-weight lowering operator. (Because of typographical difficulties the $\bar{\eth}$ of earlier references appears here as $\eth^*$.) Integration over the sphere gives

$$0=\oint W^{-1}L^2K d\Omega=\oint W^{-1}(\eth^2W)(\eth^{*2}W)d\Omega.$$

Positive definiteness of the integrand then demands that

$$\eth^2W=0. \qquad (2.4)$$

The smoothness conditions obtained in Eqs. (2.3) and (2.4) are sufficient to ensure that space time is flat, as may easily be checked by referring to the components of the Riemann tensor given in Appendix B. This establishes that there are no regular solutions of type N or type III, except for flat space.

Alternatively, we may arrive at this conclusion by means of a coordinate-independent argument on a single null hypersurface. Let Γ be any null hypersurface with topology $S^2\times E^1$ emanating out to future null infinity. The total energy and momentum P_a ($a=0, 1, 2, 3$) measured by observers at null infinity at the retarded time determined by Γ may be calculated by invariant means.[13,15] The result shows that $P_a=0$ when Γ is shear free and the coefficient $\mathrm{II}_{\mu\nu\rho\sigma}$ is absent in Eq. (2.1). The positive definiteness of the energy carried off by gravitational radiation implies that at later retarded times the total energy is either zero or negative. Hence a shear-free null hypersurface of topology $S^2\times E^1$ on which the Riemann tensor is algebraically type N or III cannot exist in any physically interesting space-time unless the space time is flat at all points future to the null hypersurface.

B. $\psi_2{}^0=-1$

In this case, the Riemann tensor is type II and the only independent scalar invariant is

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}=48/(rW)^6.$$

There is a singularity at $r=0$ unless the interior region is appropriately filled in with matter. Here we are only

[12] R. Penrose, Proc. Roy. Soc. (London) **A284**, 159 (1965).

[13] R. Penrose, in *Relativity, Groups, and Topology*, edited by C. DeWitt and B. DeWitt (Gordon and Breach, Science Publishers, Inc., New York, 1964), p. 565.

[14] E. T. Newman and R. Penrose, J. Math. Phys. **7**, 863 (1966).

[15] J. Winicour, J. Math. Phys. **9**, 861 (1968).

concerned with ruling out singularities in the neighborhood of null infinity. By combining Eqs. (1.3) and (1.4), the field equation for W may be rewritten as

$$6(W^{-2})_{,0}=W\eth^2\eth^{*2}W-(\eth^2W)(\eth^{*2}W). \qquad (2.5)$$

Robinson and Trautman based their existence theorem for nontrivial, regular, type-II solutions on the following lines.[4] At some initial retarded time let W be a smooth positive function on the sphere. Then Eq. (2.5) will propagate W smoothly for some finite time. The Schwarzschild metric is the only explicitly solved example. In this case, W is initially given as a constant on the sphere and propagates as a constant. In the remaining part of this section we will show that certain additional specializations in the geometry cannot lead to nontrivial, regular solutions of Eq. (2.5) and that any regular solution of Eq. (2.5) must be either radiative or Schwarzschild.

1. Conformally Spherically Symmetric Space Times

Any spherically symmetric geometry admits a shear-free null hypersurface. The same will be true of any conformally spherically symmetric geometry since the vanishing of shear is a conformally invariant property. Let $h_{\mu\nu}$ be spherically symmetric metric and put

$$g_{\mu\nu}=e^f h_{\mu\nu}. \qquad (2.6)$$

For any Killing vector ξ^α of the metric $h_{\mu\nu}$, we have

$$\mathcal{L}_\xi g_{\mu\nu}=g_{\mu\nu}\mathcal{L}_\xi f, \qquad (2.7)$$

where $\mathcal{L}$ symbolizes the Lie derivative. Collinson and French[16] have proved a theorem which states that for any empty space-time metric satisfying

$$\mathcal{L}_\xi g_{\mu\nu}=\varphi g_{\mu\nu},$$

the function φ must be a constant except in the type-N case. Since we are no longer concerned with type-N solutions, we may apply this theorem to Eq. (2.7) to obtain

$$\mathcal{L}_\xi f=\text{const.}$$

Single valuedness then requires that f is a constant function on the sphere. Consequently, the metric in Eq. (2.6) must be spherically symmetric. Application of the Birkhoff theorem then establishes that the only regular conformally spherically symmetric empty-space solution is the Schwarzschild solution.

2. Vanishing Radiative Riemann Tensor

A convenient choice of null tetrad,

$$g^{\mu\nu}=2l^{(\mu}n^{\nu)}-2m^{(\mu}\bar{m}^{\nu)}, \qquad (2.8)$$

for the Robinson-Trautman metric is

$$l^\mu=(0,W^{-1},0),$$
$$n^\mu=(1,\tfrac{1}{2}Wg^{11},g^{1A}), \qquad (2.9)$$
$$m^\mu=(0,0,r^{-1}t^A).$$

The radiative $O(r^{-1})$ part of the Riemann tensor ($\Psi_4{}^0$ in the Newman-Unti[17] formalism) can easily be obtained by referring to the results given in Appendix B. We find

$$\Psi_4{}^0\equiv\lim_{r\to\infty} rR_{\mu\alpha\beta\nu}n^\mu\bar{m}^\alpha\bar{m}^\beta n^\nu=-\tfrac{1}{2}W(W^{-1}\eth^{*2}W)_{,0}. \qquad (2.10)$$

The radiative component of the Riemann tensor vanishes if and only if

$$(W^{-1}\eth^2W)_{,0}=0. \qquad (2.11)$$

Let us examine the consequences of Eq. (2.11). We can set

$$\eth^2W=\alpha W, \qquad (2.12)$$

where α is some undetermined spin-weight 2 function which is constant in time. Putting this result into the propagation equation (2.5), we find

$$6(W^{-2})_{,0}=W^2\eth^2\bar{\alpha}+2W(\eth\bar{\alpha})(\eth W). \qquad (2.13)$$

Dividing by W^2 and differentiating, we now obtain

$$3(W^{-4})_{,00}=2(\eth\bar{\alpha})(W^{-1}\eth W)_{,0}. \qquad (2.14)$$

On the other hand, we may also write Eq. (2.11) in the form

$$\eth[W^2(W^{-1}\eth W)_{,0}]=0. \qquad (2.15)$$

It is now easy to see that Eqs. (2.14) and (2.15) imply that

$$\oint W^2(W^{-4})_{,00}d\Omega=0$$

or, equivalently,

$$\oint[(W^{-2})_{,00}+4W^{-4}W_{,0}W_{,0}]d\Omega=0. \qquad (2.16)$$

It is, however, clear from Eq. (2.5) that for any type-II Robinson-Trautman metric the conservation law

$$\oint(W^{-2})_{,0}d\Omega=0 \qquad (2.17)$$

must hold. We can thus conclude from Eq. (2.16) that W is independent of time, so that lack of a radiative Riemann tensor implies that the system is static. Furthermore, by dividing Eq. (2.13) by W and integrating, we obtain

$$\oint\alpha\bar{\alpha}d\Omega=0,$$

[16] C. D. Collinson and D. C. French, J. Math. Phys. 8, 701 (1967).

[17] E. T. Newman and T. W. J. Unti, J. Math. Phys. 3, 891 (1962).

so that

$$\eth^2 W = \alpha = 0.$$

Consequently we may express W in terms of an expansion in spherical harmonics,

$$W = a + \Sigma_m b_m Y_{1m},$$

which cuts off at $l=1$. By a rotation of polar-coordinate axes, we may put this in the form

$$W = a + b\cos\theta.$$

From Eq. (1.4) we see that K must be a constant and from Eq. (1.3) we find that

$$4\pi = K \oint W^{-2} d\Omega,$$

so that $K>0$ and

$$1 > L^2 \ln W.$$

In the present case, this implies that $a^2 > b^2$. This inequality is the sufficient condition for the existence of a conformal change of coordinates on the sphere[18] (corresponding to a Lorentz transformation along the polar axis) such that in the new coordinate system $b'=0$.

We can thus reduce the solution to the form $W=$ const and arrive at the following important result: *Any regular Robinson-Trautman solution with a vanishing radiative component of the Riemann tensor must be the Schwarzschild solution.*

3. *Separable Solutions*

The *ansatz*

$$W = f(u) g(x^A)$$

leads to a separation of the propagation equation [Eq. (2.5)] into two equations, one of which involves only time dependence and the other only angular dependence. It is clear, in this csae, that Eq. (2.11) holds so that $\Psi_4{}^0 = 0$. Consequently, the Schwarzschild solution is the only regular solution of this type.

4. *Twist-Free Kerr-Schild Metrics*

Kerr and Schild[8] have found a way to obtain all solutions to Einstein's equations of the form

$$g_{\mu\nu} = \eta_{\mu\nu} + k_\mu k_\nu, \qquad (2.18)$$

where $\eta_{\mu\nu}$ is the Minkowski metric and k_μ is a null vector. The vacuum field equations demand that k_μ be shear free. Hence in the twist-free case the Kerr-Schild metrics are a subclass of the Robinson-Trautman metrics being considered in this paper. We now show that there are no regular Kerr-Schild solutions other than the Schwarzschild solution. Putting the metric in the form of Eq. (1.2), we have from Eq. (2.18)

$$\eta_{\mu\nu} dx^\mu dx^\nu = H du^2 + 2W du\, dr + 2rW_{,A} du\, dx^A - r^2 q_{AB} dx^A dx^B, \qquad (2.19)$$

where H is some as yet undetermined function which makes the right-hand side a flat-space metric. The relevant components of the Riemann tensor for a general metric of the form of the right-hand side of Eq. (2.19) are given in Eq. (B2) of Appendix B. Flatness of this metric gives the conditions

$$\eth H = H_{,1} = (W^{-1} \eth^2 W)_{,0} = 0.$$

For such metrics $\Psi_4{}^0 = 0$, which establishes the proposed result.

5. *Conformally Kerr-Schild Metrics*

The success of the Kerr-Schild approach to finding new solutions suggests the possibility that the more general metric

$$g_{\mu\nu} = f\eta_{\mu\nu} + k_\mu k_\nu, \qquad (2.20)$$

where f is an arbitrary conformal factor, might also lead to interesting results. When k_μ is a twist-free and shear-free null vector, such metrics will again be a subclass of the Robinson-Trautman metrics being considered here. In that case, we have in analogy with Eq. (2.19)

$$f\eta_{\mu\nu} = H du^2 + 2W du\, dr + 2rW_{,A} du\, dx^A - r^2 q_{AB} dx^A dx^B.$$

Now the metric on the right-hand side must only be *conformally* flat. The relevant Weyl tensor components can easily be calculated from Eq. (B2) of Appendix B. Proceeding as in the Kerr-Schild case, the vanishing of the Weyl tensor implies that $\Psi_4{}^0 = 0$. Again, the only regular solution is the Schwarzschild solution.

6. *Type-D Solutions*

An important subclass of type-II geometries are the type-*D* metrices. In these geometries, besides the double null eigenvector of the Riemann tensor l_μ there is an additional one, say v_μ, satisfying

$$R_{\mu\nu\rho[\sigma} v_{\tau]} v^\nu v^\rho = 0. \qquad (2.21)$$

Let l_μ be the hypersurface orthogonal double null eigenvector given in Eq. (2.8), and let v_μ be an independent solution of Eq. (2.21) which is not necessarily twist free. By completing a null tetrad based upon these two eigenvectors, we may write

$$g^{\mu\nu} = 2l^{(\mu} v^{\nu)} - 2q^{(\mu} \bar{q}^{\nu)}. \qquad (2.22)$$

Equation (2.21) implies that

$$R_{\mu\nu\rho\sigma} v^\mu q^\nu v^\rho q^\sigma = 0, \qquad R_{\mu\nu\rho\sigma} v^\mu q^\nu v^\rho l^\sigma = 0. \qquad (2.23)$$

The null tetrad given in Eq. (2.22) must be related to the null tetrad previously adopted in Eq. (2.8) by a null

[18] R. Sachs, Phys. Rev. **128**, 2851 (1962).

rotation of the form[6]

$$v_\mu = n_\mu + c\bar{c}l_\mu + \bar{c}m_\mu + c\bar{m}_\mu ,$$
$$q_\mu = m_\mu + cl_\mu .$$

Reexpressed in terms of the original null tetrad, Eqs. (2.23) then become

$$R_{\mu\nu\rho\sigma}n^\mu m^\nu n^\rho m^\sigma = 3c^2 R_{\mu\nu\rho\sigma}n^\mu l^\nu n^\rho l^\sigma ,$$
$$R_{\mu\nu\rho\sigma}n^\mu m^\nu n^\rho l^\sigma = -\tfrac{3}{2}cR_{\mu\nu\rho\sigma}n^\mu l^\nu n^\rho l^\sigma . \tag{2.24}$$

By use of the results of Eqs. (B1) of Appendix B, Eqs. (2.24) become

$$c = -\tfrac{1}{12}\sqrt{2}rW^2\eth K - \tfrac{1}{2}\sqrt{2}\eth W ,$$
$$c^2 = \tfrac{1}{12}r^2W^4(W^{-1}\eth^2 W)_{,0} + \tfrac{1}{6}rW^3(\eth W\eth K/2W - \eth^2 K/4) + \tfrac{1}{2}(\eth W)^2 . \tag{2.25}$$

Elimination of c from Eqs. (2.25) gives

$$\eth(W^2\eth K) = 0 , \tag{2.26}$$

$$(\eth K)^2 = 6(W^{-1}\eth^2 W)_{,0} . \tag{2.27}$$

It is easy to show from the propagation equation (1.4) that Eq. (2.27) follows from Eq. (2.26). Hence Eq. (2.26) gives the necessary and sufficient conditions for a Robinson-Trautman metric to be of type D. By applying the operator $\eth^*$ to Eq. (2.26), we find

$$\eth(K^2 + W^2\eth\eth^* K) = 0 . \tag{2.28}$$

Combining Eq. (2.28) with the propagation equation, we then obtain

$$K_{,0} = \tfrac{1}{6}W^2\eth K\eth^* K . \tag{2.29}$$

We now combine Eqs. (2.26) and (2.29) to obtain

$$(W^2\eth K)_{,0} = 0 ,$$

so that if the constraint equation (2.26) is satisfied at one time, the propagation equation guarantees that it will remain satisfied. Putting

$$W^2\eth K = \eta , \tag{2.30}$$

we have

$$\eth\eta = 0 \tag{2.31}$$

and

$$\eta_{,0} = 0 . \tag{2.32}$$

Consequently, the spin-weight 1 function η is time-independent and built out of vector spherical harmonics with $l=1$. We now can rewrite the propagation equation as

$$12W_{,0} = 2\eta\eth^* W - W\eth^*\eta . \tag{2.33}$$

The problem of finding type-D Robinson-Trautman solutions now reduces to first finding initial conditions for W consistent with the constraint embodied in Eqs. (2.30) and (2.31). Then a solution may be obtained by propagating these initial data by Eq. (2.33). The constraint will be automatically propagated. Because the propagation equation (2.33) is now linear, explicit solutions can be obtained quite readily. Examples are given by Robinson and Trautman.

Despite their attractive simplicity, we now show that the only type-D solution which is regular at null infinity is the Schwarzschild solution. First, we note that

$$\eth K = W\eth^*\eth^2 W - (\eth^2 W)(\eth^* W) .$$

From this it follows that

$$W^{-4}\eta = \eth^*(W^{-1}\eth^2 W) ,$$

so that

$$\oint W^{-4}\eta\bar{\eta}d\Omega = 0 .$$

From the positive definiteness of the integrand, we conclude that $\eta = 0$. The solution is static and, by previous results, either singular at null infinity or the Schwarzschild solution.

III. PHYSICAL PROPERTIES

The results of Sec. II show that, with the exception of the Schwarzschild solution, all presently known Robinson-Trautman solutions are not sufficiently regular to represent systems with localized sources. The only additional simplification that appears to be compatible with a realistic shear-free system is axial symmetry, and this does not lead to any major simplification in the propagation equation. Notwithstanding this negative result, the existence theorem presented by Robinson and Trautman ensures that shear-free radiative exterior solutions compatible with a bounded source do exist. For these solutions the metric must deterministically evolve in time according to the propagation equation in order that the shear-free property be maintained. Therefore, "news" in the sense of Bondi *et al.*[9] is not allowed. From this it is clear that the Robinson-Trautman systems are not typical of the most general type of gravitationally radiating system. Nevertheless, they furnish a tractable model for asymptotically flat radiative metrics, and it is interesting to inquire into their physical properties.

Let us first consider the 10 constants of the motion recently discovered by Newman and Penrose.[19] These constants are constructed out of the $O(r^{-6})$ part of the Riemann tensor. Because these higher-order coefficients are all zero for the Robinson-Trautman metrics, the Newman-Penrose constants vanish. This result is a specialization to the hypersurface orthogonal case of a more general result due to Exton[20]: The Newman-Penrose constants are all zero for any algebraically special space-time.

[19] E. T. Newman and R. Penrose, Proc. Roy. Soc. (London) **A305**, 175 (1968).
[20] A. Exton (private communication).

Robinson and Trautman have pointed out the existence of a different conservation law

$$\frac{d}{du}\oint W^{-2}d\Omega=0. \tag{3.1}$$

Geometrically, this conservation law states that the surface area of a 2-sphere with metric $W^{-2}q_{AB}$ is a constant of the motion. It is a special case of a general conservation law for type-II spaces which has been discussed in detail by Jordan, Ehlers, and Sachs.[21]

For asymptotically flat systems, physical quantities which have been called the asymptotic symmetry linkages can be introduced which form a representation of the asymptotic symmetry group or Bondi-Metzner-Sachs group.[15,22] In addition to the translations and Lorentz rotations of the Poincaré group, the Bondi-Metzner-Sachs group contains an infinite-dimensional supertranslation subgroup.[17] Consequently, besides the energy, momenta, and angular momenta, the related physical quantities include an infinite set of supermomenta. The asymptotic symmetry linkages are functionals of the spherical 2-spaces Σ lying on a null hypersurface extending out to future null infinity. They are generalizations of the functionals considered by Komar,[23]

$$K_\xi(\Sigma)=\oint_\Sigma \xi^{[\mu;\nu]}dS_{\mu\nu},$$

where ξ^μ is a global Killing vector, to the case where no global symmetries but only asymptotic symmetries exist. In the present case, applying Killing's equation asymptotically, we find for the asymptotic behavior of the Bondi-Metzner-Sachs symmetry descriptors

$$\xi^\mu\sim[f(u,x^A),\ -\tfrac{1}{2}rf^B{}_{:B},\ f^A(x^B)], \tag{3.2}$$

where

$$(Wf)_{,0}=\tfrac{1}{2}Wf^B{}_{:B}-W_{,B}f^B \tag{3.3}$$

and

$$f^{(A:B)}=\tfrac{1}{2}q^{AB}f^C{}_{:C}. \tag{3.4}$$

Here the colon represents two-dimensional covariant differentiation on the unit sphere. The regular solutions of Eq. (3.4) describe the six-parameter group of conformal transformations of a sphere which is isomorphic to the orthochronous Lorentz group. The solutions of Eq. (3.3) with $f^A=0$ and $Wf=\alpha(x^B)$ describe the supertranslation subgroup. The supertranslations for which α only contains terms with $l\leq 1$ in a spherical harmonic decomposition form the four-parameter invariant-translation subgroup. So far these asymptotic symmetry descriptors are uniquely determined only at null infinity. In order to define the asymptotic symmetry linkages corresponding to some null hypersurface N, the descriptors are propagated along N by the projection of Killing's equation

$$\xi^{(\mu;\nu)}k_\nu=\tfrac{1}{2}\xi^\nu{}_{;\nu}k^\mu,$$

where k_μ is the normal to N. Carrying this out for the shear-free hypersurface $u=$ const gives

$$\begin{aligned}\xi^0&=f,\\ \xi^1&=-\tfrac{1}{2}rf^A{}_{:A}+W_{:A}f^{:A}+\tfrac{1}{2}Wf^{:A}{}_A,\\ \xi^A&=f^A-r^{-1}Wf^{:A}.\end{aligned} \tag{3.5}$$

The linkage functional for some spherical 2-space Σ lying on N is given by

$$L_\xi(\Sigma)=\oint_\Sigma(\xi^{[\mu;\nu]}-\xi^\rho{}_{;\rho}k^{[\mu}m^{\nu]})dS_{\mu\nu}, \tag{3.6}$$

where m^μ is any vector field on Σ normalized by

$$k^\mu m_\mu=-1.$$

When ξ^μ is a global Killing vector, this reduces to Komar's integral. In the present case, if we take Σ to be a surface of constant u and r, we find the particularly simple result

$$L_\xi(u,r)=(4\pi)^{-1}\oint fW^{-2}d\Omega. \tag{3.7}$$

We immediately see that our final result for the linkage functional is independent of r. This is a consequence of the general vacuum result that *the asymptotic symmetry linkages on a shear-free null hypersurface are independent of choice of spherical 2-space.* We may slide the surface of integration freely along the null hypersurface without changing the linkage integrals. (Note that Komar's functionals are completely surface-independent *in vacuo*.) In particular, the linkage through any finite sphere is equal to the total linkage through the sphere at null infinity. Inserting into Eq. (3.7) the value $f=W^{-1}$ appropriate to a time translation, we find for the total energy

$$E(u)=(4\pi)^{-1}\oint W^{-3}d\Omega. \tag{3.8}$$

Taking the time derivative by means of the propagation law, we then have for the radiative energy flux

$$dE/du=-(16\pi)^{-1}\oint W^{-1}(\eth^2W)(\eth^{*2}W)d\Omega. \tag{3.9}$$

This result demonstrates unequivocally that shear-free gravitational systems can radiate gravitational energy. The energy loss is positive definite, as it must be in general for any asymptotically flat system.

[21] P. Jordan, J. Ehlers, and R. K. Sachs, Akad. Wiss. Lit. (Mainz) **1**, 1 (1961).
[22] L. A. Tamburino and J. H. Winicour, Phys. Rev. **150**, 1039 (1966).
[23] A. Komar, Phys. Rev. **113**, 934 (1959).

The total angular momentum is not so well defined for asymptotically flat systems as it is in special relativity, where the ambiguity is of the form

$$\mathbf{L}'=\mathbf{L}+\mathbf{a}\times\mathbf{P},$$

the translations $\mathbf{a}$ mixing the momenta into the definition. In the asymptotically flat case, the supertranslations mix all of the supermomenta into the transformation properties of the angular momentum.[21] This, of course, is in addition to the usual transformation properties under the homogeneous Lorentz group. In the present case, Eq. (3.3) gives for the ξ^0 component of a rotational descriptor at retarded time u

$$Wf=[Wf]_{u=u_0}+\int_{u_0}^{u}[\tfrac{1}{2}Wf^B{}_{:B}-W_{,B}f^B]du. \quad (3.10)$$

It is clear that we can mix in the proper supertranslation to make $f=0$ at retarded time u for the spatial rotations. However, the angular components of the rotational descriptors f^A do not explicitly enter into the linkage integral, Eq. (3.7). Hence, at any retarded time u there exists a supertranslation frame for which the total angular momentum of a Robinson-Trautman system vanishes. More generally, the following conclusion follows from the results of Ref. 15. *For any shear-free null hypersurface whose normal is a double null eigenvector of the Riemann tensor, the total angular momentum can be transformed to zero.* No similar result applies to twisting shear-free congruences. For instance, the Kerr metric, which admits such a congruence, has nonvanishing total angular momentum.

The result that the linkage integrals are independent of r suggests that no incoming gravitational fields are flowing across the shear-free outgoing null hypersurfaces. In fact, Sachs's examination of algebraically special fields was physically motivated by his recognition that they could be utilized in posing covariant criteria for the presence of purely outgoing (or, in the time reversed case, purely incoming) gravitational radiation.[24] However, any definitive conclusion concerning the absence of incoming radiation must be consistent with the behavior of the Riemann tensor at past null infinity. So far, we have restricted our analysis to future null infinity exclusively. A study of past null infinity would at the least require knowledge of the behavior of the function W in the limit $u\to-\infty$. The propagation equation is sufficiently nonlinear and complicated to make this a difficult task. Basically, this equation is parabolic, like a diffusion equation. An analysis of the linearized version of this equation reveals that W smooths out its angular behavior as the system evolves into the future. By the same token, the angular behavior becomes more exaggerated as we trace it back into the past. Although the linearized version is not reliable for the propagation of a nonlinear disturbance through an infinite time interval, it does raise doubts concerning the regularity of past null infinity.

Another global property of physical interest is the existence of trapped surfaces as defined by Penrose.[25] Consider a two surface given by $u=\text{const}$, $r=f(x^A)$. The divergence of the outgoing null rays normal to the surface is

$$\rho_{\text{out}}=(Wf)^{-1}.$$

The divergence of the incoming null rays normal to the surface is

$$\rho_{\text{in}}=-(W/f)[1-2(W^3f)^{-1}-\eth\eth^*\ln f],$$

the over-all minus sign indicating that the null rays converge at large distances for reasonably smooth surfaces. However, surfaces with $f>0$ exist at small distances such that all normal null rays diverge. Such a surface is antitrapped in the sense of Penrose. In the time-reversed case, all normal null rays would converge and the surface would be trapped. An example of such an antitrapped surface is given by $r=f=\text{const}$, where

$$f<\min\{2/W^3\},$$

the minimum being taken over the sphere. The surface defined by

$$f(1-\eth\eth^*\ln f)=2/W^3 \quad (3.11)$$

has the property that all of its incoming normal null rays are divergenceless, as in the case of the Schwarzschild surface. Although we cannot solve the equation for this surface explicitly, a general idea of where such a surface occurs can be obtained by integrating Eq. (3.11) over the sphere. We find

$$\langle f\rangle_{\text{av}}=2E-\langle f^{-1}(\eth f)(\eth^*f)\rangle_{\text{av}}.$$

In the Schwarzschild case, this reduces to the usual result $r=2m$. The presence of radiation effectively reduces the average luminosity distance at which such a surface occurs. The important point worth emphasizing is that the Robinson-Trautman metrics provide a simple model for nonsingular trapped surfaces whose properties are not modified in any drastic way by the presence of gravitational radiation.

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APPENDIX A

The Foster-Newman form of the line element is

$$ds^2=[-R(\ln P^2)_{,0}+K+2\psi_2{}^0/R]du^2+2du\,dR-\tfrac{1}{2}(R^2/P^2)d\zeta\,d\bar{\zeta}, \quad (A1)$$

[24] R. Sachs, Proc. Roy. Soc. (London) **A264**, 309 (1961).

[25] R. Penrose, Phys. Rev. Letters **14**, 57 (1965).

where $P=P(u,\zeta,\bar\zeta)$, $\psi_2{}^0=0$ or -1, and

$$K=4P^2(\ln P^2)_{,\zeta\bar\zeta}. \qquad (A2)$$

The one Einstein equation remaining to be solved is

$$-3\psi_2{}^0(\ln P)_{,0}+2P^2K_{,\zeta\bar\zeta}=0. \qquad (A3)$$

Equations (A1)–(A3) completely specify the conditions for a shear-free Robinson-Trautman system. The coordinates $x^1=r$, $x^A=(\theta,\varphi)$ used in this paper are related to the Foster-Newman coordinates by

$$\zeta=\cot\tfrac{1}{2}\theta\, e^{i\varphi},$$
$$R=2\sqrt{2}rP\sin^2(\tfrac{1}{2}\theta).$$

Straightforward transformation of Eq. (A1) leads to Eq. (1.2), with

$$W=2\sqrt{2}P\sin^2(\tfrac{1}{2}\theta).$$

The most direct way of obtaining the transformed versions of Eqs (A2) and (A3) is to use the spin-weight ladder operator $\eth$ introduced by Newman and Penrose. In terms of the metric of the unit sphere q^{AB} and the complex dyad t^A,

$$q^{AB}=2t^{(A}\bar t^{B)},$$

the operator $\eth$ is defined by, say,

$$\eth(\eta_{ABC}t^At^B\bar t^C)=\sqrt{2}\eta_{ABC:D}t^At^B\bar t^Ct^D,$$

where a colon represents two-dimensional covariant differentiation on the unit sphere. Equations (A2) and (A3) can then be easily rewritten as

$$K=[4P^2/(1+\zeta\bar\zeta)^2]\eth\eth^*\ln P^2,$$
$$-3\psi_2{}^0(\ln P)_{,0}+[2P^2/(1+\zeta\bar\zeta)^2]\eth\eth^*K=0.$$

Equivalently, in the (θ,φ) coordinate system these equations give Eqs. (1.3) and (1.4). The total angular momentum operator L^2 acting on a spin-weight 0 function φ is given by

$$L^2\varphi=-\eth\eth^*\varphi=-\varphi^{:A}{}_A.$$

The operator $\eth$ is used extensively in Secs. II and III. We strictly adhere to the conventions of Newman and Penrose, so that in a polar-coordinate system

$$t^A=[-1/\sqrt{2},\ -i/(\sqrt{2}\sin\theta)],$$
$$t_A=q_{AB}t^B.$$

If $d\Omega$ is the element of surface area on the sphere, then

$$\oint\eth^*\eta\, d\Omega=0,$$

where η is a nonsingular spin-weight $+1$ function. If η has spin-weight s, then

$$(\eth^*\eth-\eth\eth^*)\eta=2s\eta.$$

For further differential and integral properties see Ref. 13.

The contravariant components of the metric for the line element in Eq. (1.2) are

$$g^{00}=g^{0A}=0,\quad g^{01}=W^{-1},$$
$$g^{11}=-1-W^{-1}\eth\eth^*W-2\psi_2{}^0/(rW^3),$$
$$g^{1A}=(rW)^{-1}q^{AB}W_{,B},$$
$$g^{AB}=-r^{-2}q^{AB}.$$

APPENDIX B

Referred to the null tetrad given in Eq. (2.9), the 10 independent tetrad components of the Riemann tensor for a type-II Robinson-Trautman metric, as displayed in Eq. (1.2) with $\psi_2{}^0=-1$, are

$$R_{\mu\nu\rho\sigma}n^\mu m^\nu n^\rho m^\sigma=W(W^{-1}\eth^2W)_{,0}/2r+(\eth W)(\eth K)/2r^2W$$
$$-\eth^2K/4r^2+3(\eth W)^2/r^3W^3,$$
$$R_{\mu\nu\rho\sigma}n^\mu m^\nu n^\rho l^\sigma=\sqrt{2}\eth K/4r^2W+3\sqrt{2}\eth W/2r^3W^3, \qquad (B1)$$
$$R_{\mu\nu\rho\sigma}(n^\mu l^\nu n^\rho l^\sigma+n^\mu l^\nu m^\rho\bar m^\sigma)=2/r^3W^3,$$
$$R_{\mu\nu\rho\sigma}l^\mu m^\nu l^\rho n^\sigma=R_{\mu\nu\rho\sigma}l^\mu m^\nu l^\rho m^\sigma=0.$$

For the more general metric given in the right-hand side of Eq. (2.19), certain relevant components of the Riemann tensor are

$$R_{0AB0}t^At^B=\tfrac{1}{4}\eth^2H-\tfrac{1}{4}rH_{,1}W^{-1}\eth^2W-\tfrac{1}{2}rW(W^{-1}\eth^2W)_{,0},$$
$$R_{010A}t^A=-\tfrac{1}{4}\sqrt{2}r\eth(r^{-1}H)_{,1},$$
$$R_{0AB1}=\tfrac{1}{2}rW^{-1}H_{,1}q_{AB},\quad R_{0101}=-\tfrac{1}{2}H_{,11},$$
$$R_{0ABC}t^At^B\bar t^C=-\tfrac{1}{4}\sqrt{2}rW^{-1}\eth H,$$
$$R_{ABCD}t^A\bar t^Bt^C\bar t^D=r^2[1+\eth\eth^*(\ln W)-W^{-2}H],$$
$$R_{1A1B}=R_{101B}=R_{10AB}=R_{1ABC}=0.$$