$$\mathcal{U}=(\tfrac{4}{5})^3(4\alpha/3\pi)^{3/2}\Big\{(-w+4m-2b+2h)\,S_l[(32/75)\alpha-(32/25)\kappa]\exp\{-[(34\alpha+48\kappa)/75](r^2+r'^2)\}$$
$$+(-3w-3m)[\alpha/(\alpha+2\kappa)]^{3/2}S_l[(32/75)\alpha]\exp[-(34/75)\alpha(r^2+r'^2)]$$
$$+(-3w-3m)\left(\frac{3\alpha}{3\alpha+2\kappa}\right)^{3/2}S_l\left(\frac{32\alpha^2+64\alpha\kappa}{75\alpha+50\kappa}\right)$$
$$\times\left[\exp\left(-\frac{34\alpha^2+28\alpha\kappa}{75\alpha+50\kappa}r^2-\frac{34\alpha^2+108\alpha\kappa}{75\alpha+50\kappa}r'^2\right)+\exp\left(-\frac{34\alpha^2+108\alpha\kappa}{75\alpha+50\kappa}r^2-\frac{34\alpha^2+28\alpha\kappa}{75\alpha+50\kappa}r'^2\right)\right]\Big\}, \quad \text{(A15)}$$

$$\mathcal{E}=(\tfrac{4}{5})^3(4\alpha/3\pi)^{3/2}S_l[(32/75)\alpha]\exp[-(34/75)\alpha(r^2+r'^2)]. \quad \text{(A16)}$$

Also, the quantity E' in Eq. (A4) is given by

$$E'=E+E_\alpha. \quad \text{(A17)}$$

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Multiple-Scattering Analysis on a Soluble Neutron-Deuteron Model*

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The multiple-scattering series for elastic scattering is investigated numerically for a model of the neutron-deuteron system, at neutron laboratory energies of 14.1, 50, and 100 MeV. The model is that of Aaron, Amado, and Yam, with spin-dependent, s-wave, separable, two-body interactions. It is found that the doublet $L=0$ series converges only slowly even at 100 MeV, and that it strongly diverges at 14.1 MeV. On the other hand, the convergence is rapid for both doublet and quartet partial waves beyond $L=2$, and for these the single-scattering plus Born-pickup terms provide an accurate approximation. Differential cross sections and partial-wave amplitudes are given for various orders of multiple scattering, and for a unitary version of the first-order approximation, and are compared with the exact results.

1. INTRODUCTION

APPROXIMATIONS based on truncated multiple-scattering series have often been used to study the scattering of elementary particles by deuterons. Of these, the most widely used is the impulse approximation,[1] which in its usual application may be regarded as the first-order contribution to the multiple-scattering series. Some attempts have been made to calculate second-[2,3] and higher-order[3] terms, but with realistic potentials even the first-order term requires approximation, and drastic simplifications must be made to calculate higher-order terms. With separable potentials, however, exact calculation to all orders becomes possible, and valuable insight into the multiple-scattering series may be gained in this way. Such studies have been made previously for the K^-—d,[4] K^+—d,[5] and Λ—d [5] systems. In this paper, we consider the n-d system, which differs in the important respect that rearrangement collisions are possible. Our main aim is to study the convergence of the multiple-scattering series of Faddeev-type[6] for this system. We hope that the results will serve as a guide to the usefulness of the multiple-scattering series for more realistic potentials.

The multiple-scattering series is derived in Sec. 2, and in Sec. 3 we briefly describe the model (the separable-potential model of Aaron, Amado, and Yam[7]). The methods used in calculating the multiple-scattering series and in analyzing the convergence are described

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[1] G. F. Chew, Phys. Rev. **80**, 196 (1950); G. F. Chew and G. C. Wick, *ibid.* **85**, 636 (1952); J. Ashkin and G. C. Wick, *ibid.* **85**, 686 (1952); G. F. Chew and M. L. Goldberger, *ibid.* **87**, 778 (1952); H. Kottler and K. L. Kowalski, *ibid.* **138**, B619 (1965). The last paper contains many references to earlier calculations with the impulse approximation.

[2] A. Everett, Phys. Rev. **126**, 831 (1962); A. K. Bhatia and J. Sucher, *ibid.* **132**, 855 (1963).

[3] N. M. Queen, Nucl. Phys. **55**, 177 (1964); **66**, 673 (1965).

[4] J. H. Hetherington and L. H. Schick, Phys. Rev. **137**, B935 (1965).

[5] J. H. Hetherington and L. H. Schick, Phys. Rev. **138**, B1411 (1965); **139**, B1164 (1965).

[6] L. D. Faddeev, Zh. Eksperim. i Teor. Fiz. **39**, 1459 (1960) [English transl.: Soviet Phys.—JETP **12**, 1014 (1961)].

[7] R. Aaron, R. D. Amado, and Y. Y. Yam, Phys. Rev. **140**, B1291 (1965).

in Sec. 4, and the results obtained for elastic scattering of 14.1-, 50-, and 100-MeV neutrons are discussed in Sec. 5. In Sec. 6, we report the results of calculations, using a unitary version of the first-order approximation.[8,9]

The main results are as follows: The multiple-scattering series converges well for the higher partial waves, and in fact for $L\geq 2$ the zero-order (Born-pickup) plus first-order (impulse) terms provide an excellent approximation at each energy. For S waves, on the other hand, the convergence is extremely slow even at the highest energy, and at 14.1 MeV the multiple-scattering series strongly diverges. Thus, at the energies considered, there would seem to be little point in explicitly calculating second- or higher-order terms. A better policy would seem to be to calculate the zero-order and first-order terms, with the low partial waves subtracted out. Then the problem reduces to finding reasonable alternative approximations for the low partial waves.

The unitary version of the first-order approximation is found to give substantially better differential cross sections than the ordinary first-order approximation, with striking improvement at the lower energies. However, the quantitative agreement is only fair, and it is clear that better methods are needed. From the partial-wave results it can be seen that the unitary correction greatly improves the S-wave amplitudes, in the sense that the gross violations of unitarity are removed. For partial waves with $L\geq 2$, however, the unitary correction tends to give a worse result, though in these cases the difference is fairly small.

2. MULTIPLE-SCATTERING SERIES

We denote the particles by 1, 2, and 3, and the interaction between 2 and 3 by $V_{23}=V_1$; correspondingly, it will sometimes be convenient to refer to 2 and 3 as the pair 1, etc. The channel in which particle 1 is free and particles 2 and 3 are bound (or correlated) is called channel 1, and, following Lovelace,[10] the three-free-particle channel is called channel 0. In this section, breakup processes are included in the discussion, and for this purpose it is convenient to introduce a fourth interaction $V_0=0$. Greek letters have the values 0, 1, 2, 3 unless stated otherwise.

The multiple-scattering series is most easily derived from the elegant formulation of the three-body problem due to Alt, Grassberger, and Sandhas.[11] The three-body transition operator for transitions from channel α to channel β is $U_{\beta\alpha}$, defined by

$$U_{\beta\alpha}(s)=(1-\delta_{\beta\alpha})(s-H_0)+\sum_{\beta\neq\gamma\neq\alpha} V_\gamma + \sum_{\beta\neq\gamma}\sum_{\delta\neq\alpha} V_\gamma G(s) V_\delta, \quad (2.1)$$

where H_0 is the three-body kinetic energy in the c.m. system, s is the complex energy parameter, and

$$G(s)=(s-H_0-V_1-V_2-V_3)^{-1}. \quad (2.2)$$

The physical amplitude for a transition from channel α to channel β is $(\Phi_\beta', U_{\beta\alpha}(E+i\epsilon)\Phi_\alpha)$, where Φ_α and Φ_β' are the initial and final channel states, satisfying

$$(H_0+V_\alpha)\Phi_\alpha=E\Phi_\alpha, \quad (2.3a)$$

$$(H_0+V_\beta)\Phi_\beta'=E\Phi_\beta'. \quad (2.3b)$$

The essential ingredients in the Faddeev approach to three-body theory are the operators T_γ, defined by

$$T_\gamma(s)=V_\gamma+V_\gamma G_0(s) T_\gamma(s), \quad (2.4)$$

where

$$G_0(s)=(s-H_0)^{-1}. \quad (2.5)$$

T_γ is essentially the two-body transition operator for the pair γ, the only difference being that G_0 includes the kinetic energy of the third particle.

In terms of the T_γ's, it is easy to show that the transition operators $U_{\beta\alpha}$ satisfy the coupled integral equations[11]

$$U_{\beta\alpha}=(1-\delta_{\beta\alpha})(s-H_0)+\sum_{\beta\neq\gamma} T_\gamma G_0 U_{\gamma\alpha}. \quad (2.6)$$

These equations express the content of three-body collision theory in remarkably concise form.

With this formulation it is a trivial matter to generate the multiple-scattering series for any type of three-body process (elastic, rearrangement, and breakup). In fact, on iterating (2.6) we obtain immediately the multiple-scattering series

$$U_{\beta\alpha}=(1-\delta_{\beta\alpha})(s-H_0)+\sum_{\beta\neq\gamma\neq\alpha} T_\gamma + \sum_{\beta\neq\gamma\neq\delta\neq\alpha}\sum T_\gamma G_0 T_\delta+\cdots. \quad (2.7)$$

For future reference, we call the successive terms of the series zero order, first order, second order, and so on.

The multiple-scattering series (2.7) does not appear to be well known, especially for the rearrangement case ($\beta\neq\alpha$; β, $\alpha=1, 2, 3$), therefore, a brief discussion of the physical interpretation of the various terms is perhaps desirable. In the following, we set $s=E+i\epsilon$.

The zero-order term of (2.7) contributes only to rearrangement collisions, for example,

$$U_{21}=(E-H_0)+\cdots. \quad (2.8)$$

[Note that in the breakup case the on-shell matrix element of $(E-H_0)$ is zero.] This term gives in fact the Born approximation for the pickup process,[12] in which the incident particle 1 collides with particle 3, and forms a bound state with it, leaving particle 2 free; for it follows from (2.3b) that the zero-order contribution to

[8] I. H. Sloan, Phys. Rev. **165**, 1587 (1968).
[9] R. W. Finkel and L. Rosenberg, Phys. Rev. **168**, 1841 (1968).
[10] C. Lovelace, in *Strong Interactions and High Energy Physics*, edited by R. G. Moorehouse (Oliver and Boyd, Ltd., London, 1964).
[11] E. O. Alt, P. Grassberger, and W. Sandhas, Nucl. Phys. **B2**, 167 (1967).
[12] G. F. Chew and M. L. Goldberger, Phys. Rev. **77**, 470 (1950).

the amplitude may be written as

$$(\Phi_2', (E-H_0)\Phi_1) = (\Phi_2', V_{13}\Phi_1). \quad (2.9)$$

From the physical interpretation, it follows that the zero-order contribution is peaked in the backward direction in the c.m. system.

Written out to first order, typical elastic, breakup, and rearrangement transition operators are

$$U_{11} = T_{12} + T_{13} + \cdots, \quad (2.10)$$

$$U_{01} = T_{12} + T_{13} + \cdots, \quad (2.11)$$

$$U_{21} = (E-H_0) + T_{12} + \cdots, \quad (2.12)$$

where for ease of interpretation we have used the alternative notation $T_3 = T_{12}$, etc. For the first two cases, the physical interpretation is clearly that the incident particle is scattered by one or other of the bound particles; this is of course the impulse approximation.[1] For the rearrangement case, the first-order term may be interpreted as a "knock-on" term, in which the incident particle 1 strikes the target particle 2 and ejects it, itself remaining behind to form a bound state with the third particle. The first-order terms may be expected to be peaked in the forward direction.

Higher-order terms are of course interpreted as multiple-scattering terms. The prescription is that only those multiple-scattering events are allowed in which successive scatterings occur between different pairs of particles, and in which the first (last) scattered pair includes the initially (finally) free particle. Queen[13] has argued that higher-order terms are increasingly isotropic, since after several scatterings the incident particle "forgets" its original direction. This conclusion is borne out strongly by our calculations.

3. MODEL

Our model is the separable-potential model of Aaron, Amado, and Yam,[7,14] of three identical nucleons interacting through spin- and isospin-dependent, s-wave, separable, two-body interactions. Thus the two-body potential is[15]

$$\langle \mathbf{q}' \mid V \mid \mathbf{q} \rangle = -\sum_{n=0}^{1} \lambda_n g_n(q') g_n(q) P_n, \quad (3.1)$$

where P_0 is the spin-isospin projection operator for the deuteron (spin 1, isospin 0), and P_1 the corresponding projection operator for the singlet state (spin 0, isospin 1). The form factors $g_n(q)$ are of the Yamaguchi[16] form,

$$g_n(q) = N_n/(q^2+\beta_n^2). \quad (3.2)$$

For the present we use units in which $\hbar = m = 1$, where m is the nucleon mass.

The triplet potential produces a bound state, the deuteron, with the wave function[16]

$$\phi_0(\mathbf{q}) = -g_0(q)/(q^2+\alpha_0^2), \quad (3.3)$$

and with binding energy $b_0 = \alpha_0^2$. We note the relation

$$g_0(q) = \langle \mathbf{q} \mid V \mid \phi_0 \rangle, \quad (3.4)$$

which follows from (3.3) and the Schrödinger equation.

On choosing N_0 to normalize the wave function, we have

$$N_0 = \pi^{-1}[\alpha_0\beta_0(\beta_0+\alpha_0)^3]^{1/2}; \quad (3.5)$$

then by fixing λ_0 to give the correct binding energy, we obtain

$$\lambda_0 = [\alpha_0(\beta_0+\alpha_0)]^{-1}. \quad (3.6)$$

It is convenient to define the singlet parameters in a similar way. Thus we introduce α_1, and define

$$N_1 = \pi^{-1}[\mid \alpha_1 \mid \beta_1(\beta_1+\alpha_1)^3]^{1/2}, \quad (3.7)$$

$$\lambda_1 = [\mid \alpha_1 \mid (\beta_1+\alpha_1)]^{-1}. \quad (3.8)$$

Since the singlet potential does not produce a bound state, it follows that $\alpha_1 < 0$.

Then the two-body scattering amplitude at energy $e+i\epsilon$ is[17]

$$\langle \mathbf{q}' \mid t(e+i\epsilon) \mid \mathbf{q} \rangle = \sum_{n=0}^{1} g_n(q') F_n(e) g_n(q) P_n, \quad (3.9)$$

where

$$F_n(e) = -\lambda_n[1-(\beta_n+\alpha_n)^2/(\beta_n-i\kappa)^2]^{-1}, \quad (3.10)$$

$$e = \kappa^2, \qquad \arg\kappa = 0 \quad \text{or} \quad \tfrac{1}{2}\pi.$$

We see that $F_0(e)$ exhibits the deuteron pole at $\kappa = i\alpha_0$ or $e = -b_0$, whereas the pole in $F_1(e)$, at $\kappa = i\alpha_1 = -i \mid \alpha_1 \mid$, is on the second sheet. The parameters α_n and β_n were fixed by fitting the scattering length and binding energy in the triplet case, and the scattering length and effective range in the singlet case, using the same data as Ref. 7.

We now turn to the three-body problem, restricting attention from here on to elastic and rearrangement processes. Since the total spin and total isospin are both conserved in this model, we work with states of definite spin ($S = \frac{1}{2}$ for the doublet case, $S = \frac{3}{2}$ for the quartet case), definite z-component of spin, isospin $T = \frac{1}{2}$, and third component of isospin $T_3 = -\frac{1}{2}$. It is convenient to denote by $\chi_{\gamma n}$ the unique three-body spin-isospin function with the above quantum numbers, for which the pair γ has the quantum numbers of the state n ($n=0$ for the deuteron, $n=1$ for the "singlet deuteron").

Three-body states in momentum space may be specified by the momentum pair $\mathbf{p}_1$ and $\mathbf{q}_1$ (or by $\mathbf{p}_2$ and $\mathbf{q}_2$, etc.), where $\mathbf{p}_1$ is the momentum of particle 1 in the c.m. system, and $\mathbf{q}_1$ is the momentum of particle 2 in the c.m. of particles 2 and 3. In this representation,

[13] N. M. Queen, Nucl. Phys. **55**, 177 (1964).

[14] Since our approach is through a potential, the model discussed here is that of Ref. 7 with the "wave-function renormalization constant" set equal to zero.

[15] Momentum states have the normalization $\langle \mathbf{q}' \mid \mathbf{q} \rangle = \delta(\mathbf{q}'-\mathbf{q})$.

[16] Y. Yamaguchi, Phys. Rev. **95**, 1628 (1954).

[17] C. Lovelace, Phys. Rev. **135**, B1225 (1964).

the matrix elements of T_γ [Eq. (2.4)] are

$$\langle \mathbf{p}_\gamma' \mathbf{q}_\gamma' \mid T_\gamma(s) \mid \mathbf{p}_\gamma \mathbf{q}_\gamma \rangle = \delta(\mathbf{p}_\gamma' - \mathbf{p}_\gamma) \langle \mathbf{q}_\gamma' \mid t_\gamma(s - \tfrac{3}{4} p_\gamma^2) \mid \mathbf{q}_\gamma \rangle. \quad (3.11)$$

We use the concise notation of Lovelace,[17] and define a ket vector $\mid \gamma n \rangle$, through

$$\langle \mathbf{q}_\gamma \mid \gamma n \rangle = \chi_{\gamma n} g_n(\mathbf{q}_\gamma). \quad (3.12)$$

Then it follows from (3.11) and (3.9) that

$$T_\gamma(s) = \sum_{n=0}^{1} \mid \gamma n \rangle \hat{\tau}_n(s) \langle \gamma n \mid, \quad (3.13)$$

where $\hat{\tau}_n$ is an operator in a two-body Hilbert space, with matrix elements

$$\langle \mathbf{p}' \mid \hat{\tau}_n(s) \mid \mathbf{p} \rangle = \delta(\mathbf{p}' - \mathbf{p}) F_n(s - \tfrac{3}{4} p^2). \quad (3.14)$$

The Lovelace[17] equations for the three-body amplitudes can now be readily derived[11] from the integral equation (2.6). On substituting (3.13) into (2.6), and multiplying on the right by $G_0 \mid \alpha n \rangle$, and on the left by $\langle \beta n' \mid G_0$, we immediately obtain the Lovelace equations

$$\hat{X}_{\beta n', \alpha n} = \hat{Z}_{\beta n', \alpha n} + \sum_\gamma \sum_{n''=0}^{1} \hat{Z}_{\beta n', \gamma n''} \hat{\tau}_{n''} \hat{X}_{\gamma n'', \alpha n}, \quad (3.15)$$

where

$$\hat{X}_{\beta n', \alpha n} = \langle \beta n' \mid G_0 U_{\beta\alpha} G_0 \mid \alpha n \rangle \quad (3.16)$$

and

$$\hat{Z}_{\beta n', \alpha n} = (1 - \delta_{\beta\alpha}) \langle \beta n' \mid G_0 \mid \alpha n \rangle. \quad (3.17)$$

Both $\hat{X}$ and $\hat{Z}$ are operators on plane-wave states $\mid \mathbf{p} \rangle$. If $\mathbf{p}$ is the momentum of the incident nucleon, then it follows from (2.5), (3.12), and (3.3) that

$$G_0 \mid \alpha 0 \rangle \mid \mathbf{p} \rangle = \Phi_{\alpha \mathbf{p}}, \quad (3.18)$$

where $\Phi_{\alpha \mathbf{p}}$ is the initial unperturbed state of the system. Thus if $\mathbf{p}$ and $\mathbf{p}'$ satisfy the energy-conservation condition,

$$E = \tfrac{3}{4} p^2 - \alpha_0^2 = \tfrac{3}{4} p'^2 - \alpha_0^2, \quad (3.19)$$

then

$$\langle \mathbf{p}' \mid \hat{X}_{\beta 0, \alpha 0} \mid \mathbf{p} \rangle = (\Phi_{\beta \mathbf{p}'}, U_{\beta\alpha} \Phi_{\alpha \mathbf{p}}). \quad (3.20)$$

In words, the on-shell matrix elements of $\hat{X}_{\beta 0, \alpha 0}$ are the physical amplitudes. Similarly,

$$\langle \mathbf{p}' \mid \hat{Z}_{\beta 0, \alpha 0} \mid \mathbf{p} \rangle = (\Phi_{\beta \mathbf{p}'}, (E - H_0) \Phi_{\alpha \mathbf{p}}), \qquad (\beta \neq \alpha) \quad (3.21)$$

so that the on-shell matrix elements of $\hat{Z}_{\beta 0, \alpha 0}$ are the zero-order (pickup) amplitudes. We note that the Eqs. (3.15) have exactly the same structure as the original equations (2.6), and, in particular, that the terms of given order in their respective iterations precisely correspond. Our purpose in following through the derivation is in fact to make this point clear.

Since the particles are identical fermions, the Eqs. (3.15) may easily be uncoupled.[17] We define

$$\hat{X}_{n', n} = \hat{X}_{1n', 1n} + \hat{X}_{2n', 1n} + \hat{X}_{3n', 1n} \quad (3.22)$$

and

$$\hat{Z}_{n', n} = 2 \hat{Z}_{2n', 1n}. \quad (3.23)$$

Then on summing over β in Eq. (3.15), and using the particle identity, we obtain

$$\hat{X}_{n', n} = \hat{Z}_{n', n} + \sum_{n''=0}^{1} \hat{Z}_{n', n''} \hat{\tau}_{n''} \hat{X}_{n'', n}, \quad (3.24)$$

which we may write as the explicit coupled integral equations

$$\langle \mathbf{p}' \mid \hat{X}_{n', n} \mid \mathbf{p} \rangle = \langle \mathbf{p}' \mid \hat{Z}_{n', n} \mid \mathbf{p} \rangle + \sum_{n''=0}^{1} \int d\mathbf{p}'' \times \langle \mathbf{p}' \mid \hat{Z}_{n', n''} \mid \mathbf{p}'' \rangle F_{n''}(s - \tfrac{3}{4} p''^2) \langle \mathbf{p}'' \mid \hat{X}_{n'', n} \mid \mathbf{p} \rangle. \quad (3.25)$$

The on-shell matrix elements of $\hat{X}_{0,0}$ are the fully antisymmetrized elastic-scattering amplitudes.

From (3.23), (3.17), and (3.12), the matrix elements of $\hat{Z}_{n', n}$ are

$$\langle \mathbf{p}' \mid \hat{Z}_{n', n} \mid \mathbf{p} \rangle = 2 J_{n'n} \frac{g_{n'}(\mid \tfrac{1}{2}\mathbf{p}' + \mathbf{p} \mid) g_n(\mid \tfrac{1}{2}\mathbf{p} + \mathbf{p}' \mid)}{s - p'^2 - \mathbf{p}' \cdot \mathbf{p} - p^2}, \quad (3.26)$$

where $J_{n'n}$ is the spin-isospin factor,

$$J_{n'n} = \chi_{2n'}^\dagger \chi_{1n}. \quad (3.27)$$

It has the values[17] $J_{00} = -\frac{1}{2}$, $J_{01} = J_{10} = J_{11} = 0$ for the quartet case, and $J_{00} = J_{11} = \frac{1}{4}$, $J_{01} = J_{10} = -\frac{3}{4}$ for the doublet case.

4. METHOD OF SOLUTION

The coupled integral equations (3.25) may be reduced to one-dimensional equations by the partial-wave expansion

$$\langle \mathbf{p}' \mid \hat{X}_{n', n} \mid \mathbf{p} \rangle = -\frac{3}{8\pi^2 k} \sum_{L=0}^{\infty} (2L+1) \times \langle p' n' \mid T_L \mid p n \rangle P_L(\hat{\mathbf{p}}' \cdot \hat{\mathbf{p}}), \quad (4.1)$$

where k is the initial momentum in the c.m. system,

$$E = \tfrac{3}{4} k^2 - \alpha_0^2.$$

We denote the on-shell partial-wave amplitude by T_L,

$$T_L = \langle k0 \mid T_L \mid k0 \rangle. \quad (4.2)$$

With the normalization given by (4.1), the partial-wave elastic unitarity relation is

$$\mathrm{Im} T_L \geq \mid T_L \mid^2, \quad (4.3)$$

corresponding to the obvious fact that the total cross section exceeds the elastic cross section.[18]

Similarly, a partial-wave expansion of the amplitudes $\langle \mathbf{p}' \mid \hat{Z}_{n', n} \mid \mathbf{p} \rangle$ [Eq. (3.26)] may be defined. The corresponding partial-wave amplitudes can be expressed in

[18] I. H. Sloan, Phys. Rev. **162**, 855 (1967).

closed form,[19] in terms of Legendre functions of the second kind. We omit the details here.

The method of exact solution of the one-dimensional analogs of (3.25) has been thoroughly discussed by Aaron and Amado.[20] The rightmost momentum is set equal to k, and n is set equal to 0, and the integration contour with respect to p'' is deformed, so as to avoid the singularities in the kernel. The contour is defined by $p''=|p''|e^{-i\phi}$, where ϕ is a phase angle; it must be chosen so that no singularities are crossed. Since we are concerned with integral equations, it is also necessary to deform the leftmost momentum p' along the same contour. Then numerical solution of the integral equation yields the values of $\langle p'n'|T_L|k0\rangle$, with $p'=|p'|e^{-i\phi}$. A further application of the integral equation is then required to obtain the on-shell amplitudes T_L.

Our main interest in this paper lies, of course, in the multiple-scattering series. In principle, the individual terms of the multiple-scattering series (2.7) are multidimensional integrals. In the present case, however, it is very much easier to generate the series by the method used in deriving the series, namely, by iteration of the integral equation. As we have already remarked, the integral equation (3.25) has precisely the same structure as the original equation (2.6), and in particular, the iterations of one correspond exactly to the iterations of the other. The great advantage of generating the series iteratively is that there is no limit, apart from numerical accuracy, to the order of multiple scattering that can be studied.

There is a technical point that requires some caution. The iterative solutions, like the exact solutions, were generated initially along the deformed contour, then the integral equation was used again to obtain the on-shell amplitudes. It is important to realize that this second application of the integral equation is in effect an additional iteration of the integral equation, thus it increases the order of multiple scattering by one.

To obtain numerical solutions, integration with respect to p'' was cut off at a large value of p'', and the integrals were replaced by sums, using Gaussian integration rules. The exact results were obtained by solving the resulting simultaneous equations by Gaussian elimination. Careful tests were made with various cutoffs and various orders of Gaussian quadrature, to ensure the accuracy of the results. It was also established that the iterative solutions were not sensitive to rounding errors, even in the cases where the series diverges.

The exact results at 14.1 MeV should agree with the corresponding results of Ref. 7. In fact, quite good agreement was obtained for $L>0$, but for $L=0$ the amplitudes were found to differ by the order of 10%. The earlier calculations did not use the contour deformation method, and it seems likely that the discrepancy arises from numerical inaccuracy in those calculations.[21]

A useful check on the calculations was obtained by evaluating the first-order (impulse) terms by explicit three-dimensional integration, and then deducing the partial-wave first-order amplitudes by numerical partial-wave analysis.[18]

In the remainder of this section, we describe the technique used to analyze the convergence of the partial-wave multiple-scattering series. We write the partial-wave equivalent of the integral equation (3.25) symbolically as

$$T(s)=B(s)+K(s)T(s).$$

Then the corresponding multiple-scattering series is

$$T=\sum_{m=0}^{\infty}K^mB,$$

and the series for the on-shell partial-wave amplitude T_L [Eq. (4.2)] is

$$T_L=\sum_{m=0}^{\infty}T_{L,m}, \tag{4.4}$$

with

$$T_{L,m}=\langle k0|K^mB|k0\rangle,$$

and $s=E+i\epsilon$.

The kernel $K(s)$ is compact for any s not on the scattering cut, and therefore has a discrete spectrum.[22] If s is real and less than $-b_0$, the kernel becomes Hermitian after a trivial similarity transformation, and therefore has a complete set of eigenfunctions. Under certain reasonable assumptions, the completeness property can be analytically continued to any s in the cut plane,[22] and in particular to $s=E+i\epsilon$.

Let the eigenfunctions and the corresponding eigenvalues of $K(E+i\epsilon)$ be denoted by $|\eta_i\rangle$ and λ_i, with $i=1,2,\cdots$. Using the completeness property, we can write

$$B|k0\rangle=\sum_i a_i|\eta_i\rangle,$$

so that $T_{L,m}$ becomes

$$T_{L,m}=\sum_i \lambda_i{}^m a_i\langle k0|\eta_i\rangle. \tag{4.5}$$

On substituting (4.5) into the multiple-scattering series (4.4) we immediately obtain the result, well known for the Born series,[22] that the series converges if and only if $|\lambda_1|<1$, where λ_1 is the eigenvalue with largest absolute value. This result was used to provide a sensitive test of the convergence or divergence of the series.

The above argument can also be used to actually

[19] R. Aaron, R. D. Amado, and Y. Y. Yam, Phys. Rev. **136**, B650 (1964).

[20] R. Aaron and R. D. Amado, Phys. Rev. **150**, 857 (1966).

[21] The inherent difficulty in obtaining accurate solutions with the method of Ref. 7 is clearly shown by Fig. 5 of Ref. 20. See also the comment at the end of p. 861, Ref. 20.

[22] R. G. Newton, *Scattering Theory of Waves and Particles* (McGraw-Hill Book Co., New York, 1966).

TABLE I. Doublet partial-wave amplitudes.

		Multiple scattering series			
E (MeV)	L	0	1	2	Exact
14.1	0	0.593	$3.094+2.147i$	$0.858+5.860i$	$-0.220+0.573i$
	1	−0.211	$0.100+0.272i$	$0.185+0.173i$	$0.151+0.189i$
	2	0.073	$0.110+0.036i$	$0.106+0.037i$	$0.106+0.037i$
	3	−0.026	$-0.021+0.005i$		$-0.021+0.005i$
	4	0.009	$0.010+0.001i$		$0.010+0.001i$
	5	−0.004			$-0.004+0.000i$
50	0	0.236	$1.341+0.923i$	$0.149+1.795i$	$0.192+0.501i$
	1	−0.120	$0.184+0.264i$	$0.295+0.228i$	$0.265+0.214i$
	2	0.054	$0.129+0.070i$	$0.124+0.068i$	$0.123+0.067i$
	3	−0.023	$-0.004+0.019i$		$-0.004+0.019i$
	4	0.010	$0.015+0.006i$		$0.015+0.006i$
	5	−0.004	$-0.003+0.002i$		$-0.003+0.002i$
	6	0.002	$0.002+0.001i$		$0.002+0.001i$
100	0	0.112	$0.684+0.357i$	$0.194+0.654i$	$0.234+0.268i$
	1	−0.066	$0.157+0.143i$	$0.227+0.130i$	$0.215+0.116i$
	2	0.034	$0.112+0.051i$	$0.107+0.049i$	$0.107+0.048i$
	3	−0.016	$0.011+0.018i$		$0.011+0.018i$
	4	0.008	$0.018+0.007i$		$0.018+0.007i$
	5	−0.004	$0.000+0.003i$		$0.000+0.003i$
	6	0.002	$0.003+0.001i$		$0.003+0.001i$

TABLE II. Quartet partial-wave amplitudes.

		Multiple scattering series			
E (MeV)	L	0	1	2	Exact
14.1	0	−1.187	$-0.625+2.123i$	$1.543+1.183i$	$0.289+0.895i$
	1	0.422	$0.468+0.260i$	$0.406+0.273i$	$0.392+0.263i$
	2	−0.146	$-0.145+0.032i$	$-0.144+0.033i$	$-0.144+0.033i$
	3	0.052	$0.051+0.004i$		$0.051+0.004i$
	4	−0.019	$-0.019+0.001i$		$-0.019+0.001i$
50	0	−0.472	$-0.041+0.806i$	$0.705+0.613i$	$0.425+0.379i$
	1	0.240	$0.345+0.218i$	$0.290+0.213i$	$0.287+0.202i$
	2	−0.108	$-0.085+0.054i$	$-0.084+0.056i$	$-0.084+0.056i$
	3	0.047	$0.052+0.014i$		$0.052+0.014i$
	4	−0.020	$-0.019+0.004i$		$-0.019+0.004i$
	5	0.009	$0.009+0.001i$		$0.009+0.001i$
	6	−0.004			$-0.004+0.000i$
100	0	−0.224	$0.064+0.329i$	$0.360+0.251i$	$0.285+0.147i$
	1	0.133	$0.238+0.127i$	$0.204+0.122i$	$0.204+0.116i$
	2	−0.068	$-0.033+0.043i$	$-0.031+0.045i$	$-0.032+0.045i$
	3	0.033	$0.045+0.015i$		$0.045+0.015i$
	4	−0.015	$-0.011+0.006i$		$-0.011+0.006i$
	5	0.007	$0.009+0.002i$		$0.009+0.002i$
	6	−0.003	$-0.003+0.001i$		$-0.003+0.001i$

calculate λ_1: It follows from Eq. (4.5) that if $a_1 \neq 0$, then

$$\lambda_1 = \lim_{m\to\infty} T_{L,m+1}/T_{L,m}.$$

That is, the largest eigenvalue λ_1 is simply the limiting ratio of successive terms in the multiple-scattering series. This method is in fact exactly equivalent to the power method for finding the largest eigenvalue of a matrix.[23] We also used the δ^2 process[23] to accelerate the convergence to the value λ_1, thus allowing λ_1 to be determined with fewer terms.

The rate of convergence of the series may be described by the number of multiple-scattering terms required to give amplitudes of specified accuracy, and results of this sort are given in Sec. 5. For a slowly convergent series, this information can be deduced from just the early terms of the series, since successive terms quickly settle down to the constant ratio λ_1.

5. RESULTS

We consider first the partial-wave results. In Tables I and II, we show the exact results at three energies, and also the results from the truncated multiple-scattering series of orders 0, 1, and 2. For visual clarity, the second-order results have been omitted where they agree to three decimal places with the first-order results.

It is immediately clear that the multiple-scattering series converges rapidly for the higher partial waves, and that for $L \geq 2$ quite good accuracy is obtained if the series is truncated at first order (giving the impulse-pickup approximation). For $L=1$, on the other hand, it is clear that the convergence is much slower, while for $L=0$ it is not at all clear that the second-order approximation is better than the first-order, except perhaps at the highest energy.

TABLE III. Convergence of the multiple-scattering series. The series converges if and only if $|\lambda_1| < 1$, where λ_1 is the largest eigenvalue of the kernel. Terms of the series up to order n are required if the multiple-scattering amplitudes are to be accurate to within 10%.

		Doublet		Quartet	
E (MeV)	L	$\lvert\lambda_1\rvert$	n	$\lvert\lambda_1\rvert$	n
14.1	0	1.47	∞	0.944	56
	1	0.46	3	0.29	2
	2	$\approx$0.1	1	$\approx$0.1	1
50	0	0.990	$\approx$300	0.63	6
	1	0.38	2	0.24	2
	2	$\approx$0.1	1	$\approx$0.1	1
100	0	0.74	10	0.48	4
	1	$\approx$0.3	2	$\approx$0.2	2
	2	$\approx$0.1	1	$\approx$0.1	1

[23] A. Ralston, *A First Course in Numerical Analysis* (McGraw-Hill Book Co., New York, 1965).

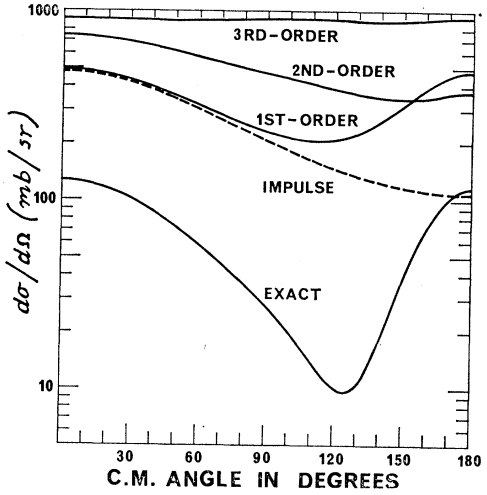

FIG. 1. Differential cross sections at 14.1 MeV. At this energy the multiple-scattering series diverges, because of the divergence of the doublet S series.

Table III contains further information on the rate of convergence of the multiple-scattering series for low partial waves. The most striking aspect of the table is the extremely slow rate of convergence of the doublet S series. At 14.1 MeV, it does not converge at all, and even at 100 MeV the convergence is so slow that multiple-scattering terms up to the tenth order are needed to give partial-wave amplitudes with 10% accuracy. The divergence at low energies is associated with the existence of a doublet S bound state, the triton. The quartet S series converges more rapidly, but the convergence is still slow, especially at the lowest energy. Table III confirms that the higher-order terms become increasingly isotropic, since the series converge rapidly for higher partial waves.

The results serve as a corrective to the belief, often implied if not stated, that the multiple-scattering series necessarily converges, and that it converges fairly rapidly. (The belief shows, for example, in the notion that second-order terms necessarily improve the first-order result.)

The results of Tables I and II, and similar results for higher orders of multiple scattering, have been combined to give differential cross sections, giving the graphs shown in Figs. 1–3. (More significant figures were carried than are shown in the table, and the tables were supplemented with higher partial-wave amplitudes calculated in the first-order or zero-order approximation.) We also show the results of the impulse approximation, because of the widespread interest in it. The impulse-approximation amplitudes were obtained by subtracting the zero-order amplitudes from the first-order (or, as noted in Sec. 4, by direct calculation).[24]

[24] It should be noted that the impulse approximation used here is that with fully off-shell antisymmetrized two-body amplitudes. In most practical calculations these amplitudes are approximated by their on-shell values.

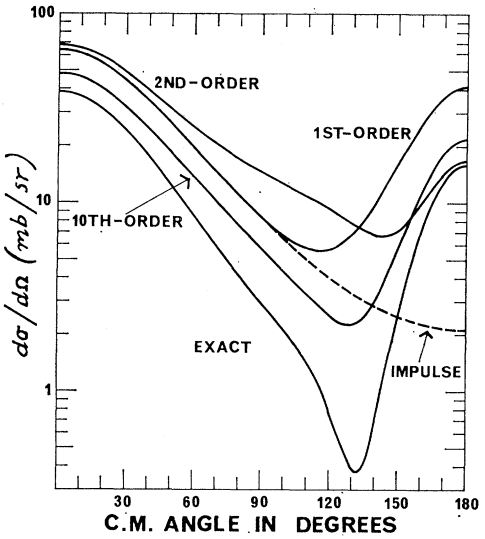

FIG. 2. Differential cross sections at 50 MeV. At this energy the multiple-scattering series converges, but the convergence is extremely slow, some hundreds of terms being required for accurate cross sections.

The zero-order (pickup) approximation is not shown, but the pickup contribution is almost entirely responsible for the backward peak in the first-order amplitudes, since the impulse term is small in backward directions. In the forward direction, the pickup contribution is small, so that the impulse and first-order curves closely agree.

From the figures, it is clear that the first-order approximation greatly improves as the energy increases, but that nevertheless a considerable discrepancy remains even at 100 MeV, especially near the deep minimum at 130°. The error is smallest in the forward direction, but still not negligible (24% at 100 MeV). It has been pointed out previously[25] that the impulse approximation appears to give significantly larger values than experiment at angles near 30°, in spite of the frequent contention that the impulse approximation is accurate in forward directions. Our results generally confirm the conclusions of Ref. 25.

Although it cannot be seriously maintained that the present model is at all realistic, nevertheless a brief comparison with the experimental differential cross sections, summarized by Bunker *et al.*,[26] is of some interest. The exact results in Figs. 1–3 agree quite well with the experimental results near the forward and backward directions.[27] Furthermore, the experimental cross sections from 20 to 100 MeV are all characterized by minima near 130°, and it can be seen that this feature is well reproduced by the model. However, at energies above 20 MeV the experimental minimum becomes progressively weaker, instead of strengthening as is the case with the model. At 50 MeV, for example, the calculated value at the minimum is about one-third of the experimental value. A possible source of the deviation is the increasing importance of higher partial waves in the two-body amplitudes as the energy increases.

To conclude this section, we remark that multiple-scattering approximations might be more effective for scattering by heavier nucleons than for deuterons,[28] provided that the target nucleus can reasonably be treated as nucleon plus core, and provided that the strong distortion by the core is treated explicitly. The possibility arises because of the strong absorption of low partial waves by the core, whereas it is precisely the low partial waves that cause the poor results for scattering by deuterons.

6. UNITARY FIRST-ORDER APPROXIMATION

The approximations derived by truncating the multiple-scattering series at first or second order have the disadvantage that the constraint of unitarity may be violated. For example, it is easy to see the first-order doublet S amplitudes in Table I all violate the con-

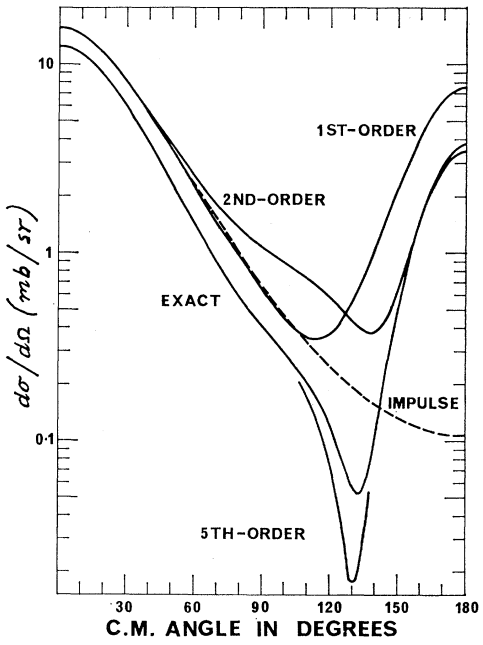

FIG. 3. Differential cross sections at 100 MeV. At this energy the multiple-scattering series slowly converges. To avoid a confused diagram, the fifth-order curve is shown only in the region of the minimum; at 0° and 180° it exceeds the exact result by 7.5 and 5%, respectively.

[25] H. Kottler and K. L. Kowalski, Phys. Rev. **138**, B619 (1965).
[26] S. N. Bunker, J. M. Cameron, R. F. Carlson, J. R. Richardson, P. Tomaš, W. T. H. Van Oers, and J. W. Verba, Nucl. Phys. **A113**, 461 (1968).
[27] In comparing the present results with experimental proton-deuteron results, the small-angle Coulomb interference region in the *p-d* curves is to be ignored.
[28] I am indebted to Professor I. E. McCarthy for this observation.

TABLE IV. Partial-wave amplitudes in the unitary first-order approximation. The ordinary first-order amplitudes and the exact amplitudes are shown for comparison.

E (MeV)	L	Doublet First order (ordinary)	Doublet First order (unitary)	Doublet Exact	Quartet First order (ordinary)	Quartet First order (unitary)	Quartet Exact
14.1	0	$3.094+2.147i$	$0.178+0.839i$	$-0.220+0.573i$	$-0.625+2.123i$	$-0.188+0.485i$	$0.289+0.895i$
	1	$0.100+0.272i$	$0.066+0.190i$	$0.151+0.189i$	$0.468+0.260i$	$0.337+0.222i$	$0.392+0.263i$
	2	$0.110+0.036i$	$0.102+0.041i$	$0.106+0.037i$	$-0.145+0.032i$	$-0.139+0.031i$	$-0.144+0.033i$
50	0	$1.341+0.923i$	$0.254+0.647i$	$0.192+0.501i$	$-0.041+0.806i$	$-0.016+0.369i$	$0.425+0.379i$
	1	$0.184+0.264i$	$0.115+0.217i$	$0.265+0.214i$	$0.345+0.218i$	$0.235+0.208i$	$0.287+0.202i$
	2	$0.129+0.070i$	$0.112+0.076i$	$0.123+0.067i$	$-0.085+0.054i$	$-0.078+0.047i$	$-0.084+0.056i$
	3	$-0.004+0.019i$	$-0.004+0.018i$	$-0.004+0.019i$	$0.052+0.014i$	$0.051+0.014i$	$0.052+0.014i$
100	0	$0.684+0.357i$	$0.301+0.409i$	$0.234+0.268i$	$0.064+0.329i$	$0.039+0.220i$	$0.285+0.147i$
	1	$0.157+0.143i$	$0.119+0.138i$	$0.215+0.116i$	$0.238+0.127i$	$0.185+0.138i$	$0.204+0.116i$
	2	$0.112+0.051i$	$0.100+0.058i$	$0.107+0.048i$	$-0.033+0.043i$	$-0.031+0.038i$	$-0.032+0.045i$
	3	$0.011+0.018i$	$0.010+0.018i$	$0.011+0.018i$	$0.045+0.015i$	$0.044+0.016i$	$0.045+0.015i$

straint (4.3), with the violation very large at the lower energies, and, in fact, this is the main reason why the first-order elastic cross sections are so large. We briefly describe an alternative first-order approximation[8,9,29] in which unitarity is always satisfied.

Using the notation of Ref. 8, we split the two-body transition operator into two parts,

$$t(s)=t^{(1)}(s)+t^{(2)}(s), \tag{6.1}$$

where $t^{(2)}$ is the δ-function contribution from the deuteron pole,

$$t^{(2)}(s)=-i\pi V\mid\phi_0\rangle\delta(s+b_0)\langle\phi_0\mid VP_0, \tag{6.2}$$

and

$$t^{(1)}(s)=t(s)-t^{(2)}(s). \tag{6.3}$$

Correspondingly, the operator T_γ has two parts,

$$T_\gamma(s)=T_\gamma^{(1)}(s)+T_\gamma^{(2)}(s). \tag{6.4}$$

An operator $U_{\beta\alpha}^{(1)}$ is introduced, defined by

$$U_{\beta\alpha}^{(1)}=(1-\delta_{\beta\alpha})(s-H_0)+\sum_{\beta\neq\gamma}T_\gamma^{(1)}G_0U_{\gamma\alpha}^{(1)} \tag{6.5}$$

$$=(1-\delta_{\beta\alpha})(s-H_0)+\sum_{\beta\neq\gamma\neq\alpha}T_\gamma^{(1)}+\cdots, \tag{6.6}$$

so that $U_{\beta\alpha}^{(1)}$ is analogous to $U_{\beta\alpha}$ [Eq. (2.6)], but with $T_\gamma^{(1)}$ replacing T_γ. Then it may be shown that $U_{\beta\alpha}$ satisfies

$$U_{\beta\alpha}=U_{\beta\alpha}^{(1)}+\sum_\gamma U_{\beta\gamma}^{(1)}G_0T_\gamma^{(2)}G_0U_{\gamma\alpha}. \tag{6.7}$$

[29] The approximation described here is that referred to as approximation I in Ref. 8.

It is easy to see, with the aid of (3.11) and (6.2), that (6.7) gives an equation for the physical (on-shell) amplitudes in terms of the corresponding on-shell amplitudes of $U_{\beta\alpha}^{(1)}$. In partial-wave form it becomes

$$T_L=T_L^{(1)}+iT_L^{(1)}T_L, \tag{6.8}$$

where $T_L^{(1)}$ is the on-shell partial-wave amplitude corresponding to $U_{\beta\alpha}^{(1)}$. This has, of course, the trivial solution

$$T_L=T_L^{(1)}/(1-iT_L^{(1)}). \tag{6.9}$$

The unitary first-order approximation is obtained by truncating the series (6.6) at first order, and then solving (6.7) exactly with the aid of (6.9). It may be shown[8] that in this approximation the unitary constraint is always satisfied. In fact, in partial-wave form the unitary condition (4.3) follows immediately from (6.9), and from the fact that $\mathrm{Im}T_L^{(1)}\geq 0$.[18] The amplitudes $T_L^{(1)}$ in this approximation are closely related to the ordinary multiple-scattering amplitudes of Tables I and II: Let $T_{L,0}$ and $T_{L,1}$ be the zero-order and first-order amplitudes from the ordinary multiple-scattering series, then

$$T_L^{(1)}=T_{L,1}-iT_{L,0}^2. \tag{6.10}$$

The second term in (6.10) is the contribution to the first-order integral from the δ function at the deuteron pole.

In Table IV, we show the partial-wave amplitudes in the unitary first-order approximation, and compare them with the exact amplitudes and the ordinary first-order amplitudes from Tables I and II. The doublet S amplitudes are seen to be greatly improved by the

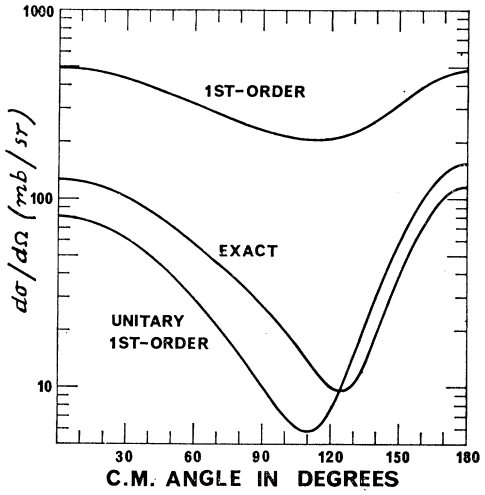

FIG. 4. Differential cross sections at 14.1 MeV in the unitary first-order approximation, compared with the ordinary first-order approximation.

unitary correction. For the quartet S case, and for both P-wave cases, it is not so clear that the unitary corrections give a systematic improvement, while for higher partial waves there is a definite tendency for the unitary correction to make matters worse, though the difference is not very large for these cases. This trend for higher partial waves is also apparent in the K^-—d results[4] analyzed in Ref. 9.

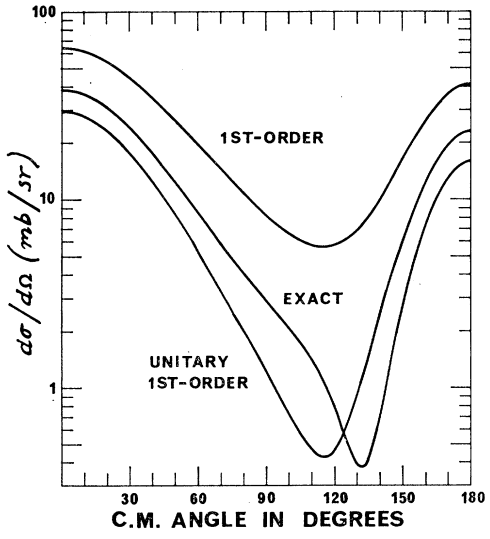

FIG. 5. Differential cross sections at 50 MeV in the unitary first-order approximation, compared with the ordinary first-order approximation.

With a separable-potential model, the residual part of the two-body amplitude $t^{(1)}$ [Eq. (6.3)] is itself separable, and it is therefore possible to generate the modified multiple-scattering series for $U_{\beta\alpha}{}^{(1)}$ using the techniques already described in Sec. 4. We have done this, and have found that at the higher energies the modified series (6.6) converges markedly more slowly than the original series (2.7), especially for the high partial waves. It is now easy to understand the situation mentioned above, that the first-order amplitudes for higher partial waves are better without the unitary correction: In these cases the ordinary multiple-scattering series has already converged, whereas the modified series is still some way from convergence.

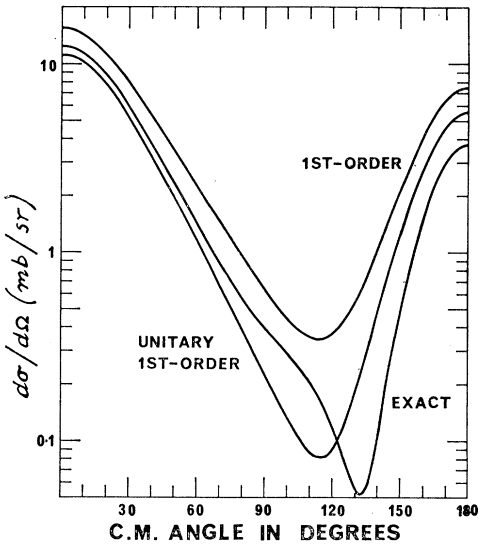

FIG. 6. Differential cross sections at 100 MeV in the unitary first-order approximation, compared with the ordinary first-order approximation.

In Figs. 4–6, we show the differential cross sections calculated in the unitary first-order approximation. It is clear that in spite of the shortcomings mentioned above the unitary corrections considerably improve the first-order results at all energies and almost all angles. The most striking aspect of the curves is that the unitary method reproduces the deep minima in the exact cross-section curves, though at somewhat too small an angle, and with somewhat different shape, whereas the ordinary first-order method fails badly in this region. Presumably, the minima should be interpreted as diffraction effects, and it is perhaps reasonable to conclude that the unitary method is quite successful in handling such effects.