Electron Bremsstrahlung in Intense Magnetic Fields

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In this paper we present detailed calculations of the most important radiation process in ponderable matter in the presence of an intense magnetic field: the electron bremsstrahlung. This differs from an ordinary bremsstrahlung process in that in a magnetic field the electrons are one-dimensional particles free to move in the direction of the field, but bound in the plane perpendicular to the magnetic field in quantized circular orbits with energy (in the nonrelativistic limit) in multiples of $11.9\times10^{-9}H$ eV, where H is the magnetic field in gauss. We have obtained the emission rate and the absorption coefficient for fields much less than 10^{13} G. This calculation is valid in the large quantum number limit.

1. INTRODUCTION

IN a recent paper,[1] we pointed out that departures from thermodynamic equilibrium will take place on the surface of a vibrating magnetic neutron star with a magnetic field in the vicinity of 10^9 G. This departure from thermodynamic equilibrium will give rise to strong radio emissions which can account for the pulsed radio emission from recently discovered pulsars. In this paper, we present detailed calculations of the chief process of emission: the electron bremsstrahlung in an intense magnetic field. Our calculation is valid for electrons of arbitrary energy and for fields $\ll 4.414\times10^{13}$ G.

2. ORBITAL QUANTIZATION AND RADIATION PROCESS

In a magnetic field the electron energy in the plane perpendicular to the magnetic field H (the latter is taken in the direction of the z axis) is quantized according to the equation

$$E(N,s,p_z,H)=\pm mc^2\left\{\left(\frac{p_z}{mc}\right)^2+\left[\left(1+\frac{H}{H_q}N\right)^{1/2}+s\frac{\alpha}{4\pi}\frac{H}{H_q}\right]^2\right\}^{1/2}, \quad (1)$$

with $N=2n+s+1$. Here p_z is the z component of the momentum of the electron. The motion of the electron along the field lines is unaffected by this quantization. $\alpha=e^2/\hbar c$, $H_q=m^2c^3/e\hbar=4.414\times10^{13}$ G, and the other symbols have their usual meanings. $N=0, 1, 2, \cdots$, characterizes the size of the electron orbit[2]; $s=\pm1$ characterizes the polarization of the electron spin with respect to the direction of the magnetic field ($s=1$ along the field, $s=-1$ against the field.) Equation (1), which includes the anomalous magnetic moment of the electron, is due to Ternov, Bagrov, and Zhukovskii.[3] Properties of an electron gas in intense magnetic fields have been studied extensively in several previous papers.[4]

The nonrelativistic limit of Eq. (1) is

$$E(N,s,p_z,H)=mc^2\left[\frac{1}{2}\left(\frac{p_z}{mc}\right)^2+\frac{1}{2}\left((2n+s+1)+\frac{s}{2\pi}\alpha\right)\frac{H}{H_q}\right]. \quad (2)$$

Therefore, in a magnetic field an electron moves in quantized orbits in the plane perpendicular to the field but is free to move in the direction of the field. This means that a "free" electron has only one degree of freedom when it is in a magnetic field. The electrons behave like one-dimensional particles, and as a consequence their radiation processes differ from those of truly free electrons.

There are three fundamental electromagnetic-radiation processes in intense magnetic fields:

(1) Spontaneous radiation. An electron can spontaneously make a transition from one state N to another state N' with $N>N'$. Such a transition corresponds to the classical synchrotron radiation. It gives rise to photons of energies at multiples of $1.16\times10^{-8}H$ eV. This will be referred to as a spontaneous magnetic transition. No continuum emission is possible because of the orbital quantization, The emitted radiation will have a finite width, usually of the order of $\Delta\lambda/\lambda\sim10^{-6}$ if the field is of the order of 10^9 G.

(2) Coulomb deexcitation through magnetic transition. This is similar to (1) except that the emission takes place in the Coulomb field of a nucleus (Z,A):

$$e^-+(Z,A)\rightarrow e^-+(Z,A)+\gamma. \quad (3)$$

The initial state of the electron, N, differs from the final state, N'. It is analogous to the Coulomb deexcitation of atomic states.

(3) One-dimensional bremsstrahlung. The reaction is similar to (3) except that the transition takes place between two electron states of the same orbit. This type

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[1] H.-Y. Chiu, V. Canuto, and L. Fassio Canuto, Nature **221**, 529 (1969).

[2] The radius R of the circular orbit can be shown to be $R^2=2(H_q/H)(\hbar/mc)N$.

[3] I. M. Ternov, V. G. Bagrov, and V. Ch. Zhukovskii, Moscow Univ. Bull. **21**, 21 (1966).

[4] V. Canuto and H.-Y. Chiu, Phys. Rev. **173**, 1210 (1968); **173**, 1220 (1968); **173**, 1229 (1968); H. -Y. Chiu and V. Canuto, Phys. Rev. Letters **21**, 110 (1968); Astrophys. J. **153**, L157 (1968).

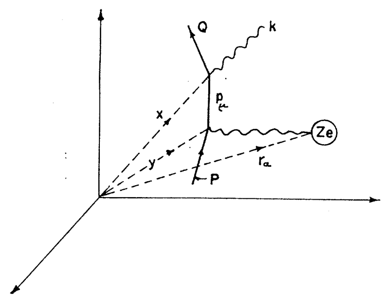

FIG. 1. Feyman diagram of electron bremmsstrahlung, illustrating the dynamical variables employed.

of transition gives rise only to a continuum emission in the radio region.

In this paper we consider process (3). Process (1) has been discussed in another paper.[5]

3. FORMULATION

The Feyman diagrams for (3) are shown in Fig. 1. We will use the S-matrix formulation to compute the transition probability.[6] We have

$$S_2=\left(\frac{-i}{\hbar c}\right)^2 \sum_{\alpha=1}^{\mathfrak{N}_i} \int\int d^4x d^4y\ T[\mathcal{L}(x)\mathcal{L}(y,r_\alpha)], \quad (4)$$

where T denotes the Wick time-ordered product, x and y are the four-coordinates of the two vertices x and y and $\mathbf{r}_\alpha$ is the position vector of the αth nucleus Ze. $\mathcal{L}(x)$ is the interaction Lagrangian of quantum electrodynamics.[7]

$$\mathcal{L}(x)=-ie\bar{\psi}(x)\gamma_\mu\psi(x)A_\mu^{(1)}(x)\,, \quad (5)$$

$$\mathcal{L}(y,r_\alpha)=-ie\bar{\psi}(y)\gamma_\nu\psi(y)A_\nu^{(2)}(y,r_\alpha)\,, \quad (6)$$

where $\psi(x)$ is the electron wave function in a magnetic field (to be discussed later) and $\bar{\psi}(x)\equiv\psi^\dagger\gamma_4$, where $\psi^\dagger$ is the Hermitian conjugate of ψ; γ_μ are the Dirac matrices and conventional meanings of e, $\hbar$, m, c are used. $A_\mu^{(1)}(x)$ is the four-potential corresponding to a real photon (photon emission) of energy E_γ $(=\hbar kc)$,

$$A_\mu^{(1)}(x)=\frac{\hbar c}{E_\gamma^{1/2}}\left(\frac{2\pi}{\Omega}\right)^{1/2}\epsilon_\mu^{(\lambda)}e^{-ik\cdot x}; \quad (7)$$

Ω is the normalization volume, $\epsilon_\mu^{(\lambda)}$ is the polarization vector, λ is the polarization index, and $k\cdot x\equiv k_\mu x^\mu$. $A_\nu^{(2)}$ is the four-potential describing the interaction between the electron and the nucleus. In the limit of static interaction, $A_\nu^{(2)}$ reduces to the Coulomb potential,[6] that is,

$$-ieA_\nu^{(2)}=-ie\delta_{\nu 4}A_4^{(2)}=e\delta_{\nu 4}A_0^{(2)}=\delta_{\nu 4}Ze^2/|\mathbf{y}-\mathbf{r}_\alpha|\,. \quad (8)$$

The Fourier representation is easily seen to be

$$-ieA_\nu^{(2)}=\delta_{\nu 4}\frac{4\pi Ze^2}{\Omega}\sum_q\left(\frac{\hbar}{q}\right)^2 e^{-i\mathbf{q}\cdot(\mathbf{y}-\mathbf{r}_\alpha)/\hbar}, \quad (9)$$

with

$$\sum_q\equiv\sum_{q_1}\sum_{q_2}\sum_{q_3}. \quad (10)$$

Using Eqs. (5)–(9), Eq. (4) becomes (we drop the subscript 2 in S)

$$S=\frac{4\pi i\alpha Ze(2\pi)^{1/2}}{(E_\gamma)^{1/2}\Omega^{3/2}}\sum_{\alpha=1}^{\mathfrak{N}_i}\sum_q\left(\frac{\hbar}{q}\right)^2 \times\int\int d^4x d^4y\ \bar{\psi}(x)\gamma_\mu\epsilon_\mu^{(\lambda)}G(x,y)\gamma_4\psi(y) \times\exp[-ik\cdot x+i\mathbf{q}\cdot(\mathbf{y}-\mathbf{r}_\alpha)/\hbar], \quad (11)$$

where $G(x,y)$ is the Green's function describing the propagation of an electron from y to x. Strictly speaking, $G(x,y)$ should be taken as the exact propagator for an electron in a magnetic field. It has a very complicated expression.[8,9] However, it can be shown that for a weak field we have

$$\begin{aligned}G(p)&=-(ip-iV+m)^{-1}\\&=-i(ip+m)^{-1}+(ip+m)^{-1}iV(ip+m)^{-1}+\cdots\\&=G_0(p)+G_1(p).\end{aligned} \quad (12)$$

We are interested in fields of the order 10^9 or greater, but much less than $H_q=4.414\times10^{13}$ G. Therefore we can[6] approximate G by G_0:

$$G_0(x,y)=\frac{i}{(2\pi)^4}\int d^4(p/\hbar)e^{ip\cdot(x-y)/\hbar}\frac{i\gamma\cdot p/\hbar-\lambda_c^{-1}}{(p/\hbar)^2+\lambda_c^{-2}}, \quad (13)$$

where p is the four-momentum of the electron in the intermediate state (see Fig. 1). From Eq. (11) we then have

$$S=S_0\sum_{\alpha=1}^{\mathfrak{N}_i}\sum_q\left(\frac{\hbar}{q}\right)^2\epsilon_i^{(\lambda)}\int dx_1dx_2dx_3dx_4\int dy_1dy_2dy_3dy_4\int dp_1dp_2dp_3dp_4 \times\psi^\dagger(x_1x_2x_3x_4)\Gamma_i\psi(y_1y_2y_3y_4)\exp[-i(k_1x_1+k_2x_2+k_3x_3+k_4x_4)] \times\exp[ip_1(x_1-y_1)+ip_2(x_2-y_2)+ip_3(x_3-y_3)+ip_4(x_4-y_4)]\exp[i(q_1y_1+q_2y_2+q_3y_3)/\hbar-i\mathbf{q}\cdot\mathbf{r}_\alpha/\hbar], \quad (14)$$

[5] H.-Y. Chiu and L. Fassio-Canuto, following paper, Phys. Rev. **185**, 1614 (1969).
[6] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill Book Co., New York, 1964), pp. 120–127.
[7] The Greek indices run over 1, 2, 3, 4; the Roman indices run over 1, 2, 3 (spacial components) and correspond to the x, y, z axes in a Cartesian coordinate system. The summation convention for repeated indices is employed. The magnetic field is in the z direction (3 axis). The notation is as follows: $q=(q_k,q_4)=(q_k,iq_0)$.
[8] R. Kaitna and P. Urban, Nucl. Phys. **56**, 518 (1964).
[9] J. Schwinger, Phys. Rev. **82**, 664 (1951).

where

$$\rho = \frac{p}{\hbar} \quad \text{and} \quad S_0 = \frac{4\pi e\alpha Z(2\pi)^{1/2}}{E_\gamma^{1/2}\Omega^{3/2}(2\pi)^4}. \tag{15}$$

S_0 has the dimension of (length)$^{-4}$. We now invoke the Coulomb gauge condition $\epsilon_4^{(\lambda)}=0$, so that the index i in $\epsilon_i^{(\lambda)}$ runs only from 1 to 3. The matrix Γ_i is given by

$$\Gamma_i = \gamma_4\gamma_i \frac{i\gamma\cdot\rho - \lambda_c^{-1}}{\rho^2+\lambda_c^{-2}}\gamma_4. \tag{16}$$

4. WAVE FUNCTION $\psi(x)$

The Dirac equation for a free electron in a constant and homogeneous external field,

$$(\gamma_\mu\partial_\mu + \lambda_c^{-1})\psi + \frac{ie}{\hbar c}\gamma_\mu \mathfrak{A}_\mu\psi = 0,$$

was solved four decades ago.[10–12] We will choose the magnetic field to be in the z direction (corresponding to the coordinate index 3). The vector potential $\mathfrak{A}_\mu$ will be chosen to be of the following gauge:

$$\mathfrak{A}_1 = -yH, \quad \mathfrak{A}_2 = \mathfrak{A}_3 = \mathfrak{A}_4 = 0. \tag{17}$$

The initial-state wave function is[11]

$$\psi_i = (L_1L_3)^{-1/2}\exp(iP_1y_1\hbar^{-1} + iP_3y_3\hbar^{-1} + iP_4y\hbar^{-1} - \tfrac{1}{2}\xi^2) \times \begin{bmatrix} C_1\bar{H}_N(\xi) \\ C_2\bar{H}_{N-1}(\xi) \\ C_3\bar{H}_N(\xi) \\ C_4\bar{H}_{N-1}(\xi) \end{bmatrix}, \tag{18}$$

where

$$C_1 = \alpha A_s, \quad C_2 = s\alpha B_s, \quad C_3 = s\beta A_s, \quad C_4 = \beta B_s,$$
$$\alpha^2 = \tfrac{1}{2}(1+\epsilon^{-1}), \quad \beta^2 = \tfrac{1}{2}(1-\epsilon^{-1}),$$
$$A_s^2 = \tfrac{1}{2}[1+sx(x^2+2N\Theta)^{-1/2}], \tag{19}$$
$$B_s^2 = \tfrac{1}{2}[1-sx(x^2+2N\Theta)^{-1/2}],$$
$$x = P_3/mc, \quad s = \pm 1, \quad \xi = y_2\gamma^{1/2} + P_1\hbar^{-1}\gamma^{-1/2}, \tag{20}$$
$$\gamma = (H/H_q)\lambda_c^{-2}, \quad \epsilon^2 \equiv \epsilon^2(N,x) = 1+x^2+2N\Theta,$$
$$\Theta = H/H_q, \quad N = 0, 1, 2, \cdots, \infty,$$

and

$$\bar{H}_N(\xi) = [\gamma^{1/4}/\pi^{1/4}2^{N/2}(N!)^{1/2}]H_N(\xi). \tag{21}$$

The final-state wave function is the same as the initial-state wave function with the appropriate variables.

[10] I. I. Rabi, Z. Physik **49**, 507 (1928).

[11] N. P. Klepikov, Zh. Eksperim. i Teor. Fiz. **26**, 19 (1954).

[12] In practice the inhomogeneity and time variation of a magnetic field are macroscopic in nature. Unless the field changes substantially over a time interval $\sim 10^{-23}$ sec or in a distance λ_d = de Broglie wavelength of a particle, the field can be regarded as constant and homogeneous. The time variation can then be treated by adiabatic perturbation methods. If the field is generated by a macroscopic current, as it is in sunspots and in neutron stars, then it is impossible for the field to change substantially in a time $\sim 10^{-23}$ sec or over a distance $\sim$de Broglie wavelength.

The classical expression for the energy is

$$\epsilon^2 = 1 + \left(\frac{P_1}{mc}\right)^2 + \left(\frac{P_2}{mc}\right)^2 + \left(\frac{P_3}{mc}\right)^2. \tag{22}$$

Comparing (22) with the expression of ϵ^2 in (20), we see that the quantization replaces the energy due to the momentum perpendicular to the field, $(P_\perp/mc)^2 = (P_1^2+P_2^2)/(mc)^2$, by $2n+s+1$.[13] If there is no interaction, then in the ground state with $n=0$, $s=-1$, there is no momentum perpendicular to the field, and hence $P_1=P_2=0$. In the case of interaction the electron will acquire a recoil momentum, but we can, without loss of generality, assume that the recoil takes place in the y direction (index 2), and put $P_1=Q_1=0$, where Q_1 is the coorespondent of P_1 for the final state.

5. EVALUATION OF S

We now substitute Eq. (18) into Eq. (14) to evaluate S. Integrations over x_1, y_1, x_3, y_3, x_4, y_4, ρ_1, ρ_3, and ρ_4 can be carried out sequentially, giving

$$S = S_0(2\pi)^3L\epsilon_i^{(\lambda)}\int\int d\rho_2 d\left(\frac{q_2}{\hbar}\right)\delta(k_4+Q_4/\hbar-P_4/\hbar) \times e^{-iq\cdot r\hbar^{-1}}\frac{F_i(\rho_2,q_2)}{k_1^2+(q_2/\hbar)^2+(k_3+Q_3/\hbar-P_3/\hbar)^2}, \tag{23}$$

where $(q_1=k_1,\ q_3=k_3+Q_3/\hbar-P_3/\hbar)$

$$F_i = \Phi_f^\dagger(k_2-\rho_2)\Gamma_i\Phi_i(q_2/\hbar-\rho_2) \tag{24}$$

and $\Phi(k_2-\rho_2)$ and $\Phi(q_2/\hbar-\rho_2)$ are spinors whose elements are Fourier-transformed with respect to $k_2-\rho_2$ and $q_2/\hbar-\rho_2$, respectively. The typical integral is of the form

$$I = \int dy\, e^{iay}H_N(y\gamma^{1/2}) = \left(\frac{2\pi}{\gamma}\right)^{1/2} i^N e^{-a^2/2\gamma}H_N(a/2\gamma^{1/2}). \tag{25}$$

For weak magnetic fields ($H \ll H_q$), $\gamma \ll 1$, the exponential factor is of the order of 10^{-4}, and

$$I = 2\pi[e^{-a^2/2\gamma}/(2\pi\gamma)^{1/2}]H_N(a/2\gamma^{1/2}) \to 2\pi\delta(a)H_N(0) \to 2\pi\delta(a), \tag{26}$$

where we have used the following representation for $\delta(x)$:

$$\delta(x) = \lim_{b\to 0}\frac{e^{-x^2/b}}{(b\pi)^{1/2}}. \tag{27}$$

[13] See Paper I of Canuto and Chiu (Ref. 4).

The S matrix is now

$$S=2(2\pi)^5 S_0\left(\frac{\gamma}{\pi}\right)^{1/2} 2^{-(N+N')/2}(N!N'!)^{-1/2}L$$

$$\times\int\int a\rho_2 a(q_2/\hbar)\delta(\rho_2-k_2)\delta(q_2/\hbar-\rho_2)$$

$$\times\delta(k_4+Q_4/\hbar-P_4/\hbar)\frac{\epsilon_i^{(\lambda)}F_i}{(B+\rho_2^2)(C+k_2^2)}, \quad (28)$$

$$F_1=(A_2-A_1)k_1-(M_1+M_2)+ik_1(A_1+A_2), \quad (29)$$

$$F_2=(A_1-A_2)k_1-ik_2(A_1+A_2)-i(M_1-M_2), \quad (30)$$

$$F_3=\frac{k_1}{k_3+Q_3}(A_3+A_6)$$

$$+i\left((k_3+Q_3)(A_1+A_2)+(k_0+Q_0)(A_9-A_{11})\right.$$

$$\left.+A_{12}-A_{10}-\frac{k_2}{k_3+Q_3}(A_6-A_3)\right), \quad (31)$$

with the following definitions:

$$\begin{aligned}
&A_1=C_1C_1'+C_3C_3', && A_2=2(NN')^{1/2}(C_2C_2'+C_4C_4'),\\
&A_3=(2N)^{1/2}(k_3+Q_3)(C_2C_1'+C_4C_3'), && A_4=(2N)^{1/2}(k_0+Q_0)(C_2C_3'+C_4C_1'),\\
&A_5=(2N)^{1/2}(C_4C_1'-C_2C_3'), && A_6=(2N')^{1/2}(k_3+Q_3)(C_1C_2'+C_3C_4'),\\
&A_7=(2N')^{1/2}(k_0+Q_0)(C_1C_4'+C_3C_2'), && A_8=(2N')^{1/2}(C_3C_2'-C_1C_4'),\\
&A_9=C_1C_3'+C_3C_1', && A_{10}=C_1C_3'-C_3C_1',\\
&A_{11}=2(NN')^{1/2}(C_2C_4'+C_4C_2'), && A_{12}=2(NN')^{1/2}(C_2C_4'-C_4C_2'),\\
&M_1=A_3+A_4+A_5, && M_2=A_6-A_7-A_8.
\end{aligned} \quad (32)$$

The transition probability is proportional to $|S|^2$. Squaring Eq. (28), we obtain

$$\left(\frac{S}{S_0}\right)^2=8(2\pi)^9\gamma L^2\frac{T_c}{2\pi}$$

$$\times\frac{\delta(k_4+Q_4/\hbar-P_4/\hbar)|\epsilon_i^{(\lambda)}F_i|^2\mathfrak{N}_i}{2^{(N+N')/2}(N!N'!)^{1/2}(B+k_2^2)^2(C+k_2^2)^2}, \quad (33)$$

with

$$B=k_1^2+(k_3+Q_3/\hbar)^2+(k_4+Q_1/\hbar)^2+\lambda_c^{-2},$$

$$C=k_1^2+(k_3+Q_3/\hbar-P_3/\hbar)^2.$$

The number of ions $\mathfrak{N}_i$ appears because it can be shown that

$$\left\langle\sum_{\alpha=1}^{\mathfrak{N}_i}\sum_{\alpha'=1}^{\mathfrak{N}_{i'}}\exp[i\mathbf{q}\cdot(\mathbf{r}_\alpha-\mathbf{r}_{\alpha'})]\right\rangle=\mathfrak{N}_i\left[1+\frac{\mathfrak{N}_i}{\Omega}g(q)\right], \quad (34)$$

where $g(q)$ is the Fourier transform of the pair distribution function.[14] The symbol $\langle\cdots\rangle$ means average over the canonical ensemble. In the range of densities and temperatures we are interested in, the ion-ion interaction can be neglected and the preceding formula simply gives $\mathfrak{N}_i$.

The time T appears because $[\delta(E)]^2\to(T/2\pi\hbar)\delta(E)$.

6. RADIATION RATE

The transition probability W (per electron per quantum state per unit time per unit volume) is obtained from the usual expression:

$$W=|S|^2/\Omega T. \quad (35)$$

Substituting Eq. (33) into (35), we obtain

$$W=|S|^2/\Omega T=8\gamma(2\pi)^8(c/L)S_0^2\delta(k_4+Q_4/\hbar-P_4/\hbar)$$

$$\times\frac{\mathfrak{N}_i|\epsilon_i^{(\lambda)}F_i|^2}{(B+k_2^2)^2(C+k_2^2)^2 2^{(N+N')/2}(N!N'!)^{1/2}}. \quad (36)$$

Restoring the dimensional factors in S_0 from Eq. (15), and letting $\omega=E_\gamma/mc^2$, we obtain

$$W=4(4\pi)^3\gamma\alpha^3Z^2\hbar c^2(E_\gamma\Omega)^{-1}(\mathfrak{N}_i/\Omega)(\lambda_c/L\Omega)\delta(\epsilon-\epsilon'-\omega)$$

$$\times\frac{|\epsilon_i^{(\lambda)}F_i|^2}{(B+k_2^2)^2(C+k_2^2)^2 2^{(N+N')/2}(N!N'!)^{1/2}}. \quad (37)$$

The energy radiation rate I is obtained by multiplying W by the energy of the outgoing photon, summing $\sum_f$ over the final states of the photon and the electron, and averaging $\frac{1}{2}\sum_i{}^{(e)}$ over the initial states. We have

$$I=\tfrac{1}{2}\sum_i{}^{(e)}\sum_f E_\gamma\frac{|S|^2}{\Omega T} \quad (38)$$

and

$$\sum_f=\sum_f{}^{(\gamma)}\sum_f{}^{(e)},$$

$$\sum_f{}^{(\gamma)}=\sum_\lambda(2\pi)^{-3}\Omega k^2dkd\Omega_\gamma$$

$$=\sum_\lambda(2\pi)^{-3}\Omega\lambda_c^{-3}\omega^2d\omega d\Omega_\gamma. \quad (39)$$

[14] T. L. Hill, *Statistical Mechanics* (McGraw-Hill Book Co., New York, 1956), pp. 179–188.

The summing over the initial states is performed by summing over all electron states in a gas. This means that[13]

$$\sum_i{}^{(e)}\rightarrow\frac{L}{2\pi\lambda_c}\sum_{N,S}\int_{-\infty}^{+\infty}f(x)dx, \tag{40}$$

$$\sum_f{}^{(e)}\rightarrow\frac{L}{2\pi\lambda_c}\int_{-\infty}^{+\infty}dx'\Theta\Omega^{2/3}(2\pi\lambda_c^2)^{-1}\times\sum_{N',S'}[1-f(x')], \tag{41}$$

where $f(x)=\{1+\exp[(\epsilon-\mu)/kT]\}^{-1}$ is the Fermi distribution function. Here μ is the chemical potential plus the rest mass of the electron in units of mc^2 and T is the temperature. Finally,

$$\sum_i{}^{(e)}\sum_f{}^{(e)}\rightarrow\frac{L\Omega}{\lambda_c}(2\pi)^{-1}\sum_N\sum_{N'}\int_{-\infty}^{+\infty}dx\,f(x)\times\left[\frac{1}{4\pi^2}\Theta\lambda_c^{-3}\right]\int_{-\infty}^{+\infty}dx'[1-f(x')]. \tag{42}$$

We are interested in the spectrum of the radiation. The spectrum is obtained by differentiating I with respect to ω and Ω, and therefore we have ($N_i=\mathfrak{N}_i/\Omega$ is the density of the nuclei Z)

$$dI=\frac{8}{\pi}\Theta\frac{\alpha^3Z^2\hbar c^2\lambda_c^{-5}N_i}{2^{(N+N')/2}(N!N'!)^{1/2}}\sum_N\sum_{N'}\int_{-\infty}^{+\infty}dx\,f(x)\times\left[\frac{1}{4\pi^2}\Theta\lambda_c^{-3}\right]\int_{-\infty}^{+\infty}dx'\,[1-f(x')]\delta(\epsilon-\epsilon'-\omega)\times\frac{\omega^2d\omega d\Omega_\gamma\sum_\lambda|\epsilon_i^{(\lambda)}F_i|^2}{(B+k_2^2)^2(C+k_2^2)^2}. \tag{43}$$

We will now reduce Eq. (43) into a usable form. From Eqs. (36) and (40), we find

$$(B+k_2^2)^{-2}(C+k_2^2)^{-2}=\lambda_c^{-8}f(\epsilon,\epsilon',\Theta), \tag{44}$$

where

$$\begin{aligned}f^{-1/2}=&[(\epsilon-\epsilon')^2+\epsilon'^2-\epsilon^2+1-a_{N'}^2+2\omega\cos\theta\,(\epsilon'^2-a_{N'}^2)^{1/2}]\\&\times\{(\epsilon-\epsilon')^2+[(\epsilon^2-a_N^2)^{1/2}-(\epsilon'^2-a_{N'}^2)^{1/2}]^2+2\omega\cos\theta\,[(\epsilon^2-a_N^2)-(\epsilon'^2-a_{N'}^2)^{1/2}]^2\}\\=&(2\epsilon'^2-2\epsilon\epsilon'-N'\Theta)\{[(\epsilon-\epsilon')+(x-x')]^2-2(\epsilon-\epsilon')(x-x')(1-\cos\theta)\},\end{aligned} \tag{45}$$

$$a_N^2=1+2N\Theta.$$

We now sum over the polarization vector $\epsilon_i^{(\lambda)}$ in the quantity $|\epsilon_i^{(\lambda)}F_i|^2$ in Eq. (33):

$$\sum_\lambda\epsilon_i^{(\lambda)}\epsilon_j^{(\lambda)}=\delta_{ij}-\frac{k_ik_j}{k^2}. \tag{46}$$

Using polar coordinates for the photon vector $\mathbf{k}$ with the polar axis aligned with the field, we obtain, after integrating over the azimuthal angle,

$$\sum_\lambda\frac{1}{2\pi}\int_0^{2\pi}d\phi|\epsilon_i^{(\lambda)}F_i|^2=\frac{1}{2\pi}|H_1|^2+\frac{1}{2\pi}|H_2|^2, \tag{47}$$

where

$$\begin{aligned}|H_1|^2=&\omega^2(A_1^2+A_2^2)+(\epsilon'^2-a_{N'}^2)(A_1+A_2)^2+\epsilon'^2(A_9-A_{11})^2+2\epsilon'(\epsilon'^2-a_{N'}^2)^{1/2}\\&\times(A_1+A_2)(A_9-A_{11})+(A_{12}-A_{10})^2+2\epsilon'(\epsilon'^2-a_{N'}^2)^{1/2}(A_1+A_2)(A_9-A_{11})\\&\times(A_{12}-A_{10})-2A_1A_2\omega^2\sin^2\theta\cos^2\theta,\end{aligned} \tag{48}$$

$$|H_2|^2=\Gamma_1(1+\cos^2\theta)+\Gamma_2\omega^2\sin^4\theta+2\Gamma_3\omega\sin^2\theta\cos\theta, \tag{49}$$

$$\begin{aligned}\Gamma_1=&2(N+N')\{[\omega\cos\theta+Q_3)A_1+(\omega+\epsilon')A_9]^2+A_{10}^2+2(\omega+\epsilon')D_1+2(\omega\cos\theta+Q_3)D_2\},\\\Gamma_2=&2(N+N')A_1^2-\tfrac{5}{2}(NN')^{1/2},\\\Gamma_3=&2(N+N')[(\omega\cos\theta+Q_3)^2A_1^2+(\omega+\epsilon')A_1A_9+D_2].\end{aligned} \tag{50}$$

Both $|H_1|^2$ and $|H_2|^2$ have to be averaged over the initial spin and summed over the final spin. The average and sum of the spin states is extended to all combinations of A's. The results for the averages are

$$\begin{aligned}&\sum_s A_1^2=\alpha^2\alpha'^2+\beta^2\beta'^2+2\alpha\beta\alpha'\beta'xx'\times(x^2+2N\Theta)^{-1/2}(x'^2+2N'\Theta)^{-1/2},\\&\sum_s A_2^2=4NN'\sum_s A_1^2,\quad\sum_s A_1A_{11}=\sum_s A_2A_9=0,\\&\sum_s A_1A_2=4(NN')^{1/2}(\alpha\alpha'+\beta\beta')\left(1-\frac{x^2}{x^2+2N\Theta}\right)^{1/2}\times\left(1-\frac{x'^2}{x'^2+2N'\Theta}\right)^{1/2},\\&\sum_s A_9^2=\alpha^2\beta'^2+\beta^2\alpha'^2+2\alpha\alpha'\beta\beta'xx'\times(x^2+2N\Theta)^{-1/2}(x'^2+2N'\Theta)^{-1/2},\\&\sum_s A_{11}^2=4NN'\sum_s A_9^2,\quad\sum_s A_{10}A_{12}=-\sum_s A_9A_{11},\\&\sum_s A_1A_9=\alpha\beta x(x^2+2N\Theta)^{-1/2}+\alpha'\beta'x'(x'^2+2N'\Theta)^{-1/2},\\&\sum_s A_9A_{11}=4(NN')^{1/2}\alpha\alpha'\beta\beta'\left(1-\frac{x^2}{x^2+2N\Theta}\right)^{1/2}\times\left(1-\frac{x'^2}{x'^2+2N'\Theta}\right)^{1/2},\end{aligned} \tag{51}$$

$$\sum_s A_2A_{11}=-4NN'\sum_s A_1A_9,\quad \sum_s A_{12}^2=4NN'\sum_s A_9^2,$$

$$\sum_s A_{10}^2=\alpha^2\beta'^2+\beta^2\alpha'^2-2\alpha\alpha'\beta\beta'xx'\times(x^2+2N\Theta)^{-1/2}(x'^2+2N'\Theta)^{-1/2},$$

$$\sum_s D_1=\alpha'^2\beta^2-\alpha^2\beta'^2,$$

$$\sum_s D_2=-x(x'^2+2N'\Theta)^{-1/2}\alpha\beta\epsilon'^{-1}-x'(x^2+2N\Theta)^{-1/2}\alpha'\beta'\epsilon^{-1}.$$

7. NONRELATIVISTIC LIMIT

In the nonrelativistic and weak-field limit, where $w=\frac{1}{2}x^2+N\Theta\ll1$, we then find ($w\equiv\epsilon-\epsilon'$)

$$f^{-1/2}(\epsilon,\epsilon',\Theta)=(2\omega+N'\Theta)[(\omega+x-x')^2-2\omega(s-x')(1-\cos\theta)],\tag{52}$$

$$\epsilon\to1+w,\quad w=\tfrac{1}{2}x^2+N\Theta,\quad \alpha\to1,\quad \beta\to\tfrac{1}{2}w,\tag{53}$$

$$\begin{aligned}
&\sum_s A_1^2\to1,\quad \sum_s A_2^2\to4NN',\quad \sum_s A_1A_{11}=\sum_s A_2A_9=0,\\
&\sum_s A_1A_2\to4(ww')^{-1/2}NN', &&\sum_s A_9A_{11}\to2NN'\Theta,\\
&\sum_s A_9^2\to\tfrac{1}{2}(w+w'+xx'), &&\sum_s A_2A_{11}\to-2NN'(x+x'),\\
&\sum_s A_{11}^2\to2NN'(w+w'+xx'), &&\sum_s A_{10}A_{12}\to-2NN'\Theta,\\
&\sum_s A_1A_9\to\tfrac{1}{2}(x+x'), &&\sum_s A_9A_{11}\to2NN'\Theta,\\
&\sum_s A_{10}^2\to\tfrac{1}{2}(w+w'-xx'), &&\sum_s A_{10}A_{12}\to-2NN'\Theta,\\
&\sum_s A_{12}^2\to2NN'(w+w'+xx'), &&\sum_s D_2\to-\tfrac{1}{2}(x+x'),\\
&\sum_s D_1\to\tfrac{1}{2}(w-w').
\end{aligned}\tag{54}$$

Therefore, we have

$$|H_1|^2=\{\xi\omega^2+2x'^2[\xi+4NN'(ww')^{-1/2}]+\xi(w+w')\}\times\sin^2\theta-8NN'(ww')^{-1/2}\omega^2\sin^2\theta\cos^2\theta,\tag{55}$$

where $\xi=1+4NN'$ and

$$\Gamma_1=2(N+N')(\omega^2\cos^2\theta+w+w'+x'^2),\tag{56}$$

$$\Gamma_2=2(N+N')-5NN'\Theta(ww')^{-1/2},\tag{57}$$

$$\Gamma_3=2(N+N')(\omega^2\cos^2\theta+x'^2).\tag{58}$$

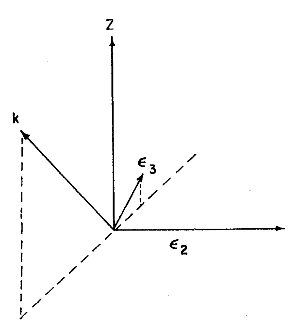

FIG. 2. Polarization vectors $\boldsymbol{\varepsilon}_2$ and $\boldsymbol{\varepsilon}_3$ and their relation to the wave vector **k**.

We finally have

$$|H_2|^2=2(N+N')\{(\omega^2\cos^2\theta+w+w'+x^{2'})(1+\cos^2\theta)+[1-5NN'\Theta(NN'ww')^{-1/2}]\omega^2\sin^4\theta+2(\omega^2\cos^2\theta+x'^2)\omega\sin^2\theta\cos\theta\}.\tag{59}$$

8. POLARIZATION

Equation (43) has been obtained after summing over all polarization states of the emitted radiation. In order to study the polarization effects, we will go back to Eq. (28). In general, the quantity ϵ_iF_i can be written in the following form:

$$\epsilon_iF_i\equiv\sum_{i=1}^{3}\epsilon_iF_i\equiv\boldsymbol{\varepsilon}\cdot\mathbf{F},\tag{60}$$

with the obvious definitions of the F_i's. In order to study *linear* polarization we can decompose $\boldsymbol{\varepsilon}$ into two mutually perpendicular components (which are perpendicular to the direction of propagation as required by the transversality of the photon). Let these two components be ϵ_2 and ϵ_3 as shown in Fig. 2. Then we have[15]

$$\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}_2+\boldsymbol{\varepsilon}_3=\beta_2q_2+\beta_3g_3,\tag{61}$$

[15] A. A. Sokolov, U. S. Atomic Energy Commission Translation Series No. AEC-tr-4322, 1960. (Available from: Office of Technical Services, Dept. of Commerce, Washington 25, D. C.).

with

$$\mathfrak{g}_2=\frac{\mathbf{k}\times\hat{z}/|\mathbf{k}|}{[1-(\mathbf{k}\cdot\hat{z})^2/|\mathbf{k}|^2]^{1/2}}, \tag{62}$$

$$\mathfrak{g}_3=\frac{\mathbf{k}(\mathbf{k}\cdot\hat{z})/|\mathbf{k}|^2-\hat{z}}{[1-(\mathbf{k}\cdot\hat{z})^2/|\mathbf{k}|^2]^{1/2}}, \tag{63}$$

$$g_s g_{s'}=\delta_{ss'} \quad (S,S'=2,3), \tag{64}$$

where $\hat{z}$ is a unit vector in the direction of the magnetic field. $\boldsymbol{\varepsilon}_2$ is in the plane perpendicular to $\mathbf{k}$ and $\hat{z}$, and $\boldsymbol{\varepsilon}_3$ is in the plane defined by $\mathbf{k}$ and $\hat{z}$, as shown in Fig. 2. For classical synchrotron radiation the polarization may be similarly analyzed into components $\boldsymbol{\varepsilon}_2$ and $\boldsymbol{\varepsilon}_3$; the polarization of the $\boldsymbol{\varepsilon}_2$ component is $\frac{7}{8}$ and that of the $\boldsymbol{\varepsilon}_3$ component is $\frac{1}{8}$.[15] As we shall see, in the nonrelativistic case the bremsstrahlung radiation will be almost 100% linearly polarized in the $\boldsymbol{\varepsilon}_3$ component, that is, it will be almost 100% polarized in the plane defined by $\mathbf{k}$ and $\hat{z}$.

Upon setting $\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}_2$, we obtain the polarization in the 2-direction from Eq. (43):

$$\frac{(dI/d\omega d\theta)_2}{(dI/d\omega d\theta)_{\text{tot}}}=\frac{1}{1+Y^2}, \tag{65}$$

and, analogously, we have the polarization in the 3-direction:

$$\frac{(dI/d\omega d\theta)_3}{(dI/d\omega d\theta)_{\text{tot}}}=\frac{Y^2}{1+Y^2}, \tag{66}$$

where Y is a complicated function of ϵ and ϵ'. In the nonrelativistic limit, $Y\rightarrow\infty$, and the radiation is almost 100% linearly polarized in the plane defined by $\mathbf{k}$ and $\hat{z}$.

In order to study the circular polarization, the vector $\boldsymbol{\varepsilon}$ may be decomposed in the following way:

$$\boldsymbol{\varepsilon}=\sum_{l=\pm1}\mathfrak{g}_l g_l, \tag{67}$$

$$\mathfrak{g}_l=2^{-1/2}(\mathfrak{g}_2+il\mathfrak{g}_3), \tag{68}$$

$$g_l g_{l'}=\delta_{ll'}. \tag{69}$$

$l=+1$ (-1) corresponds to right-hand (left-hand) circular polarization. Substituting the $l=+1$ component into Eq. (43), we obtain

$$\frac{(dI/d\omega d\theta)_l}{(dI/d\omega d\theta)_{\text{tot}}}=\frac{1}{2}-l\frac{Y}{Y^2+1}. \tag{70}$$

In the nonrelativistic limit there is thus no circular polarization.

9. DISCUSSION

As discussed before, the use of the free-particle Green's function makes the computation valid only for large quantum numbers, i.e., small magnetic fields. There are, however, cases in which the main contribution is given by the low-quantum-number region. As an example we quote the thin plasma layer at the surface of a neutron star, in which a low-density (nonrelativistic) electron gas is imbedded in the strong neutron star magnetic field. The plasma effect ignored in the present paper should also be included. This full problem is now being investigated by one of the authors and the results will shortly be submitted for publication.

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