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Invariances of Approximately Relativistic Lagrangians and the Center-of-Mass Theorem. I *

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In Newtonian mechanics the ten classical integrals of the equations of motion of a system of interacting point particles can be related to the invariance of the corresponding Lagrangian under the ten infinitesimal transformations of the Galilei group. Systems described by approximately relativistic equations [such as the Darwin equations in special relativity and the Einstein-Infeld-Hoffmann (EIH) equations in general relativity] also possess ten integrals of equivalent physical significance; previous work has established similar relations with invariance properties of the Lagrangian for only seven of these, but not for the three expressing the uniform motion of the center of mass. It is shown here that for any Lagrangian whatever which is a function of the particle positions and velocities alone, and which is invariant under the infinitestimal time and space translations, it is possible to find an additional exact invariance under a three-parameter set of infinitesimal transformations (which, in general, depends on a functional rather than a function). The transformations define a velocity which for approximately relativistic systems can be interpreted as that of their center of mass; for such systems the three conservation laws following from this transformation express the constancy of this velocity. A number of examples are given; for the Darwin and EIH equations, the conservation laws agree with those previously obtained directly from these equations.

I. INTRODUCTION

ANY Newtonian system of point particles with gravitational interactions possesses ten integrals which are algebraic functions of the positions and velocities, the so-called "classical integrals."[1] One of these corresponds to the conservation of energy, three to the conservation of linear momentum, three to the conservation of angular momentum, and three express the uniform motion of the center of mass (c.m.). Similar integrals exist for any interactions derivable from a potential depending only on the mutual separations.

These integrals can be readily obtained by direct integration of Newton's equations of motion. However, new aspects of their significance emerge through their derivation, on the basis of Noether's theorem,[2,3] from the invariance properties of the Lagrangian under the infinitesimal transformations of the inhomogeneous Galilei group.[3,4] The integrals are related, respectively, to the time translation, space translations, space rotations, and the Galilei transformations between inertial systems in relative motion.

The Lorentz group is a ten-parameter group, as is the Galilei group. If the equations of motion of a special relativistic system of point particles can be derived from a "Fokker-type" variational principle[5–7] describing direct particle interactions, this system too possesses ten integrals which are constants of the motion.[8] However, the forces acting in such a system do not depend on the simultaneous positions of the particles only; thus the constants of the motion do not depend on the position and velocities of the particles at any single time, in sharp contrast to the Newtonian conserved quantities, but involve integrals extended over the world lines of the particles. It has been shown that (except for the trivial case of no interaction) these constants of the motion of special relativistic dynamics must always be of this type.[9]

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[1] For an elementary derivation see, e.g., R. Kurth, *Introduction to the Mechanics of Stellar Systems* (Pergamon Press, Inc., New York, 1957), Chap. III. A detailed discussion is given by A. Wintner, *The Analytical Foundations of Celestial Mechanics* (Princeton University Press, Princeton, N. J., 1947), Chap. V.

[2] E. Noether, Nachr. Akad. Wiss. Göttingen, II Math.-Physik. Kl. 235 (1918).

[3] A simplified discussion with examples is given by E. L. Hill, Rev. Mod. Phys. **23**, 253 (1951); for a more general discussion see A. Trautman, in *Gravitation*, edited by L. Witten (Wiley-Interscience, New York, 1962), Chap. 5, or in *Brandeis Summer Institute in Theoretical Physics, 1964* (Prentice-Hall, Inc., Englewood Cliffs, N. J., 1965), Vol. 1, Chap. 7.

[4] E. Bessel-Hagen, Math. Ann. **84**, 258 (1921). This paper includes a slightly more general formulation of Noether's theorems than stated in Ref. 2; it, too, is credited to Noether. We shall use this formulation throughout.

[5] First found for point changes by A. D. Fokker, Z. Physik **58**, 386 (1929). It involves half-retarded plus half-advanced interactions, and integrals over functions of two invariant parameters such as proper times. It is these features which we consider characteristic of "Fokker-type" principles, in agreement with common usage in special relativity.

[6] For the case of mesic interactions, see P. Havas, Phys. Rev. **87**, 309 (1952); **91**, 997 (1953), and references given there.

[7] Such a variational principle can be obtained also in a Lorentz-invariant approximation method in the *general* theory of relativity, as shown by P. Havas and J. N. Goldberg, Phys. Rev. **128**, 398 (1962).

[8] For the explicit form of these integrals see J. W. Dettman and A. Schild, Phys. Rev. **95**, 1057 (1954).

[9] For a discussion of this problem see P. G. Bergmann, in the article on Special Relativity, in *Encyclopedia of Physics*, edited by S. Flügge (Springer-Verlag, Berlin, 1962), Vol. III, Part B1.

However, there are other ways of describing relativistic direct particle interactions which are more closely analogous to the Newtonian description; in particular, in expressing all quantities entering the equations of motion at a single time.[10] Such equations can be either exact ones,[11,12] or approximate ones obtained in the course of an expansion of a set of exact equations (which might be those of a theory of direct particle interactions or of a field theory with sources) in powers of v/c, such as those studied by Darwin in electrodynamics[13] or by Einstein, Infeld, and Hoffmann (EIH) in the general theory of relativity.[14,15] The exact equations known so far either do not follow from a variational principle in a simple and direct manner,[11] or involve a Lagrangian including derivatives of all orders.[12] Nothing is known about their integrals and we shall not consider them here. The approximate equations, on the other hand, are known to follow from variational principles involving the particle positions and velocities only.[13,15,16]

The question thus arises whether these approximately relativistic variational principles, or indeed also some Newtonian variational principles involving more general forces than those considered above, allow ten conservation laws similar to those of the standard Newtonian theory. This question has been partly answered in the affirmative a long time ago; it was found that the approximately relativistic Lagrangian for the EIH equations as well as its generalization to rotating bodies[17,18] is invariant under time translations as well as space translations and space rotations, which by Noether's theorem implies conservation of energy and of linear and angular momentum.[15,17] However, these Lagrangians are not exactly invariant under either the usual Galilei transformations or the Lorentz transformations relating different intertial systems; thus the c.m. theorem was first obtained by Fichtenholz[16,19] as a direct consequence of the approximate equations of motion rather than as a consequence of any invariance properties of the Lagrangian.

Recently, the question of these invariance properties was reconsidered[20] and a rather complex set of finite transformations (not forming a group) leaving the equations of motion invariant was suggested. However, no attempt was made to use the corresponding infinitesimal transformations to obtain the c.m. theorem from the Lagrangian.

In the case of the seven conservation laws known to be a consequence of Noether's theorem for the approximately relativistic Lagrangians, the conservation laws follow (as they should) from an *exact* invariance under infinitesimal transformations, which happen to be the *same* transformations as those familiar from Newtonian theory, and generate a group of finite transformations which forms a subgroup of the Galilei group as well as the Lorentz group. One would not expect the remaining three transformations to belong to the Galilei group, and one might think that they should be related to the Lorentz group. In studying this question, we have found that for *any* Lagrangian which is a function of the particle positions and velocities alone, and which is invariant under the infinitesimal time and space translations (but not necessarily the rotations), it is possible to find an additional *exact* invariance under a three-parameter set of infinitesimal transformations. These are *not* general transformations like the others considered above, but *specific* ones dependent on the particular Lagrangian under consideration and which define a velocity interpretable as that of the c.m. of the system, which need not form the generators of a finite group of transformations. The three conservation laws following from these transformations are similar to the Newtonian c.m. theorem.

As noted before, the ten exact general constants of the motion cannot be functions of a single time only. Thus, for any systematic approximation method such as those considered above, conceivably it will no longer be possible at some stage of the approximation procedure to obtain such functions. However, it is still possible for the first seven conservation laws at the stage considered in the examples of Sec. V containing interactions. For the remaining three it is also possible in general if we only consider the c.m. *theorem*; for the c.m. *coordinate*, on the other hand, it is possible only because certain expressions happen to be integrable, if use is made of the approximate equations of motion in all the cases for which such equations are known.

II. NOETHER'S THEOREM

The fundamental connection between the invariance properties of a variational principle and the conserva-

[10] For a brief review of the various methods see P. Havas, in *Statistical Mechanics of Equilibrium and Non-Equilibrium*, edited by J. Meixner (North-Holland Publishing Co., Amsterdam, 1965), p. 1.

[11] D. G. Currie, Phys. Rev. **142**, 817 (1966); R. N. Hill, J. Math. Phys. **8**, 201 (1967). Variational principles for such equations can be obtained by methods such as those proposed by P. Havas, Nuovo Cimento Suppl. **5**, 363 (1957) and Bull. Am. Phys. Soc. **1**, 337 (1956); see also E. H. Kerner, Phys. Rev. Letters **16**, 667 (1966).

[12] D. Hargreaves, Trans. Cambridge Phil. Soc. **22**, 191 (1917); see also E. H. Kerner, J. Math. Phys. **3**, 35 (1962).

[13] C. G. Darwin, Phil. Mag. **39**, 537 (1920).

[14] A. Einstein, L. Infeld, and B. Hoffmann, Ann. Math. **39**, 66 (1938).

[15] For a detailed review see L. Infeld and J. Plebański, *Motion and Relativity* (Pergamon Press, Inc., New York, 1960) and, from a different point of view, V. Fock, *The Theory of Space Time and Gravitation* (Pergamon Press, Inc., New York, 1959).

[16] I. G. Fichtenholz, Zh. Eksperim. i. Teor. Fiz. **20**, 233 (1950).

[17] W. Tulczyjew, Acta Phys. Polon. **18**, 37 (1959); V. Fock, in Ref. 15.

[17] W. Tulczyjew, Acta Phys. Polon. **18**, 37 (1959); V. Fock, in Ref. 15.

[18] R. Michalska, Bull. Acad. Polon. Sci., Cl. III, **8**, 237 (1960).

[19] I. G. Fichtenholz, Dokl. Akad. Nauk SSSR **64**, 325 (1949). The case of only two particles was treated earlier by V. A. Fock, *ibid.* **32**, 28 (1941).

[20] S. Chandrasekhar and G. Contopolous, Proc. Roy. Soc. (London) **A298**, 123 (1967).

tion laws has been expressed in two theorems by Noether,[2] referring to the invariance under a finite and an infinite continuous group, respectively. In the present context we are only concerned with the first case of a group G_p depending on p parameters. Furthermore, we shall restrict ourselves to the consideration of an integral

$$I=\int L(t,x^i,v^i)dt\,,\quad i=1\cdots N\,,\quad v^i\equiv\frac{dx^i}{dt}\,,\tag{1}$$

extended over a one-dimensional region only, and not containing higher derivatives of the functions $x^i(t)$ than the first. In the following, t is always interpreted as the time. As usual, we introduce the Lagrangian derivatives

$$L_i\equiv\frac{\partial L}{\partial x^i}-\frac{d}{dt}\frac{\partial L}{\partial v^i}.\tag{2}$$

We now consider the infinitesimal transformations

$$\begin{aligned}t'&=t+\delta t\,,\\ x'^i&=x^i+\delta x^i\,,\end{aligned}\tag{3}$$

where δt, δx^i are functions of t, x^i, v^i depending linearly on the infinitesimal parameters of G_p.

Then Noether's first theorem states[4,21]: If I is invariant under the infinitesimal transformations of G_p up to a divergence, there exist precisely p linearly independent combinations of the Lagrangian derivatives which are divergences. Conversely, if p such combinations are divergences, there exists a set of p linearly independent infinitesimal transformations which leave I invariant up to a divergence. These transformations generate a G_p provided the δx^i depend on the v^i at most linearly, and on no derivatives of the v^i.

Since we are concerned with the one-dimensional case only, "divergence" here is equivalent to a total time derivative. In particular, then, if

$$L'dt'=Ldt+\frac{dC}{dt}dt\,,\tag{4}$$

where L' is the *same* function of the transformed variables (3) as L is of the untransformed ones, we have

$$\frac{d}{dt}\left[\left(\frac{\partial L}{\partial v^i}v^i-L\right)\delta t-\frac{\partial L}{\partial v^i}\delta x^i+C\right]=L_i(\delta x^i-v^i\delta t)\,,\tag{5}$$

where summation over i is understood.

It should be noted that the statement of the theorem concerns the expressions (5), which are a direct consequence of the invariance (up to a divergence) of (1) under (3), and *not* any conservation laws; such laws only follow if the Euler-Lagrange equations

$$L_i=0\tag{6}$$

are satisfied. This, from (5), indeed leads to conservation laws. However, Noether's theorem does *not* claim to make a statement on the total number of conservation laws following from invariance under G_p, but (as explicitly noted by Noether herself) only on the number of linear combinations of the L_i which become divergences. This distinction has been almost universally overlooked in the literature of physics, which has led to numerous misunderstandings.

In the usual statement of Noether's theorem for an integral of the form (1), the infinitesimal transformations are taken to be functions of the variables at time t only. However, nothing in the derivation of the theorem has to be changed if the infinitesimal transformations are functionals, i.e., involve integrals over the time,[22] and thus do not generate a G_p in general, as long as these transformations are linearly independent. The use of such transformations will lead to expressions (5) which involve these same functionals. In general, one might expect that this will also be the case for the conservation laws following from (5) by imposition of the Euler-Lagrange equations (6). However, situations may arise in which the use of (6) will reduce the functionals to functions of a single time, and thus lead to conservation laws of the usual type. This is indeed the case for the approximately relativistic equations considered in Sec. V.

III. GALILEI GROUP AND NEWTONIAN CONSERVATION LAWS

We now consider the case of Newtonian point mechanics in the Lagrangian form.[23] We restrict ourselves to Cartesian coordinates and denote the position vector of the kth particle by $\mathbf{r}_k$, with components $x_k{}^i$, and we now use $x_k{}^i$ ($i=1, 2, 3$, $k=1\cdots n$) in place of the N quantities x^i. Then

$$\begin{aligned}L&=T-V\,,\\ T&=\tfrac{1}{2}\sum_{k=1}^{n}m_k\mathbf{v}_k{}^2\,,\quad \mathbf{v}_k=\frac{d\mathbf{r}_k}{dt}\,,\\ V&=\tfrac{1}{2}\sum_{i\neq k}\sum V_{ik}(r_{ik})\,,\quad r_{ik}=|\mathbf{r}_i-\mathbf{r}_k|\,.\end{aligned}\tag{7}$$

The infinitesimal transformations of the Galilei group are of the form (3), and the customary choice of

[21] The derivation and statement of this theorem in Ref. 4 are entirely correct; however, Eq. (4) is stated incorrectly and Eq. (5) incompletely. For the correct forms see Refs. 3, which unfortunately do not give a complete statement of the theorem.

[22] Transformations involving functionals and the related conservation laws have been considered independently in a different context by J. N. Goldberg and E. T. Newman, J. Math. Phys. **10**, 369 (1969).

[23] For the specific case of Newton's law of gravitation, the method and results of this section are due to Bessel-Hagen (Ref. 4) and can also be found in Hill (Ref. 3). The generalization to arbitrary potentials of the form (7) is trivial.

ten independent transformations is

$$\delta t=\tau\,,\quad \delta x_k{}^i=0\,, \tag{8}$$

$$\delta t=0\,,\quad \delta x_k{}^i=\alpha^i\,, \tag{9}$$

$$\delta t=0\,,\quad \delta x_k{}^i=\beta_l{}^i x_k{}^l\,,\quad (\beta_l{}^i=-\beta_i{}^l)\,, \tag{10}$$

$$\delta t=0\,,\quad \delta x_k{}^i=\gamma^i t\,, \tag{11}$$

where τ, α^i, β^i, γ^i are ten independent infinitesimal parameters. The transformations (8)–(10) clearly leave L invariant, whereas for (11) we obtain

$$L'dt'=Ldt+\frac{dC}{dt}dt\,,\quad C=\sum_{i,k} m_k\gamma^i x_k{}^i. \tag{12}$$

Using Eq. (5) together with (6), we get the conservation laws corresponding to (8)–(11)

$$\frac{dE}{dt}=0\,,\quad E\equiv\sum_k \mathbf{p}_k\cdot\mathbf{v}_k-L\,,\quad p_k{}^i\equiv\frac{\partial L}{\partial v_k{}^i}, \tag{13}$$

$$\frac{d\mathbf{P}}{dt}=0\,,\quad \mathbf{P}\equiv\sum_k \mathbf{p}_k\,, \tag{14}$$

$$\frac{d\mathbf{J}}{dt}=0\,,\quad \mathbf{J}\equiv\sum_k \mathbf{r}_k\times\mathbf{p}_k\,, \tag{15}$$

$$\frac{d\mathbf{M}}{dt}=0\,,\quad \mathbf{M}\equiv\sum_k m_k\mathbf{r}_k-\mathbf{P}t. \tag{16}$$

The first three relations of course express the conservation of energy and of linear and angular momentum. Equation (16) can be rewritten, introducing the (constant) total mass M and the c.m. coordinate $\mathbf{R}$,

$$\frac{d\mathbf{M}}{dt}=0\,,\quad \mathbf{M}=M\mathbf{R}-\mathbf{P}t\,,$$
$$M\mathbf{R}\equiv\sum_k m_k\mathbf{r}_k\,,\quad M\equiv\sum_k m_k\,, \tag{17}$$

or, using (14),

$$\frac{d\mathbf{R}}{dt}\equiv\mathbf{V}=\frac{\mathbf{P}}{M}=\text{const}\,, \tag{18}$$

which is the familiar form of the c.m. theorem. However, it should be noted that it is the form (16) which is a consequence of Noether's theorem, and the forms (17) and (18) have been arrived at by using the *definitions* for M and $\mathbf{R}$. Other definitions would have led to other, equally valid, forms; in particular, M and thus $\mathbf{V}$ could differ from the values indicated by constants.

IV. CENTER-OF-MASS THEOREM FOR GENERAL VELOCITY-DEPENDENT LAGRANGIANS

We now consider instead of (7) a general Lagrangian

$$L=L(x_i{}^l-x_k{}^l,\, v_k{}^l)\,. \tag{19}$$

From its form it is clear that it is invariant under the transformations (8) and (9), and thus we still have the laws of conservation of energy and linear momentum (13) and (14). To maintain the law of conservation of angular momentum (15) we would have to require that $x_i{}^l-x_k{}^l$ and $v_k{}^l$ only enter (19) in a form invariant under (10), i.e., only in combinations involving scalars (or pseudoscalars) constructed from $\mathbf{r}_i-\mathbf{r}_k$ and $\mathbf{v}_k$. While this will be the case for all the examples to be considered, this requirement is not needed for the derivation of the results of this section.

Instead of the Galilei transformations (11) we now consider a 3-parameter set of infinitesimal transformations, parametrized by $\boldsymbol{\varepsilon}$,

$$\delta t=\frac{\boldsymbol{\varepsilon}}{c^2}\cdot\mathbf{R}\,,\quad \delta\mathbf{r}_k=\boldsymbol{\varepsilon}t\,,\quad \mathbf{R}(t)\equiv\int^t \mathbf{V}[\mathbf{r}_k(\bar{t})]d\bar{t}\,, \tag{20}$$

where $\mathbf{V}$ is a velocity vector which depends on the time *only through the particle variables* in a manner to be determined; the position vector $\mathbf{R}$ then is a function or a functional of these variables depending on whether $\mathbf{V}d\bar{t}$ is a perfect differential or not. The constant velocity c will be taken to be the velocity of light in the later examples, but regardless of this interpretation it is needed for dimensional reasons if $\mathbf{R}$ is to represent a position.

Clearly $x_i{}^l(t)-x_k{}^l(t)$ is unchanged under the transformation (20). For the transformed velocities we have

$$v_k{}'^l\equiv\frac{dx_k{}'^l}{dt'}=\frac{dx_k{}'^l}{dt}\frac{dt}{dt'}=\frac{d(x_k{}^l+\epsilon^l t)}{dt}\left(1+\frac{\boldsymbol{\varepsilon}}{c^2}\cdot\mathbf{V}\right)^{-1}$$
$$=v_k{}^l+\epsilon^l-v_k{}^l\frac{\boldsymbol{\varepsilon}}{c^2}\cdot\mathbf{V} \tag{21}$$

to first order in $\boldsymbol{\varepsilon}$.

In addition, we must take into account the change in the "volume element" dt:

$$dt'=dt\left(1+\frac{\boldsymbol{\varepsilon}}{c^2}\cdot\mathbf{V}\right). \tag{22}$$

Because of the infinitesimal character of $\boldsymbol{\varepsilon}$ the changes due to (21) and to (22) are additive, i.e.,

$$\delta L\equiv L'dt'-Ldt=(L'-L)dt+L\frac{\boldsymbol{\varepsilon}}{c^2}\cdot\mathbf{V}dt. \tag{23}$$

Furthermore, L' differs from L only in the terms due to (21) which arise from the terms in L containing $\mathbf{v}_k$, and which involve $\boldsymbol{\varepsilon}$ only linearly, since it is infinitesimal; thus

$$L'-L=\sum_k\frac{\partial L}{\partial v_k{}^l}\left(\epsilon^l-v_k{}^l\frac{\boldsymbol{\varepsilon}}{c^2}\cdot\mathbf{V}\right). \tag{24}$$

Inserting this into (23) we get

$$\delta L=\left[\left(L-\sum_k\frac{\partial L}{\partial v_k{}^l}v_k{}^l\right)\frac{\boldsymbol{\varepsilon}}{c^2}\cdot\mathbf{V}+\sum_k\frac{\partial L}{\partial v_k{}^i}\epsilon^i\right]dt. \tag{25}$$

If we wish the transformation (20) to leave the integral I invariant, this expression must vanish, and because the three components of $\boldsymbol{\varepsilon}$ are independent, this must be the case for each factor of ϵ^i separately, or

$$\frac{1}{c^2}\left[L-\sum_k \frac{\partial L}{\partial v_k{}^l}v_k{}^l\right]V^i+\sum_k \frac{\partial L}{\partial v_k{}^i}=0. \qquad (26)$$

Because of the *definitions* (not the constancy) of E and $\mathbf{P}$ given in Eqs. (13) and (14), this can be written

$$\frac{E}{c^2}\mathbf{V}-\mathbf{P}=0. \qquad (27)$$

We recognize this as being of the form (17) of the c.m. theorem (with M replaced by E/c^2); however, it is not a theorem, but a *condition* determining the transformation (20) in such a way as to make δL vanish. Although in general we cannot integrate it explicitly we can still define $\mathbf{R}$ as

$$\mathbf{R}(t)=\int^t \frac{\mathbf{P}[\mathbf{r}_k(\bar{t})]c^2}{E[\mathbf{r}_k(\bar{t})]}d\bar{t}, \qquad (28)$$

which is to be evaluated along an arbitrary path in configuration space. However, no use need be made of this expression at any stage. As noted in Sec. II, Eq. (5) still holds for such a functional.

To obtain the c.m. *theorem*, we substitute (20) into (5) and get, using the definitions (13) and (14) again,

$$\frac{d\mathbf{M}}{dt}=0, \quad \mathbf{M}=\frac{E}{c^2}\mathbf{R}-\mathbf{P}t, \qquad (29)$$

which because of the *constancy* of E and $\mathbf{P}$ expressed in (13) and (14) is equivalent to (27). However, the conservation law (29), unlike the condition (27), depends on the Euler-Lagrange equations of motion (6), and thus Eq. (28) must be integrated along a path allowed by these equations. Treating both $\mathbf{P}$ and E as constant, we of course simply obtain (29), but do not obtain an expression for $\mathbf{R}$ in terms of the particle variables. To get such an expression, we must keep $\mathbf{P}$ or E or both in the form (13) and (14) under the integral in (28) and use (6) in the integration only.

In all the applications considered in this paper, it is sufficient, though not necessary, to treat only E as constant and to evaluate

$$\mathbf{R}(t)=\frac{c^2}{E}\int^t \mathbf{P}[\mathbf{r}_k(\bar{t})]d\bar{t} \qquad (30)$$

with the help of the equations of motion.

It might appear strange that we obtained the c.m. theorem in this section from an exact invariance of I under the transformation (20), whereas in the preceding section we obtained the corresponding theorem (16) from an invariance up to a divergence [Eq. (12)] under the transformation (11). However, we can obtain Eqs. (17) or (18) by the method of this section by simply replacing the Lagrangian (7) by $L-\sum_k m_kc^2$. This does not change either the equations of motion or any of the theorems (13)–(16); it only changes the *explicit* form of E in (13) by the additional constant term $\sum_k m_kc^2$ (irrelevant in the Newtonian context), and the definition of M to $\sum_k m_k+E_N/c^2$, where E_N is the Newtonian energy. Nevertheless, clearly this procedure is artificial for Lagrangians which are Galilei-invariant (up to a divergence); our procedure, though formally general, is natural and convenient only in the context of approximately relativistic equations such as those treated in the next section. We shall return to this point in Sec. V A.

We also note that formally we could have maintained a closer analogy with the procedure of the preceding section by maintaining the transformations (11) and defining the expression C as a functional rather than a function; indeed this would be the functional $\boldsymbol{\varepsilon}\cdot\int\mathbf{P}[\mathbf{r}(\bar{t})]d\bar{t}$, i.e., the expression (30), apart from a constant factor. Then by Eq. (15) we would arrive at a conservation law involving this functional, which reduces to a function if Eq. (30) is integrable by use of the equations of motion. Although this procedure formally maintains all ten Galilei transformations, it appears less natural than the one used in this section, which leads to a c.m. velocity determined by Eq. (27) whether or not Eq. (30) is integrable. Furthermore, it is also less general, since it does not yield the form (28).

The essential point of our considerations is that, as noted in Sec. III, the usual transformation (11) leads to a conservation law (16) expressed in terms of the individual coordinates of the particles and their derivatives; the c.m. theorem (17) is arrived at by *defining* the c.m. in terms of these coordinates. Our transformation (20), in contrast to this, *directly* yields a conservation law (29) involving a "c.m." coordinate; the expression for this quantity in terms of the individual coordinates and their derivatives requires an additional integration (28) or (30).

It should also be noted that in going from the form (16) to the c.m. theorem (17) or (18), use had to be made of the law of conservation of mass, which has no connection with any invariance properties of the Lagrangian. The form (29), on the other hand, is equivalent to (27) by virtue of the law of conservation of energy, which does follow from an invariance property of the Lagrangian.

V. APPLICATIONS

In this section, we first discuss the case of noninteracting particles in subsection A, a case which can be treated exactly. We then expand the results obtained in powers of $(v/c)^2$ and briefly discuss the problem of systematic approximations. The other subsections treat cases of interacting particles to order $(v/c)^2$, for special-relativistic equations in subsections B and C, for general-relativistic ones in subsections D and E. Included are all cases of nonrotating interacting particles whose approximate equations of motion have been discussed in the literature. The c.m. theorem is found to have the same form in all cases, as discussed at the end of this section.

A. Noninteracting Particles

We first consider the exact special relativistic Lagrangian appropriate for the form (1) for n free particles with rest masses m_k,

$$L=-\sum_k m_k c^2[1-(v_k/c)^2]^{1/2}, \qquad (31)$$

where $v_k\equiv|\mathbf{v}_k|$. This is clearly invariant under the transformations (8)–(10), implying the conservation laws (13)–(15). The transformation (20) leads to the conservation law (29), where,

$$\mathbf{P}=\sum_k m_k\mathbf{v}_k/[1-(v_k/c)^2]^{1/2}, \qquad (32)$$

$$E=\sum_k m_k c^2/[1-(v_k/c)^2]^{1/2}. \qquad (33)$$

The equations of motion (6) imply

$$d\mathbf{v}_k/dt=0; \qquad (34)$$

thus Eq. (28) or (30) can readily be integrated to give

$$\mathbf{R}=\frac{\sum_k m_k\mathbf{r}_k/[1-(v_k/c)^2]^{1/2}}{\sum_k m_k/[1-(v_k/c)^2]^{1/2}}, \qquad (35)$$

which is the obvious relativistic generalization of Eq. (17), with all Newtonian masses replaced by the relativistic ones. For a single particle, $\mathbf{R}$ coincides with the position $\mathbf{r}$, and $d\mathbf{R}/dt$ with $\mathbf{v}$, as in the Newtonian case. While these properties are physically desirable, we note that just as in the Newtonian case, there is an element of arbitrariness in the form (35); it would be different if L and thus E were changed by the addition of an arbitrary constant, which of course would not influence the equations of motion (6).

It is instructive for our later examples to consider the approximate Lagrangians obtained by expanding (31) in powers of $(v/c)^2$ and keeping terms up to order l. We then have

$$L_l=-c^2\sum_k m_k\left\{1+\sum_{j=1}^{l}(-1)^j\binom{\frac{1}{2}}{j}\left(\frac{v_k}{c}\right)^{2j}\right\}, \qquad (36)$$

where $\binom{i}{j}$ are the usual binomial coefficients, and thus

$$\mathbf{P}_l=\sum_k m_k\sum_{j=1}^{l}2j(-1)^{j+1}\binom{\frac{1}{2}}{j}\left(\frac{v_k}{c}\right)^{2(j-1)}\mathbf{v}_k \qquad (37)$$

and

$$E_l=c^2\sum_k\sum_{j=0}^{l}m_k(2j-1)(-1)^{j+1}\binom{\frac{1}{2}}{j}\left(\frac{v_k}{c}\right)^{2j}. \qquad (38)$$

Equation (37) can be rewritten, using the relation between binomial coefficients, as

$$\mathbf{P}_l=\sum_k m_k\sum_{j=0}^{l-1}(2j-1)(-1)^{j+1}\binom{\frac{1}{2}}{j}\left(\frac{v_k}{c}\right)^{2j}\mathbf{v}_k. \qquad (37')$$

Because Eq. (34) still holds, Eq. (30) can be integrated as before to yield

$$\mathbf{R}_l=\frac{\sum_k m_k\sum_{j=0}^{l-1}(2j-1)(-1)^{j+1}\binom{\frac{1}{2}}{j}\left(\frac{v_k}{c}\right)^{2j}\mathbf{r}_k}{\sum_k m_k\sum_{j=0}^{l}(2j-1)(-1)^{j+1}\binom{\frac{1}{2}}{j}\left(\frac{v_k}{c}\right)^{2j}}. \qquad (39)$$

It should be noted that while this expression is an *exact* consequence of the invariance of L under the transformation (20), the numerator and the denominator are of different orders in $(v/c)^2$; thus it would be more in the spirit of an approximation procedure to remedy this by terminating the series in the denominator at $j=l-1$. This would have the advantage that $\mathbf{R}_1$ would agree with the definition (17) and that $\mathbf{R}_l$ would represent the generalization to order l of that definition, with all Newtonian masses replaced by the approximate relativistic masses to that order, and that in the case of a single particle $\mathbf{R}$ and $\mathbf{V}$ would then coincide with $\mathbf{r}$ and $\mathbf{v}$, in agreement with the exact result just discussed.

Precisely the same difference in powers of c will arise in all further examples, and has already been encountered at the end of Sec. IV. It always arises, although the expressions for $\mathbf{P}$ and E entering the conservation law (29) [as obtained from the conservation laws (13) and (14)] are of the *same* order in $(v/c)^2$, because it is not E itself, but the total mass E/c^2, which appears in (29) and in the expression (30) for the c.m. coordinate $\mathbf{R}$. Thus, if we terminate the series in E/c^2 at $j=l-1$, we will in all subsequent examples *with* interactions regain the Newtonian result (17) in lowest order ($l=1$), as we should, *without* the need for a redefinition of M noted in the previous section. This indicates that the method outlined there, while perfectly general, is particularly appropriate within the framework of an approximation procedure.

We also note that because (39) and similar expressions in the examples below are approximations, it would be appropriate to replace the ratio of the two power series by a single such series. However, in addition to being cumbersome, the resulting expression would be far less instructive, and therefore we shall not carry out the division.

B. Electromagnetic Interactions

We now consider the case of n particles of masses m_k and electric charges e_k. If their interaction is half-retarded plus half-advanced, the exact equations of motion can be derived either from a manifestly Lorentz-invariant variational principle,[5] or, as shown by Hargreaves,[12] from one which involves an integral over a single time just as for Newtonian mechanics. If only terms up to $(v/c)^2$ are kept in the latter variational principle, it reduces to that obtained later independently by Darwin.[13] To this order the equations are the same whether we consider time-symmetric interactions, or purely retarded or purely advanced ones, as radiation terms are of higher order. We then have

$$L_D=L_2-V_C+I_D\,,\quad V_C\equiv\frac{1}{2}\sum_{i\neq k}\sum\frac{e_ie_k}{r_{ik}},\tag{40}$$

where L_2 is given by (36) with $l=2$, and

$$I_D=\frac{1}{4c^2}\sum_{i\neq k}\sum\frac{e_ie_k}{r_{ik}}\left[\mathbf{v}_i\cdot\mathbf{v}_k+\frac{(\mathbf{v}_i\cdot\mathbf{r}_{ik})(\mathbf{v}_k\cdot\mathbf{r}_{ik})}{r_{ik}^2}\right].\tag{41}$$

From the invariance of L_D under (8) and (9) we get

$$\mathbf{P}=\mathbf{P}_2+\frac{1}{2c^2}\sum_{i\neq k}\sum\frac{e_ie_k}{r_{ik}}\left[\mathbf{v}_k+\frac{(\mathbf{v}_i\cdot\mathbf{r}_{ik})\mathbf{r}_{ik}}{r_{ik}^2}\right]\tag{42}$$

and

$$E=E_2+V_C+I_D\,,\tag{43}$$

where $\mathbf{P}_2$ and E_2 are given by (37) and (38) with $l=2$. The equation of motion are, from (2) and (6),

$$\frac{d\mathbf{p}_k}{dt}=-e_k\nabla_k\sum_{i\neq k}\frac{e_i}{r_{ik}}\left\{1-\frac{1}{2c^2}\left[\mathbf{v}_i\cdot\mathbf{v}_k+\frac{(\mathbf{v}_i\cdot\mathbf{r}_{ik})(\mathbf{v}_k\cdot\mathbf{r}_{ik})}{r_{ik}^2}\right]\right\},$$

$$\mathbf{p}_k\equiv m_k\left[1+\tfrac{1}{2}\left(\frac{v_k}{c}\right)^2\right]\mathbf{v}_k+\frac{1}{2c^2}\sum_{i\neq k}\frac{e_ie_k}{r_{ik}}\left[\mathbf{v}_i+\frac{(\mathbf{v}_i\cdot\mathbf{r}_{ik})\mathbf{r}_{ik}}{r_{ik}^2}\right].\tag{44}$$

We now have to apply Eq. (30) to Eq. (42) to obtain **R**. Unlike the case of noninteracting particles, the expression for **P** cannot be readily recognized as a total derivative; however, we can easily verify that **R** is given by[24]

$$\frac{E}{c^2}\mathbf{R}=\sum_k\left\{m_k\left[1+\tfrac{1}{2}\left(\frac{v_k}{c}\right)^2\right]+\frac{1}{2c^2}\sum_{i\neq k}\frac{e_ie_k}{r_{ik}}\right\}\mathbf{r}_k.\tag{45}$$

Differentiating this expression with respect to t we obtain

$$\sum_k\left\{m_k\left[1+\tfrac{1}{2}\left(\frac{v_k}{c}\right)^2\right]+\frac{1}{2c^2}\sum_{i\neq k}\frac{e_ie_k}{r_{ik}}\right\}\mathbf{v}_k$$
$$+\frac{1}{c^2}\sum_k\left[m_k\mathbf{v}_k\cdot\frac{d\mathbf{v}_k}{dt}-\tfrac{1}{2}\sum_{i\neq k}\frac{e_ie_k}{r_{ik}^3}\mathbf{r}_{ik}\cdot(\mathbf{v}_i-\mathbf{v}_k)\right]\mathbf{r}_k.$$

The second of these sums can be rewritten, using Eq. (44) for $m_k\,d\mathbf{v}_k/dt$. Since we need retain only terms of order $(v/c)^2$, it is sufficient to insert the Newtonian approximtion of Eq. (44). We then get

$$-\frac{1}{2c^2}\sum_{i\neq k}\sum\frac{e_ie_k}{r_{ik}^3}[2\mathbf{v}_k\cdot\mathbf{r}_{ik}+(\mathbf{v}_i-\mathbf{v}_k)\cdot\mathbf{r}_{ik}]\mathbf{r}_k\,,$$

or

$$-\frac{1}{2c^2}\sum_{i\neq k}\sum\frac{e_ie_k}{r_{ik}^3}[(\mathbf{v}_i+\mathbf{v}_k)\cdot\mathbf{r}_{ik}]\mathbf{r}_k\,,$$

which because of the double summation and the definition of $\mathbf{r}_{ik}$ can be rewritten

$$-\frac{1}{4c^2}\sum_{i\neq k}\sum\frac{e_ie_k}{r_{ik}^3}\{[(\mathbf{v}_i+\mathbf{v}_k)\cdot\mathbf{r}_{ik}]\mathbf{r}_k+[(\mathbf{v}_i+\mathbf{v}_k)\cdot\mathbf{r}_{ki}]\mathbf{r}_i\}$$
$$=-\frac{1}{4c^2}\sum_{i\neq k}\sum\frac{e_ie_k}{r_{ik}^3}[(\mathbf{v}_i+\mathbf{v}_k)\cdot\mathbf{r}_{ik}]\mathbf{r}_{ki}$$
$$=\frac{1}{2c^2}\sum_{i\neq k}\sum\frac{e_ie_k}{r_{ik}^3}(\mathbf{v}_i\cdot\mathbf{r}_{ik})\mathbf{r}_{ik}.$$

Thus indeed we obtain Eq. (42) and the expression (45) is correct to the order required.

It should be noted that while in the presence of interactions the contribution of the potential energy to the *total* energy (43) is to be expected, the fact that the equivalent contribution to the total mass can be split into contributions localized at the positions $\mathbf{r}_k$ (which can be interpreted as due to the potential energy of the kth particle in the field of all others) as indicated in (45) could not be readily anticipated. However, we shall encounter this feature in all other cases as well, as discussed at the end of this section.

C. Other Special Relativistic Interactions

Some time ago Bagge[25] considered the problem of finding a Lagrangian for N particles correct to order $(v/c)^2$ which would allow arbitrary two-body interactions and would take account of the binding energy of the system. With some simplifications and a slight change in notation his Lagrangian can be written

$$L_B=L_2-V+I_B\,,\quad V\equiv\tfrac{1}{2}\sum_{i\neq k}\sum V_{ik}(r_{ik})$$
$$I_B=\frac{1}{4c^2}\sum_{i\neq k}\sum\left[V_{ik}\mathbf{v}_i\cdot\mathbf{v}_k-\frac{(\mathbf{v}_i\cdot\mathbf{r}_{ik})(\mathbf{v}_k\cdot\mathbf{r}_{ik})}{r_{ik}}\frac{dV_{ik}}{dr_{ik}}\right],\tag{46}$$

where V_{ik} is an arbitrary function of its argument. For special choices of V_{ik} this Lagrangian reduces to L_D or to the Lagrangian appropriate for a vector meson field; in general it gives the interactions to order $(v/c)^2$ following from a field theory described by a four-vector,

[24] For a different derivation of Eq. (45) see L. Landau and E. Lifshitz, *Classical Theory of Fields* (Addison-Wesley Publishing Co., Reading, Mass., 1962), 2nd English ed., p. 194. We shall discuss the method used in Sec. VI.

[25] E. Bagge, Z. Naturforsch. **1**, 361 (1946).

as noted by Bopp (according to Bagge).[25] The transformations (8) and (9) lead to

$$\mathbf{P}=\mathbf{P}_2+\frac{1}{2c^2}\sum_{i\neq k}\sum\left[V_{ik}\mathbf{v}_k-\frac{dV_{ik}}{dr_{ik}}\frac{(\mathbf{v}_i\cdot\mathbf{r}_{ik})\mathbf{r}_{ik}}{r_{ik}}\right] \quad (47)$$

and

$$E=E_2+V+I_B. \quad (48)$$

The equations of motion follow from (2) and (6) as before, but as we shall only require the Newtonian approximation, we shall not give them here. In analogy to Eq. (45) we now define

$$\frac{E}{c^2}\mathbf{R}=\sum_k\left\{m_k\left[1+\left(\frac{v_k}{c}\right)^2\right]+\frac{1}{2c^2}\sum_{i\neq k}V_{ik}\right\}\mathbf{r}_k. \quad (49)$$

To verify that this expression satisfies Eq. (30) we differentiate it with respect to t, and as in Subsec. B obtain two sums; the first of these contains V_{ik} instead of e_ie_k/r_{ik} and the second one, after insertion of the Newtonian approximation for $d\mathbf{v}_k/dt$, becomes a sum of terms proportional to $r_{ik}^{-1}\,dV_{ik}/dr_{ik}$ rather than to $-e_ie_k/r_{ik}^3$. Otherwise, the calculations are identical, and lead to the required Eq. (47).

For interactions to order $(v/c)^2$ following from a field theory described by a scalar rather than a four-vector the Lagrangian corresponding to Bopp's is

$$L_S=L_2-V+I_S, \quad (50)$$

$$I_S=\frac{1}{4c^2}\sum_{i\neq k}\sum\left\{[v_i^2+v_k^2-\mathbf{v}_i\cdot\mathbf{v}_k]V_{ik}-\frac{(\mathbf{v}_i\cdot\mathbf{r}_{ik})(\mathbf{v}_k\cdot\mathbf{r}_{ik})}{r_{ik}}\frac{dV_{ik}}{dr_{ik}}\right\} \quad (51)$$

with V and V_{ik} as above, as will be shown elsewhere. The transformation (8) and (9) leads to the same momentum (47) as before, and to

$$E=E_2+V+I_S; \quad (52)$$

thus we obtain the same expression (49) for the c.m. as before.

D. Gravitational Interactions

In the general theory of relativity, the equations of motion are not independent of the field equations. The exact form of the equations of motion for a system of particles is not known, but we can only proceed by approximation methods. Because we are concerned here with equations resembling the Newtonian ones, we shall only consider the slow-motion approximation for monopole particles, which leads to the well-known EIH equations in the first post-Newtonian order in v/c.[14,15,26] These can be derived from a variational principle[15,16]; we shall use a Lagrangian of the form[27]

$$L_G=L_2-V_G+I_G,$$

$$V_G=-\tfrac{1}{2}\sum_{i\neq k}\sum\frac{Gm_im_k}{r_{ik}}\left(1-\frac{G}{2c^2}\sum_{j\neq k}\frac{m_j}{r_{jk}}\right), \quad (53)$$

$$I_G=\frac{1}{4c^2}\sum_{i\neq k}\sum\frac{Gm_im_k}{r_{ik}}\times\left\{6v_k^2-\left[7\mathbf{v}_i\cdot\mathbf{v}_k+\frac{(\mathbf{v}_i\cdot\mathbf{r}_{ik})(\mathbf{v}_k\cdot\mathbf{r}_{ik})}{r_{ik}^2}\right]\right\}, \quad (54)$$

where G is the constant of gravitation.

From the invariance under (8) and (9) we get

$$\mathbf{P}=\mathbf{P}_2-\frac{G}{2c^2}\sum_{i\neq k}\sum\frac{m_im_k}{r_{ik}}\left\{\mathbf{v}_k+\frac{(\mathbf{v}_i\cdot\mathbf{r}_{ik})\mathbf{r}_{ik}}{r_{ik}^2}\right\} \quad (55)$$

and

$$E=E_2+V_G+I_G. \quad (56)$$

We note that, apart from the different coupling constants, the structure (55) of $\mathbf{P}$ is the same as in the electromagnetic case (42). This suggests that we can take instead of (45)

$$\frac{E}{c^2}\mathbf{R}=\sum_k\left\{m_k\left[1+\tfrac{1}{2}\left(\frac{v_k}{c}\right)^2\right]-\frac{G}{2c^2}\sum_{i\neq k}\frac{m_im_k}{r_{ik}}\right\}\mathbf{r}_k. \quad (57)$$

Apart from the different coupling constants, the proof that this is indeed the correct expression is identical with that presented in Subsec. B. The conservation law (29) with (57) was obtained first by direct integration of the EIH equations by Fichtenholz, taking E/c^2 initially[19] as $\sum_k m_k$, and later[16] as

$$\sum_k\left\{m_k\left[1+\tfrac{1}{2}\left(\frac{v_k}{c}\right)^2\right]-\frac{G}{2c^2}\sum_{i\neq k}\frac{m_im_k}{r_{ik}}\right\}, \quad (58)$$

i.e., the expression following from Eq. (53) if terms of order $(v/c)^4$ are omitted, in agreement with our discussion in Sec V A. Since both these expressions are constant it follows from our considerations in Secs. III and V A that both are correct mathematically, although they imply slightly different velocities $\mathbf{V}$ of the c.m., and there are physical reasons to prefer the form (58).

We also note that the expansion of the equations of motion of the Lorentz-invariant approximation

[26] The history of the slow-motion approximation is very complex, and will be described elsewhere by one of us (P.H.). We only note here that whereas the correct form of the EIH equations was first given in Ref. 14, and the corresponding Lagrangian in Ref. 16, the equations (except for an insignificant mistake) were already obtained by W. de Sitter, M. N. R. A. S. **77**, 155 (1916), and the correct Lagrangian for N nonrotating droplets (from which both the particle Lagrangian and the EIH equations follow trivially) by H. A. Lorentz and J. Droste, Verslag. K. Akad. Wetensch. Amsterdam **26**, 392 (1917).

[27] We use the form given originally by Fichtenholz (Ref. 16), rather than that given in Ref. 15. The same form can also be found in Landau and Lifshitz, Ref. 24, § 105 (the Lagrangian given in the 1948 Russian edition in incorrect, as noted in Ref. 16, as is the one given in the 1st English edition, 1951).

method[7,28] in powers of (v/c) [29] leads to the same conservation law (29) with (57) and (58) as the EIH equations to the order considered here, as will be shown elsewhere.

E. Combined Gravitational and Electromagnetic Interactions

The general relativistic equations of motion of charged particles were obtained by Bażański[30] on the basis of the slow-motion approximation.[31] His Lagrangian[32] can be written

$$L_{GE}=L_2-V_G-V_C-V_{GE}+I_D+I_G, \qquad (59)$$

$$V_{GE}=\frac{G}{2c^2}\sum_{i\neq k}\sum\left[\frac{(2e_ie_k+e_i^2)m_k}{r_{ik}^2}+\sum_{j\neq i,k}e_ie_km_j\times\left(\frac{1}{r_{ij}r_{ik}}+\frac{1}{r_{ik}r_{jk}}-\frac{1}{r_{ij}r_{jk}}\right)\right]. \qquad (60)$$

We therefore have

$$\mathbf{P}=\mathbf{P}_2+\frac{1}{2c^2}\sum_{i\neq k}\sum\frac{e_ie_k-Gm_im_k}{r_{ik}}\left[\mathbf{v}_k+\frac{(\mathbf{v}_i\cdot\mathbf{r}_{ik})\mathbf{r}_{ik}}{r_{ik}^2}\right] \qquad (61)$$

and

$$E=E_2+V_G+V_C+V_{GE}+I_D+I_G. \qquad (62)$$

In complete analogy to the cases treated earlier, we can take

$$\frac{E}{c^2}\mathbf{R}=\sum_k\left\{m_k\left[1+\tfrac{1}{2}\left(\frac{v_k}{c}\right)^2\right]+\frac{1}{2c^2}\sum_{i\neq k}\frac{e_ie_k-Gm_im_k}{r_{ik}}\right\}\mathbf{r}_k; \qquad (63)$$

the proof that the time derivative of this quantity equals $\mathbf{P}$ is identical to the one presented in Subsec. B.

To summarize the results of this section, all the cases considered were found to lead to a conservation law (29) with c.m. coordinates determined by

$$M\mathbf{R}=\sum_k M_k\mathbf{r}_k,\quad M\equiv\sum M_k, \qquad (64)$$

with

$$M_k=\frac{m_k}{[1-(v_k/c)^2]^{1/2}} \qquad (65)$$

for free particles exactly, and

$$M_k=m_k\left[1+\tfrac{1}{2}\left(\frac{v_k}{c}\right)^2\right]+\frac{1}{c^2}V_k \qquad (66)$$

for interacting particles to order $(v/c)^2$, where V_k is the *Newtonian* potential energy of the kth particle in the field of all others. Equations (64) and (66) are the natural generalizations of the Newtonian result (17) for the gravitational case (59), as was already noted by Fichtenholz[16]; so are Eqs. (64) and (65) for free particles. For the special-relativistic cases considered in Subsecs. B and C, however, no general argument is known which would lead us to expect that we can express $M\mathbf{R}$ as a sum of localized contributions $M_k\mathbf{r}_k$, as noted above, and of course the equations of motion (6) are *not* reducible to those of particles of mass M_k; on the other hand, the form of M simply expresses the equivalence of the total mass and energy of the system.

VI. DISCUSSION

In Sec. IV it was shown that for any velocity-dependent Lagrangian (19) invariant under the infinitesimal time and space translations (8) and (9), we have an additional invariance under a set of infinitesimal 3-parameter transformations (20), depending on a functional $\mathbf{R}(t)$ which can be expressed in the forms (28) or (30). $\mathbf{R}$ is a functional of a velocity $\mathbf{V}$ which by Eq. (27) is related to the total energy and momentum in a way appropriate for the c.m. theorem. These relations as well as the invariance are properties of the Lagrangian whether or not the Euler-Lagrange equations (6) are satisfied; if they are, E, $\mathbf{P}$, and $\mathbf{V}$ are all constant. They are all determined only up to additive constants, as noted in Secs. III–V.

Thus we have a constant velocity $\mathbf{V}$ associated with the system described by the Lagrangian (19) regardless of the particular form of the latter function, and in this sense we always have a c.m. theorem for a system whose Lagrangian only depends on $x_i^l-x_k^l$ and v_k^l. However, to enable us to talk of a c.m. we should be able to associate $\mathbf{V}$ with a particular point, not just with the system as a whole, i.e., the position vector $\mathbf{R}$ defined in (20) should be a *function* of the particle variables rather than a functional, at least when it enters the conservation law (29).

All other infinitesimal transformations (8)–(10) considered here depend on functions rather than functionals, whether or not the equations of motion (6) are satisfied; if they are, the corresponding conservations laws (13)–(15) also only depend on functions. On the other hand, the transformations (20) always depend on functionals; nevertheless the corresponding conservation law (27) always depends on functions, just like all the others. It is only when we wish to define c.m. coordinates $\mathbf{R}$ that we encounter a problem different from the customary one. However, in all the

[28] P. Havas, Phys. Rev. **108**, 1351 (1957).
[29] S. F. Smith and P. Havas, Phys. Rev. **138**, 3495 (1965).
[30] S. Bażański, Acta. Phys. Polon. **15**, 363 (1966).
[31] For earlier slightly incorrect treatments based on these approximations see P. R. Wallace, thesis, University of Toronto (unpublished); Am. J. Math. **63**, 729 (1941); and B. Bertotti, Nuovo Cimento **2**, 231 (1955). The results differ from those of Ref. 30 only in the term I_{GE} given by Eq. (60). The difference is irrelevant for our result (61), and also for (63) if only terms up to order $(v/c)^2$ are kept in E/c^2.
[32] S. Bażański, Acta Phys. Polon. **15**, 423 (1957) and in *Recent Developments in General Relativity* (Pergamon Press, Ltd., London, 1962), p. 137.

examples considered in Sec. V [which include all cases of equations of motion correct to order $(v/c)^2$ considered in the literature] $\mathbf{R}$ can be expressed as a function of the particle variables in the conservation law (29) to that order provided only that the equations of motion (6) are satisfied. It can *not* be so expressed in the transformations (20) [since the Lagrangian must be invariant whether or not (6) holds] but this is irrelevant for the validity (or derivation) of the conservation laws.

As noted in Sec. IV, in all the examples considered it is sufficient to consider the form (30) [rather than the more general one (28)] for $\mathbf{R}$ to be able to obtain an expression depending on the particle variables alone. Indeed, in all cases in Secs. V B–V E involving interactions we are always concerned with the same functional form (47) of the total momentum in our integral (30) in spite of the significant differences in form of the Lagrangians. It is easy to give examples of Lagrangians (19) which lead to total momenta of a form differing from (47) and which do *not* allow expression (30) to be integrated; thus our distinction between the forms (27) and (29) of the conservation law is not trivial. Whether *any* approximately relativistic equations correct to $(v/c)^2$ and derivable from a Lagrangian must lead to a functional form (47) and allow (30) to be integrated, is an open question.

It is remarkable that to the order considered there exists such a formal similarity between the conservation laws of the special relativistic systems considered in Secs. V B and V C and those of the general relativistic ones (Secs. V A and V E), in particular of the law (29) with (64) and (66). This formal similarity masks a fundamental physical difference, however; the special relativistic equations must be interpreted in a flat space, whose metric properties are independent of the system of particles under consideration, whereas the general relativistic ones hold in a curve space whose metric is a function of the particle variables to the corresponding order of approximation.[15,20]

Although the physical reason for the formal similarity is not clear, the mathematical one is easy to trace and might be worth noting. All the approximately relativistic Lagrangians of Sec. V are of the form

$$L=L_2-V+I,\quad V\equiv\sum_{i\neq k}\sum V_{ik}(r_{ik})+V_2(r_{ik}),\quad (67)$$

with

$$I=\frac{1}{4c^2}\sum_{i\neq k}\sum\left\{[A(\mathbf{v}_i-\mathbf{v}_k)^2+\mathbf{v}_i\cdot\mathbf{v}_k]V_{ik}-\frac{(\mathbf{v}_i\cdot\mathbf{r}_{ik})(\mathbf{v}_k\cdot\mathbf{r}_{ik})}{r_{ik}}\frac{dV_{ik}}{dr_{ik}}\right\},\quad (68)$$

where $A=0$ in Eqs. (41) and (46), $A=1$ in Eq. (51), and $A=-3$ in Eq. (60). V_2 is of order $(v/c)^2$ and appears only in the general relativistic Lagrangians (53) and (59). As it does not depend on the velocities, it does not enter into the definitions of $\mathbf{p}_k$ and $\mathbf{P}$; being of higher order, it does not enter the Newtonian equations of motion, which are used in proving the various forms of $\mathbf{R}$. Thus it has no relevance for the c.m. theorem to order $(v/c)^2$. The term proportional to A enters $\mathbf{p}_k$ in the form

$$\frac{A}{2c^2}\sum_{i\neq k}(\mathbf{v}_k-\mathbf{v}_i)V_{ik}\qquad (69)$$

and thus does not contribute to the total momentum $\mathbf{P}$. Furthermore, it too does not enter the Newtonian equations of motion, being of higher order.

The general form (67) with (68) could, of course, have simply been postulated as the Lagrangian appropriate for the description of a system of interacting particles to order $(v/c)^2$. For the values of A indicated above, it does indeed represent an approximation to that order of exact (special or general) relativistic variational principles. Whether this is the case for other values of A, and whether all such approximations must be of this form, has not been established.

The question arises how our 10-parameter set of infinitesimal transformations (8)–(10), (20) is related to the Galilei or Lorentz group. The seven transformations (8)–(10) are common to all three sets. The Galilei group allows the three further transformations (11) which are the same at all points of space and independent of the system under consideration. The corresponding infinitesimal Lorentz transformations are of the form

$$\delta t=\frac{1}{c^2}\gamma^i x_k{}^i,\quad \delta x_k{}^i=\gamma^i t\qquad (70)$$

and thus, while still independent of the system under consideration, are *not* the same at all points of space. On the other hand, our transformations (20) are the same at all points of space just like the Galilei ones, but do depend on the system under consideration through the functional dependence of $\mathbf{V}$ and thus $\mathbf{R}$ on the particle variables implied by Eqs. (26) or (27). However, as noted in Sec. IV, it is possible to modify our procedure to maintain the form (8–11) of the Galilei transformations (though losing some generality); thus to some extent it is at our disposal to choose transformations more resembling one or the other of the groups in question. Indeed, as we shall show in a later paper, it is also possible to put the c.m. conserved vector in a form in which it generates a group of canonical transformations in phase space which coincide with the Lorentz transformations to order $(v/c)^2$.

In our initial attempts to obtain the c.m. theorem by using Noether's theorem we applied the transformations (70) to some of the Lagrangians considered in Sec. V. However, they were found not to be invariant under these transformations. Thanks to Noether's theorem, we only had to consider the infinitesimal transformations of the Lagrangians (19). In a study of the EIH equations, Chandrasekhar and Contopoulos[20] considered the *finite* transformations of the Lagrangian given by Eqs. (50) and (51) to allow transformations to the c.m. frame, i.e., the frame of reference in which the total momentum vanishes. Their rather lengthy calculations lead to a transformation which contains all the terms of the finite Lorentz transformation to order $(v/c)^2$ plus additional ones. For our present considerations it is only necessary to consider the last two sections of their paper. They quote the well-known theorem that the equations of motion have the same form in two sets of variables if the two Lagrangians differ by the total derivative of some function; then they state that this is the case for the two Lagrangians related by their transformation, or even by the truncated Lorentz transformation alone. However, it can be readily seen that in their demonstration that the Lagrangians differ by a total derivative use has to be made of the approximate equations of motion in one set of variables; these are needed even if the transformations are infinitesimal. Noether's theorem, on the other hand, requires that the two Lagrangians differ by a total derivative whether or not the equations of motion are satisfied, as discussed in Sec. II. Thus it seems that the transformation and invariances considered by Chandrasekhar and Contopoulos have no direct bearing on those needed to obtain the conservation laws by the Lagrangian approach used here.[33]

It seems clear, however, that the conservation laws for those Lagrangians, such as that of Darwin (see Sec. V B), which are derivable as approximations from an exact Lorentz-invariant Lagrangian involving interactions between particles and fields could be derived in a different way from invariance considerations. The invariance of this Lagrangian under transformations of the 10-parameter inhomogeneous Lorentz group leads, via the Noether theorem, to the usual ten conserved quantities written in terms of both particle and field variables.[34] Expansion of the field variables in terms of the particle variables to the order in v/c required (omitting infinite self-action terms) would then lead to conservation laws valid to appropriate order when these expansions are inserted into the conservation laws. Although Landau and Lifshitz[24] did not derive the c.m. result for a system of charged particles interacting with the electromagnetic field from invariance considerations, they did carry out the expansion procedure on the rather obvious expression to be expected in order to derive the c.m. result for the Darwin approximation (Sec. V B).

Similarly, one could approach the problem by starting from the exact Lorentz-invariant conservation laws following from "Fokker-type" variational principles,[8] and expanding them in powers of v/c. Furthermore, the interaction terms in the equation of motion for the kth particle following from such principles can be expressed in terms of "adjunct fields" (which are integrals over the entire motion of all but the kth particle); similarly, the conservation laws can be expressed in terms of an "adjunct field theory" as sums over contributions from each particle separately plus those arising from adjunct field tensors.[35,36] These conservation laws are equivalent on the one hand to those written directly in terms of particle variables alone, and on the other to those of the corresponding field theory with sources in which infinite self-action terms have been omitted. Thus it can be expected that both in field theory and in the theory of direct particle interactions we must also arrive at the same approximate conservation laws.

We might contrast the approaches just sketched and those of this paper by saying that in this paper we deal with exact symmetries of approximate Lagrangians, while the other approaches would use approximations to the consequences of symmetries of exact Lagrangians. Some aspects of the alternative approaches are currently being investigated, in part in collaboration with H. Woodcock.

It is possible to generalize the method of this paper, both to Lagrangians (19) describing systems of particles with forces depending on higher orders of v/c (such as the "post-post-Newtonian" equations of motion in general relativity) or on derivatives of the velocities[12,36] and to Lagrangians describing continua.[36,37] For the particular case of the post-Newtonian general relativistic perfect fluid in adiabatic motion, the c.m. theorem (whose form follows from the work of Fock[15]) has been derived from transformations analogous to (20); this work will be described in a paper with T. Pascoe. The form of the c.m. theorem has not been established in any of the other cases mentioned in this paragraph.

[33] The invariance of the conservation law (29) with (57) and (58) under the truncated finite Lorentz transformations was already noted by Fichtenholz (Ref. 16). He also obtained the relations between the right-hand sides of Eq. (57) in two frames of reference connected by these transformations, which agree with those of Ref. 20.

[34] Bessel-Hagen (Ref. 4) derived the ten conservation laws for the Maxwell field from the Lagrangian using Noether's theorem, and wrote them down for the field interacting with ponderable matter, but did not use a particular model for the matter (such as charged particles) and so did not derive the conservation laws for the interacting case from a Lagrangian.

[35] In electrodynamics this was shown first by J. Frenkel, Z. Phys. **32**, 518 (1925) and J. L. Synge, Trans. Roy. Soc. Can. **34**, 1 (1940); an alternate form of the energy-momentum tensor was suggested by J. A. Wheeler and R. P. Feynman, Rev. Mod. Phys. **21**, 425 (1949).

[36] For Lagrangians such as that considered in Ref. 12 describing electromagnetic interactions as infinite series involving the $\mathbf{v}_k$ and their derivatives, no such derivatives appear to order $1/c^2$, and thus to that order the Lagrangians are of the type considered in this paper.

[37] The seven conservation laws corresponding to the transformations (8)–(10) have been derived by Noether's theorem for the post-Newtonian general relativistic perfect fluid in adiabatic motion by T. Pascoe and J. Stachel, Bull. Am. Phys. Soc. **14**, 69 (1969). The results agree with those obtained by S. Chandrasekhar [Astrophys. J. **141**, 1488 (1965)] by direct integration of the equations of motion.

In this paper we have used Noether's theorem to derive the c.m. theorem from the invariance of the Lagrangian under a 3-parameter set of infinitesimal transformations which will not, in general, generate a finite group of transformations in configuration space. We could use instead the Hamiltonian formalism and consider canonical transformations in phase space. Then it proves possible to define a 3-parameter Abelian group of canonical transformations in phase space (existing independently of any particular Hamiltonian, of course) whose generators, for Hamiltonians conserving total energy and momentum, reduce to the generators of symmetry transformations corresponding to the ones we have used in the Lagrangian approach. These transformations, as well as the canonical symmetry transformations generated by the c.m. constants of the motion considered here will be discussed in a separate paper.

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Shear-Free Gravitational Radiation

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It is shown that, with the exception of the Schwarzschild metric, gravitational systems described by suitably regular vacuum solutions of Einstein's equations admitting a shear- and twist-free congruence of diverging null rays must be radiative. Robinson and Trautman have demonstrated the existence of type-II solutions of this kind, which describe radiating gravitational systems with bounded sources. However, attempts to display an explicit radiative solution through specilaization to conformally spherically symmetric, Kerr-Schild, conformally Kerr-Schild, and type-*D* space times lead to singular metrics. Finally, important physical properties of these systems, including energy, angular momentum, radiation flux, and trapped surfaces, are discussed.

I. INTRODUCTION

MANY of the known solutions to Einstein's equations are shear free.[1] The plane-wave type, with divergence-free ray vectors, clearly represent an excessively unphysical extrapolation of gravitational systems with sources confined to a bounded region. Robinson and Trautman[2,3] were the first to investigate systematically shear-free vacuum metrics with diverging ray vectors. This paper is concerned with the question: Which types of gravitational systems with bounded sources can be described by the Robinson-Trautman metrics? The Schwarzschild metric provides an important example and suggests the possible existence of other cases of physical interest. Robinson and Trautman confined their original analysis to hypersurface-orthogonal shear-free metrics. Although they later generalized their approach to include twisting solutions,[4] such as Kerr's, we will restrict our attention here to the twist-free case.

Since we are concerned with vacuum solutions exterior to some bounded region containing sources, the Goldberg-Sachs[5] theorem is applicable: Shear-free vacuum metrics are algebraically special. This means that the Weyl tensor for such solutions has particularly simple algebraic properties. Correspondingly, there exists a coordinate system in which the metric is also algebraically simpler than usual. For this purpose, Robinson and Trautman used null coordinates based upon the shear-free family of diverging null hypersurfaces. Their work showed that in such a coordinate system, the analytical properties of the metric also simplify considerably. This feature can best be appreciated in terms of some work by Newman and Unti[6] concerning the Lienard-Wiechert potentials of an accelerating charged particle in the context of special relativity. In terms of a null coordinate system based upon the shear-free family of null cones, $u=\text{const}$, emanating from the world line of the accelerating particle, the description of the electromagnetic field becomes especially simple. A gauge can be found in which the vector potential satisfies

$$A_\alpha = A u_{,\alpha}. \qquad (1.1)$$

This algebraic statement is unusual from the customary point of view of describing the radiation field as trans-

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