Effective $\alpha+\alpha$ and $\alpha+N$ Potentials from Resonating-Group Calculations*†

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The method of the resonating-group structure is used to study the $\alpha+\alpha$ and $\alpha+N$ systems. With a nucleon-nucleon potential of a near-Serber exchange mixture, it is found that good agreement with the empirical phase-shift data can be obtained. Using the resultant radial scattering functions, effective potentials between the clusters are derived, which yield the same phase shifts as does the resonating-group calculation. These potentials are quite similar to the phenomenological potentials obtained by Ali and Bodmer, and Darriulat *et al.* in the $\alpha+\alpha$ case and by Swan and Pearce in the $\alpha+N$ case. They consist of a hard core with a radius which is weakly energy-dependent but strongly l-dependent and an attractive long-range part which has only a weak dependence on the relative orbital angular momentum and the energy in both even- and odd-l states. In addition, it is found that the long-range part has an odd-even feature which has previously been noted by Gammel and Thaler in their phenomenological study of the $\alpha+p$ scattering data, but has not been commonly considered in the usual local-potential approach to a scattering problem. Based on the experience here, an effective potential is also proposed for those cases where a straightforward application of the resonating-group method is impractical.

I. INTRODUCTION

PHENOMENOLOGICAL, effective $\alpha+N$,[1] $\alpha+\alpha$,[2–4] and $O^{16}+O^{16}$ [5,6] potentials have recently been constructed by various authors. In all these potentials, one of the common features is that they contain a short-range repulsive part. In the $\alpha+\alpha$ case, where a very careful phenomenological analysis[2] using the experimentally determined phase-shift data has been made, it was found in addition, that the repulsive part is angular-momentum-dependent (l-dependent), while the attractive long-range part can be made both l- and energy-independent in the low-energy region below reaction thresholds.

In a recent study,[7] the $He^3+\alpha$ elastic scattering problem was considered with the method of resonating-group structure which employs a completely antisymmetric wave function and a nucleon-nucleon potential. From this study, an effective potential between the He^3 and the α clusters was derived which, in fact, contains nearly all the features mentioned above for the phenomenological $\alpha+\alpha$ potential. In this calculation, we use the same method to study the $\alpha+\alpha$ and $\alpha+N$ problems. It is hoped that by considering various cases, where different composite particles are involved, a better understanding of the nature of the effective potential and the origin of the short-range repulsive core may be obtained.

Resonating-group calculations,[8] including the $\alpha+\alpha$ [9] and $\alpha+N$ cases,[10] have been carried out by a number of authors to obtain scattering phases; hence, only a brief description of the formulation will be given in Sec. II. In Sec. III, we present the results for the phase shifts in the $\alpha+\alpha$ and $\alpha+N$ cases; these results are somewhat different from those obtained previously,[9,10] which is due to the fact that in our present calculation a different cluster size or a different nucleon-nucleon potential has been chosen. In Sec. IV, a study of the effective potentials is presented; here also, we compare the resultant potentials with those determined phenomenologically and discuss features which should be present in a description of the interaction between two composite particles using local potentials. Finally, in Sec. V, we summarize the results of this investigation and discuss a possible extension to more complicated

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† A preliminary account of this work was presented at the American Physical Society, Miami Beach Meeting, 1968; Bull. Am. Phys. Soc. **13**, 1401 (1968).

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[7] R. E. Brown and Y. C. Tang, Phys. Rev. **176**, 1235 (1968).

[8] D. R. Thompson and Y. C. Tang, Phys. Rev. **159**, 806 (1967); Phys. Rev. Letters **19**, 87 (1967); Nucl. Phys. **A106**, 591 (1968); Phys. Letters **26B**, 194 (1968). References to earlier resonating-group calculations are contained therein.

[9] E. Van der Spuy, Nucl. Phys. **11**, 615 (1959); A. C. Butcher and J. M. McNamee, Proc. Phys. Soc. (London) **74**, 529 (1959); E. W. Schmid and K. Wildermuth, Nucl. Phys. **26**, 463 (1961); S. Okai and S. C. Park, Phys. Rev. **145**, 787 (1966); R. Tamagaki (to be published).

[10] S. Hochberg, H. S. W. Massey, and L. H. Underhill, Proc. Phys. Soc. (London) **A67**, 957 (1954); S. Hochberg, H. S. W. Massey, H. H. Robertson, and L. H. Underhill, *ibid.* **A68**, 746 (1955).

problems where a straightforward application of the resonating-group method is impractical.

II. BRIEF FORMULATION

The wave functions for the $\alpha+\alpha$ and $\alpha+N$ systems are assumed to be

$$\Psi_{\alpha\alpha}=\mathcal{A}[\phi_1\phi_2F(\mathbf{R}_1-\mathbf{R}_2)\xi(\sigma,\tau)] \quad (1)$$

and

$$\Psi_{\alpha N}=\mathcal{A}[\phi_1F(\mathbf{R}_1-\mathbf{r}_5)\xi(\sigma,\tau)], \quad (2)$$

respectively, where $\mathcal{A}$ is an antisymmetrization operator and ξ denotes the appropriate charge-spin function. The functions ϕ_1 and ϕ_2 describe the spatial behavior of the α clusters; they are given by

$$\phi_1=\exp[-\tfrac{1}{2}\alpha\sum_{i=1}^{4}(\mathbf{r}_i-\mathbf{R}_1)^2] \quad (3)$$

and

$$\phi_2=\exp[-\tfrac{1}{2}\alpha\sum_{i=5}^{8}(\mathbf{r}_i-\mathbf{R}_2)^2], \quad (4)$$

where $\mathbf{R}_1$ and $\mathbf{R}_2$ are the position vectors of the c.m. of the two α clusters, respectively. The function $F(\mathbf{r})$ describes the relative motion of the clusters and is determined from the variational principle

$$\delta\int\Psi^*(H-E')\Psi d\tau=0, \quad (5)$$

where H is the Hamiltonian of the system, and E' is the total energy.

The nucleon-nucleon potential is chosen as

$$V_{ij}=-V_0\exp(-\kappa r_{ij}^2)(w+mP_{ij}^r+bP_{ij}^\sigma-hP_{ij}^\tau)+(e^2/4r_{ij})(1+\tau_{iz})(1+\tau_{jz}), \quad (6)$$

with $V_0=72.98$ MeV, $\kappa=0.46$ F^{-2}, and the constants w, m, b, and h satisfying the equations

$$w+m+b+h=1 \quad (7)$$

and

$$w+m-b-h=0.63. \quad (8)$$

As in previous resonating-group calculations,[7,8] we have written V_{ij} as

$$V_{ij}=yV_{\text{Serber}}+(1-y)V_{\text{symmetric}}, \quad (9)$$

where V_{Serber} is given by Eq. (6) with $w=m$ and $b=h$, and $V_{\text{symmetric}}$ is given by Eq. (6) with $m=2b$ and $h=2w$.

Using Eq. (5), an integrodifferential equation of the form

$$\left[\frac{\hbar^2}{2\mu}\left(\frac{d^2}{dr^2}-\frac{l(l+1)}{r^2}\right)+E-V_D(r)-V_C(r)\right]f_l(r)=\int_0^\infty k_l(r,r')f_l(r')dr' \quad (10)$$

can be derived, with $f_l(r)$ defined by the equation

$$F(\mathbf{r})=\sum_l[f_l(r)/r]P_l(\cos\theta). \quad (11)$$

In Eq. (10), μ represents the reduced mass, while E represents the relative energy of the two clusters in the c.m. system.[11] The explicit forms of $V_D(r)$, $V_C(r)$, and $k_l(r,r')$ are given in the Appendix.[12]

By solving Eq. (10) with the proper boundary conditions, phase-shift values can be obtained. Using the resultant function $f_l(r)$, we can then construct an effective local E- and l-dependent potential

$$V_l^*(r)=V_D(r)+[f_l(r)]^{-1}\int_0^\infty k_l(r,r')f_l(r')dr' \quad (12)$$

between the clusters. Clearly, because of the particular way of construction, the potential V_l^* will yield the same values for δ_l as those calculated with the resonating-group method.

With the two-body potential of Eq. (6), it can be easily shown[13] that the binding energy of an α particle is much larger than that found experimentally, while the rms radius is much smaller. Evidently, this comes from the fact that this particular potential, being purely attractive in the relative s state, is too simple to give a detailed account of the features of a tightly bound system, such as the α particle. To remedy this situation, it is clear that one needs to use a more realistic potential, including at least a repulsive component.[14] This would, however, complicate the numerical computation immensely. In this calculation, therefore, we shall adopt the crude procedure of fixing the width parameter α of the internal functions ϕ_1 and ϕ_2 such that the experimentally determined value of 1.48 F for the α-particle rms radius[15] is given correctly. In this way, we obtain $\alpha=0.514$ F^{-2}. Using this value, the expectation value of the α-particle Hamiltonian obtained with the function ϕ_1 of Eq. (3) is -27.79 MeV, which is quite close to the experimental value of -28.3 MeV.

The use of a one-channel approximation in this study means that the specific distortion effect, i.e., the distortion effect over and above that already implicitly given by the antisymmetrization procedure, is not properly considered. To crudely compensate for this omission, we shall treat the quantity y in Eq. (9) as an adjustable parameter, with its optimum value determined from a best fit to the experimental data.

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[11] All energies will be in the c.m. system unless otherwise specified.

[12] Since the Coulomb interaction between two protons is long-ranged, the exchange contribution to $V_C(r)$ can be omitted for simplicity. This will cause a slight overestimate of y defined in Eq. (9) and prevent the occurrence of redundant solutions of the type discussed in Ref. 8.

[13] Y. C. Tang and R. C. Herndon, Nucl. Phys. **A93**, 692 (1967).

[14] Y. C. Tang, E. W. Schmid, and R. C. Herndon, Nucl. Phys. **65**, 203 (1965); I. R. Afnan and Y. C. Tang, Phys. Rev. **175**, 1337 (1968).

[15] R. F. Frosch, J. S. McCarthy, R. E. Rand, and M. R. Yearian, Phys. Rev. **160**, 874 (1967); Y. C. Tang and R. C. Herndon, Phys. Letters **18**, 42 (1965).

As has been discussed in previous calculations,[7,16] this is a reasonably valid procedure when the clusters involved have a low compressibility. Further, it should be noted that for the sake of simplifying the calculations, we have employed a nucleon-nucleon potential with no repulsive core and equal range in the triplet and singlet spin states. It is hoped that an adjustment in the value of y may also partially correct the defects caused by the adoption of such a simple potential.

We should mention here that there is a consistency check on the above-mentioned procedure of varying y. The value of y determined by the best-fit criterion should be close to 1, since it is known that the experimental two-nucleon scattering data favor a near-Serber exchange mixture for the nucleon-nucleon potential. If the resultant value for y should turn out to be quite different from 1, it would be a rather clear indication that this crude procedure is not accurate enough, and a better way of including the specific distortion effect, together with the adoption of a more realistic nucleon-nucleon potential, must be considered.

III. PHASE SHIFTS

A. $\alpha+\alpha$ System

Calculated phase shifts for $l=0$, 2, and 4 are shown as a function of E in Figs. 1–3, using $\alpha=0.514$ F^{-2} and various values of y. In these figures, the empirical phase shifts of Heydenburg and Temmer[17] in the energy range 0.2–1.5 MeV, of Tombrello and Senhouse[18] in the energy range 1.9–5.9 MeV, of Nilson *et al.*,[19] and of Werner and Zimmerer[20] in the energy range 5.0–11.5 MeV are also shown. From these figures, it can be seen that the optimum value of y depends somewhat upon the value of the relative orbital angular momentum, although a single value of y equal to about 0.95 does yield a reasonably good fit to the empirical data. Using the best-fit criterion, the y values for $l=0$, 2, and 4 are equal to 0.89, 0.92, and 1.03, respectively; these values are rather close to 1, indicating that the procedure of y variation is a fairly satisfactory one.

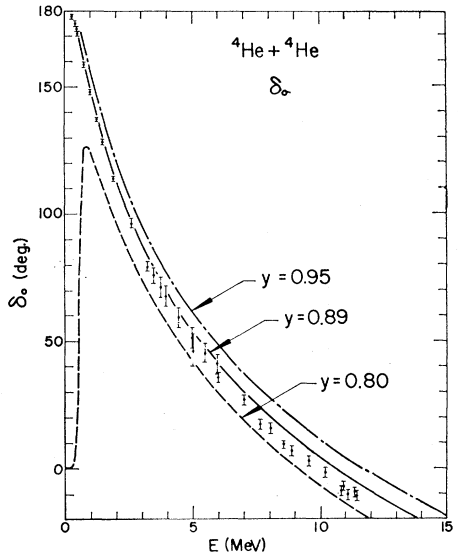

FIG. 1. Calculated phase shift δ_0 for $\alpha+\alpha$ scattering as a function of the c.m. energy. The experimental values are those given in Refs. 17–20.

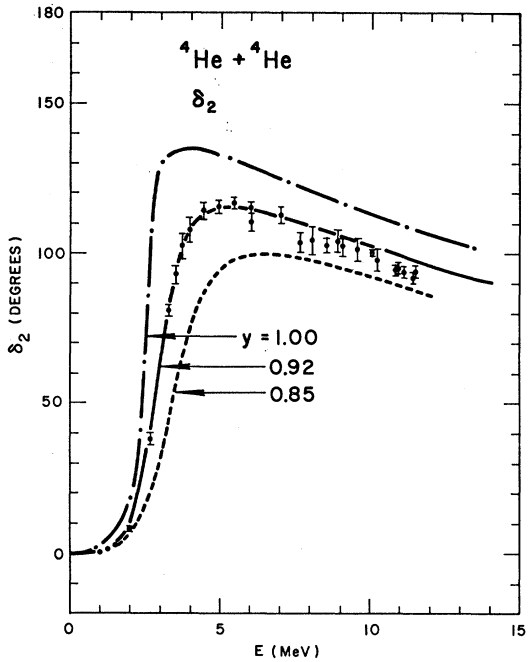

FIG. 2. Calculated phase shift δ_2 for $\alpha+\alpha$ scattering as a function of the c.m. energy. The experimental values are those given in Refs. 17–20.

Using $y=0.89$, the resonant energy in the $l=0$ state is 0.077 MeV,[21] which is quite close to the experimentally determined value of 0.092 MeV.[22] This shows that with a single value of y, the behavior of the δ_0 phases can be described satisfactorily from about 10 MeV to as low as 0.1 MeV. From this, we infer that the specific distortion effect in the case of $\alpha+\alpha$ scattering must be relatively unimportant, since it seems unlikely that this effect, if important, could be so energy-independent over such a wide energy range.

[16] D. R. Thompson and Y. C. Tang, Bull. Am. Phys. Soc. **13**, 99 (1968); Phys. Rev. **179**, 971 (1969).

[17] N. P. Heydenburg and G. M. Temmer, Phys. Rev. **104**, 123 (1956).

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[21] If y is changed by 0.01, the resonant energy is changed by 0.065 MeV.

[22] J. Benn, E. B. Dally, H. H. Muller, R. E. Pixley, H. H. Staub, and H. Winkler, Phys. Letters **20**, 43 (1966).

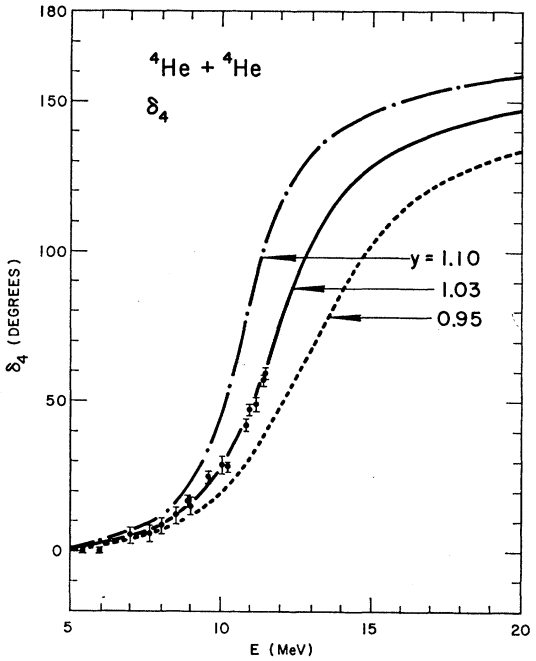

FIG. 3. Calculated phase shift δ_4 for $\alpha+\alpha$ scattering as a function of the c.m. energy. The experimental values are those given in Refs. 17–20.

Thus, the variation of the optimum value of y with l, mentioned in the previous paragraph, is probably not caused by the omission of the specific distortion effect, but can be attributed mainly to the fact that an overly simple two-nucleon potential has been used in our calculation.

Since the main concern of this investigation is to study the behavior of the effective potentials between composite particles, we shall not insist on using a single y value for all angular momentum states, as was done in previous resonating-group calculations,[7,8,16] but shall use the optimum value for each l state in the following discussions.

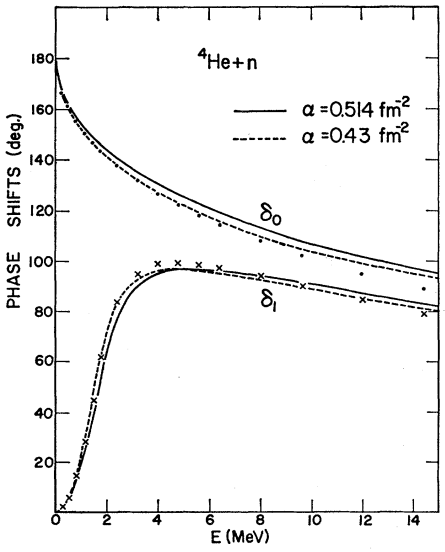

FIG. 4. Calculated phase shifts δ_0 and δ_1 for $\alpha+n$ scattering as a function of the c.m. energy. The data points shown are obtained by using the central potential given in Eq. (8) of Ref. 23.

B. $\alpha+N$ System

In Fig. 4, the solid lines represent the $l=0$ and 1 phases for $\alpha+n$ scattering, calculated with $\alpha=0.514$ F^{-2} and $y=0.95$. The dots and crosses represent the phase-shift values obtained by using the central potential of Eq. (8) published in a recent paper by Satchler *et al.*,[23] who did a careful phenomenological analysis of the

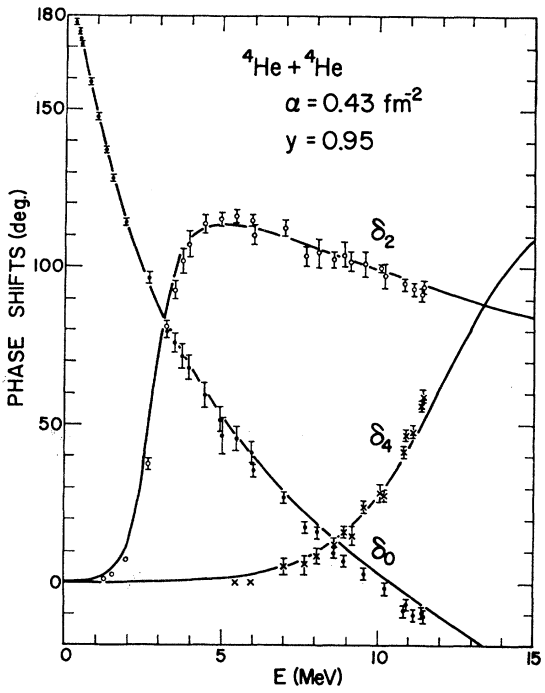

FIG. 5. Comparison of the calculated phase shifts for $\alpha+\alpha$ scattering with the experimental phase shifts of Refs. 17–20. The parameters used are $\alpha=0.43$ F^{-2} and $y=0.95$.

$\alpha+N$ scattering data in the energy region below the reaction threshold. Here it is seen that the agreement between calculated and empirical values is only fair. As in the $\alpha+\alpha$ case, one can again attempt to improve the fit by varying the parameter y. Unfortunately, however, this procedure leads to a rather small value of y, equal to about 0.6, in the $l=0$ state. On the other hand, if y is kept fixed at 0.95, then one finds that a decrease in α from 0.514 to 0.43 F^{-2} can produce good fits to the empirical data in both the $l=0$ and 1 states; this is shown by the dashed lines in Fig. 4.

It is somewhat surprising that even in the $\alpha+\alpha$ case the phases calculated with $\alpha=0.43$ F^{-2} and $y=0.95$ (see Fig. 5) agree very well with those empirically

[23] G. R. Satchler, L. W. Owen, A. J. Elwyn, G. L. Morgan, and R. L. Walter, Nucl. Phys. **A112**, 1 (1968).

determined. This does not mean, however, that one should attach more physical meaning to this new value of α. Clearly, with $\alpha=0.43$ F^{-2}, the asymptotic behavior of $\Psi_{\alpha\alpha}$ or $\Psi_{\alpha N}$ is not given correctly. Rather, one should take the viewpoint that the adjustment in the value of α is just another crude procedure to compensate for the lack of certain features in our nucleon-nucleon potential.

In spite of the above discussion, it still seems that, since our major emphasis is on the effective potential, we should use a combination of α and y values which yield the best phase-shift set. Thus, in the following, the discussion for the $\alpha+N$ case will be based entirely upon computations using $\alpha=0.43$ F^{-2} and $y=0.95$.

The $\alpha+p$ phases for $l=0$ to 6 in the energy region 15–76 MeV are tabulated in Table I. From this table, we conclude that, except for the $l=1$ level in the low-excitation region, there is no other resonant level in Li^5 (or He^5) which has an $\alpha+p$ (or $\alpha+n$) cluster structure and is narrow enough to be detected experimentally. The levels known to exist at 16.65 and 20.0 MeV [24] must have a cluster structure which is distinctly different from the $\alpha+p$ cluster structure.[25]

TABLE I. Calculated phase shifts, in degrees, for the $\alpha+p$ system. The parameters used are $\alpha=0.43$ F^{-2} and $y=0.95$.

E (MeV)	δ_0	δ_1	δ_2	δ_3	δ_4	δ_5	δ_6
15	97.55	81.50	2.82	1.46	0.06	0.02	0
20	88.53	74.21	5.98	2.74	0.21	0.05	0
25	81.40	67.90	9.32	4.24	0.48	0.12	0.01
31.84	73.45	60.63	13.38	6.48	1.07	0.28	0.04
40	65.79	53.66	17.14	9.09	2.01	0.58	0.12
50	58.30	47.10	20.35	11.79	3.35	1.13	0.30
60	52.39	42.22	22.42	13.76	4.74	1.82	0.56
76	45.28	36.80	24.03	15.63	6.84	3.09	1.16

At $E=31.84$ MeV, Giamati and Thaler[26] have performed a phase-shift analysis using both differential cross section and polarization data.[27,28] The spin averaged values of the real parts of their resultant phases are

$$\delta_0=66.5°,\quad \delta_1=60.9°,\quad \delta_2=16.9°,$$
$$\delta_3=10.4°,\quad \delta_4=2.1°,\quad \delta_5=1.2°. \qquad (13)$$

Comparing with our calculated values given in Table I, we note that there is a reasonable agreement,[29] indicating that even at relatively high energies our calculated values can still be used as starting values for a detailed phase-shift analysis. In fact, even the differential cross section at 31.84 MeV obtained with our purely real phases seems to agree fairly well with experiment (see Fig. 6).

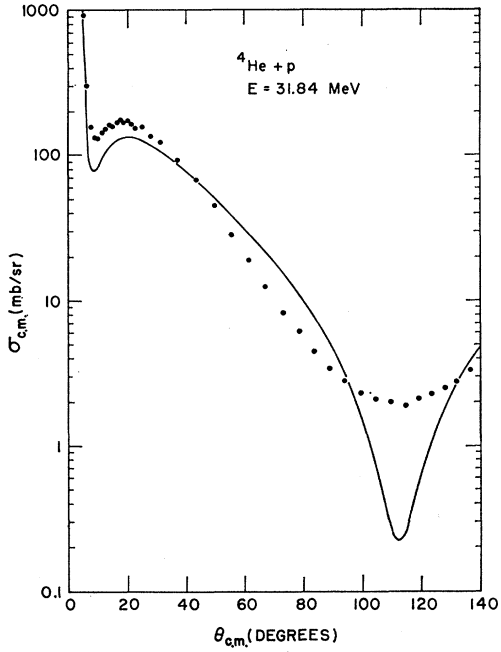

FIG. 6. Comparison of the calculated differential cross section for $\alpha+p$ scattering with experimental data at 31.84 MeV. The experimental data are those of Ref. 27.

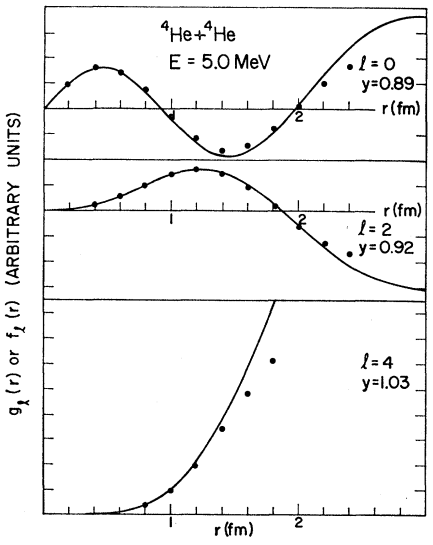

FIG. 7. The radial functions f_l and g_l at 5.0 MeV in the $\alpha+\alpha$ case. The curves represent the functions f_l as calculated from Eq. (10), while the solid dots represent the functions g_l as calculated from Eq. (14).

[24] T. Lauritsen and F. Ajzenberg-Selove, Nucl. Phys. **78**, 1 (1966).
[25] L. D. Pearlstein, Y. C. Tang, and K. Wildermuth, Phys. Rev. **120**, 224 (1960).
[26] C. C. Giamati and R. M. Thaler, Nucl. Phys. **59**, 159 (1964).
[27] M. K. Brussel and J. H. Williams, Phys. Rev. **106**, 286 (1957).
[28] C. F. Hwang, D. H. Nordby, S. Suwa, and J. H. Williams, Phys. Rev. Letters **9**, 104 (1962).
[29] C. C. Giamati, V. A. Madsen, and R. M. Thaler [Phys. Rev. Letters **11**, 163 (1963)] have also performed an analysis using only real phases. The values they obtained are quite similar to those given in Eq. (13).

IV. EFFECTIVE POTENTIALS

A. $\alpha+\alpha$ System

The radial functions $f_0(r)$, $f_2(r)$, and $f_4(r)$ for $E=5.0$ MeV are shown by the solid lines in Fig. 7. For comparison, we have also shown the values, represented by solid dots, of the functions $g_l(r)$ given by

$$g_0(r)=c_0 r[1-(8/3)\alpha r^2+(16/15)\alpha^2 r^4]\exp(-\alpha r^2),$$
$$g_2(r)=c_2 r^3[1-(4/7)\alpha r^2]\exp(-\alpha r^2), \quad (14)$$
$$g_4(r)=c_4 r^5\exp(-\alpha r^2),$$

with $\alpha=0.514$ F^{-2}. With these expressions for $g_l(r)$, the functions

$$\Psi_{lm}=\mathcal{A}[\phi_1\phi_2 r^{-1}g_l(r)Y_{lm}(\Omega)\xi(\sigma,\tau)] \quad (15)$$

are the usual shell-model wave functions describing the lowest configuration $(1s)^4(1p)^4$ in an oscillator well with width parameter α. The quantities c_0, c_2, and c_4 in Eq. (14) are normalization constants; they are chosen such that $f_0=g_0$ and $f_2=g_2$ at the positions of the first maximum, and $f_4=g_4$ at $r=1.0$ F. From this figure, it can be immediately seen that the functions f_l and g_l are quite similar to each other for r less than about 2 F, which is approximately the value of the rms radius of the folded matter distribution of the two α clusters. In particular, the nodes of the two functions nearly coincide. Further, it should be mentioned that the positions of the nodes of the radial scattering functions are only weakly energy-dependent[30]; for example, at $E=2$, 5, and 12 MeV, the outermost node of f_0 occurs at $r=2.00$, 1.98, and 1.93 F, respectively, while that of g_0 occurs at 1.99 F.

The above-mentioned features about the positions of the outermost nodes in the radial scattering functions, occurring at $r=r_{l0}$,[31] are very useful when one attempts to construct an effective potential between two more complicated clusters, such as between two O^{16} clusters or between a proton and a Pb^{208} nucleus. In these latter situations, these positions will be hard to find, since it will obviously be an impractical matter to set up the integrodifferential equation using the resonating-group method and then solve for f_l; however, they can be easily found by considering the function g_l in the oscillator shell model.

From Fig. 7, It is seen that the functions f_0 and f_2 take on rather large values at $r\lesssim 2$ F. As has been pointed out previously in an analogous case of He^3+He^4 scattering,[7] this certainly does not mean that the two α clusters have a large probability of being close together. Rather, it should be realized that these functions occur under the antisymmetrization operator and, hence, have no clear interpretation in terms of cluster separation when the clusters are close to each other.

The effective potentials $V_l^*(r)$, as defined in Eq. (12), are shown in Fig. 8 for $l=0$ and 2 at 5.0 MeV. Because of the nonlocal nature of the interaction between the clusters, singularities occur in the effective potentials at values of r for which $f_l(r)=0$. It should be noted that such singularities in V_l^* would not appear in situations where there are open reaction channels which are properly accounted for by a more detailed calculation, such as a many-channel calculation.[7,32] In fact, in those cases where the reaction probabilities are large, the resultant potential V_l^* may even turn out to be a purely attractive, smoothly varying potential.[33] In our present case, however, we expect the singularities to play a more prominent role, since there is no reaction channel open below about 17 MeV. Thus, it will not be possible to describe the features of the $\alpha+\alpha$ phase-shift results with a purely attractive and smooth potential, which is indeed in agreement with the findings of many authors who have performed phenomenological analyses on $\alpha+\alpha$ scattering data with effective potentials.[2–4,34]

The fact that V_l^* contains singularities in the region where the clusters overlap strongly means that this effective potential has more academic than practical interest. Thus, it seems desirable to construct another effective potential $\tilde{V}_l$ which deemphasizes this particular region. This can be done by defining $\tilde{V}_l$ in the

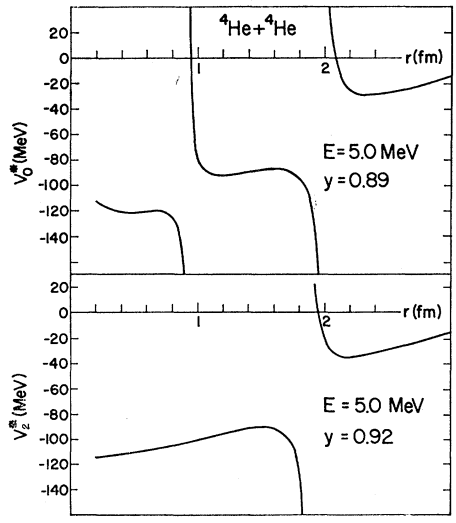

FIG. 8. Effective potentials $V_l^*(r)$ for $l=0$ and 2 in the $\alpha+\alpha$ case.

[30] This has also been pointed out by R. Tamagaki and H. Tanaka, Progr. Theoret. Phys. (Kyoto) **34**, 191 (1965); S. Okai and S. C. Park, Phys. Rev. **145**, 787 (1966).

[31] If the node at the origin is also counted, then the outermost node in the $\alpha+\alpha$ problem is the third node in the $l=0$ case, second node in the $l=2$ case, and first node in the $l=4$ case. For these nodes, the values of r_{l0} at $E=5$ MeV are equal to 1.98, 1.88, and 0 F for $l=0$, 2, and 4, respectively.

[32] K. Wildermuth and W. McClure, *Cluster Representation of Nuclei* (Springer-Verlag, Berlin, 1966).

[33] The potential V_l^* will be complex; here, we are referring to the real part only.

[34] R. Haefner, Rev. Mod. Phys. **23**, 228 (1951); E. van der Spuy and H. J. Pienaar, Nucl. Phys. **7**, 397 (1958); A. R. Bodmer and S. Ali, *ibid.* **56**, 657 (1964); Y. C. Tang and R. C. Herndon, Phys. Rev. **138**, B637 (1965).

following way[35]:

$$\tilde{V}_l(r)=\infty \qquad r\leq r_{l0}$$
$$=V_l^*(r), \qquad r>r_{l0} \qquad (16)$$

where a hard core of radius r_{l0} has been used to represent the much more intricate $\alpha+\alpha$ interaction in the interior region. Clearly, such a potential will yield exactly the same phase shifts as those obtained using the nonlocal potential in the resonating-group method.

The potentials $\tilde{V}_0$, $\tilde{V}_2$, and $\tilde{V}_4$ are shown in Fig. 9 at $E=2$ and 12 MeV.[36] Also, as a comparison, we have shown in the same figure the values of the phenomenological energy-independent potentials determined by Ali and Bodmer[2,37] (solid dots), and Darriulat *et al.*[3] (crosses). From this figure, it is seen that the potential $\tilde{V}_4$ is quite energy-insensitive and very similar to the potentials of the latter authors. In particular, it is noted that our potential $\tilde{V}_4$ has no repulsive core, and indeed Ali and Bodmer have found that no repulsive

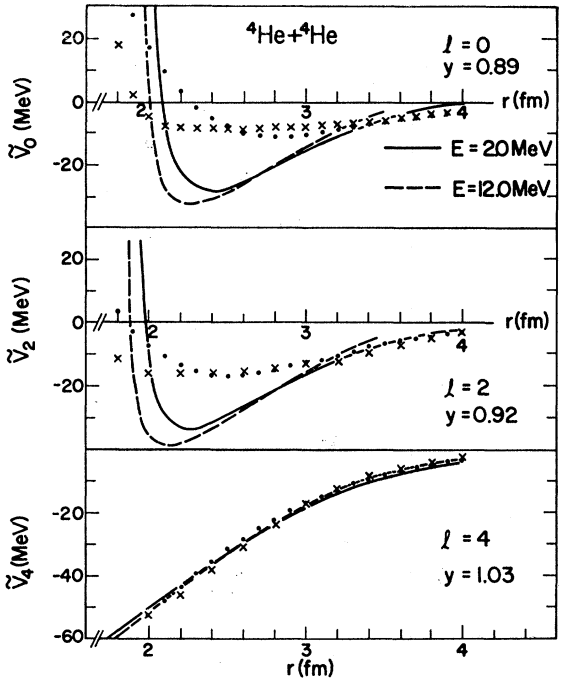

FIG. 9. Effective potentials $\tilde{V}_l(r)$ for $l=0$, 2, and 4 in the $\alpha+\alpha$ case. The solid dots represent the values of the phenomenological potentials of Ali and Bodmer (Ref. 2), while the crosses represent those of Darriulat *et al.* (Ref. 3).

[35] It is appropriate to mention that the potentials $\tilde{V}_l(r)$ do not allow the existence of spurious bound $l=0$ and 2 states in the $\alpha+\alpha$ system.

[36] The potentials $\tilde{V}_l$ have also very narrow singularities in the surface region where the average potential depth is small. As has been mentioned before (Ref. 7), these singularities have very little effect on the scattering cross sections, since their spatial extent is small compared to the wavelength λ of relative motion of the clusters in the energy region which is of interest to us. Thus, for all practical purposes, these narrow surface singularities can be replaced by smoothed potentials.

[37] The potentials of Ali and Bodmer (Ref. 2) are their potentials d_0, d_2, and d_4.

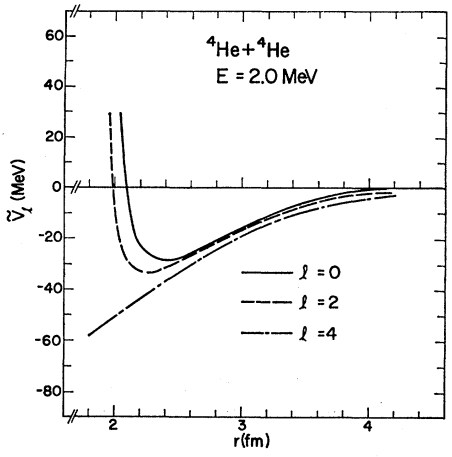

FIG. 10. Effective potentials $\tilde{V}_l$ for $l=0$, 2, and 4 at 2.0 MeV in the $\alpha+\alpha$ case.

core of any significant strength is permissible in their $l=4$ potential. On the other hand, it seems that, at first sight, the potentials $\tilde{V}_0$ and $\tilde{V}_2$ are quite different from the phenomenological potentials in the region where $r\lesssim 3$ F. But this is almost entirely due to the fact that different ways of parametrization have been utilized for the repulsive parts; thus, while the potentials $\tilde{V}_l$ have a hard repulsive core, the potentials of Ali and Bodmer and of Darriulat *et al.* have a soft repulsive core. In fact, upon closer examination, we do find that there are a number of relevant quantities, namely, the classical turning distances and the intrinsic ranges of the repulsive parts, where our potentials $\tilde{V}_l$ and the phenomenological potentials yield similar results. For instance, the classical turning distances, computed with the Coulomb and centrifugal potentials also taken into account, are equal to 2.00 and 1.90 F at 12 MeV for $l=0$ and 2, respectively, which compare very favorably with the corresponding values of 2.10 and 1.98 F obtained with the potentials of Ali and Bodmer.

The particular way in which the hard-core radius changes with energy is also interesting. When the energy is changed from 2 to 12 MeV, r_{00} changes from 2.00 to 1.93 F, while r_{20} changes from 1.89 to 1.83 F. The fact that both these radii decrease with energy indicates that, in a crude way, the slightly energy-dependent hard cores used in this investigation can be replaced by energy-independent soft cores. Together with the observation from Fig. 9 that the long-range parts of $\tilde{V}_l$ are only weakly energy-dependent, this therefore explains the fact that, even with an hypothesis of energy independence, Ali and Bodmer and Darriulat *et al.* were still able to obtain good fits to the empirical phase-shift results.

The l dependence of $\tilde{V}_l(r)$ is shown in Fig. 10. Here, the interesting point to note is that the variation of the effective potentials with l is very much the same as that found phenomenologically by Ali and Bodmer

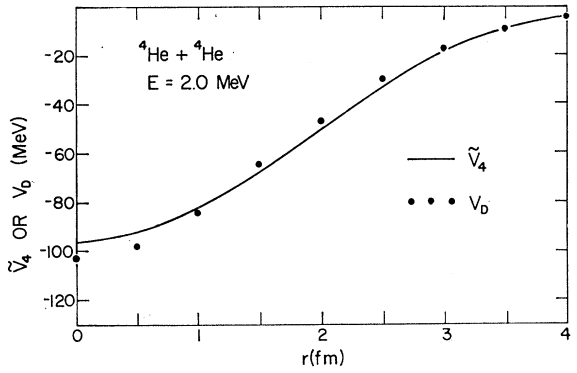

FIG. 11. Comparison of the effective potential $\tilde{V}_4$ and the direct potential V_D in the $\alpha+\alpha$ case.

and by Darriulat *et al.* In particular, it is seen that the l dependence manifests itself mainly in the short-range part, and that the tails of the potentials for all three l values are quite similar.

In Fig. 11, we show a comparison between the direct potential $V_D(r)$ and the effective potential $\tilde{V}_4(r)$.[38] From this figure, we see that especially in the region where $r \gtrsim 2$ F, there is a close resemblance. This indicates that in the surface region where the effective potential is only weakly energy- and l-dependent, the antisymmetrization procedure seems to be of only minor importance. It should be emphasized, however,

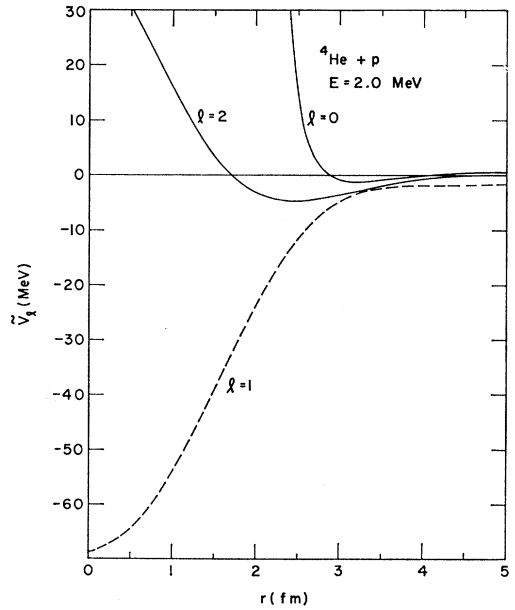

FIG. 12. Effective potentials $\tilde{V}_l$ for $l=0$, 1, and 2 at 2.0 MeV in the $\alpha+p$ case.

[38] It is interesting to note that if one uses an unantisymmetrized wave function for the $\alpha+\alpha$ or $\alpha+N$ system and a two-body potential given by Eq. (6), but with P_{ij}^r replaced by $-P_{ij}^\sigma P_{ij}^\tau$ then the effective potential between the clusters will just be given by the direct potential V_D.

that this latter observation is a particular feature of the $\alpha+\alpha$ scattering problem. In the $\alpha+N$ case to be discussed below, we do find that even in the long-range part the antisymmetrization procedure plays a much more important role.

B. $\alpha+N$ System

The features of the radial scattering functions and the effective potentials V_l* in the $\alpha+N$ case are very similar to those in the $\alpha+\alpha$ case; hence, they will not be further discussed here. In Fig. 12, we show the effective potentials $\tilde{V}_l$ for $l=0$, 1, and 2 at 2 MeV in the $\alpha+p$ system. From this figure, it is seen that the $l=2$ potential has no hard core, but does become moderately repulsive for small values of r. We should mention, however, that this latter type of repulsion

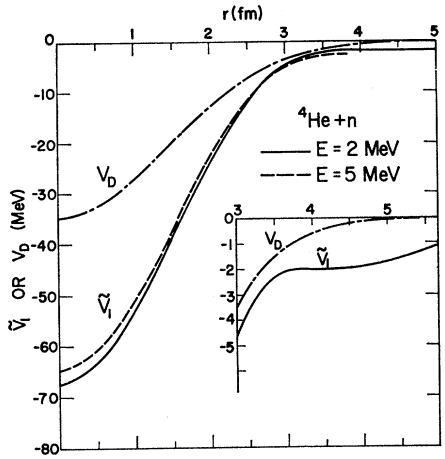

FIG. 13. Comparison of the effective potential $\tilde{V}_1$ and the direct potential V_D in the $\alpha+n$ case.

will have little influence on the scattering phases, since the centrifugal potential has a much larger magnitude in this particular region.

The potential $\tilde{V}_0$ contains a hard core of radius equal to 2.21 F,[39] followed by a very weak attractive component, while the potential $\tilde{V}_1$ is purely attractive. Both of these potentials are quite similar to the phenomenological potentials derived by Swan and Pearce[1] using the empirical phase-shift data.

The odd-even feature in the long-range part of the effective potential, noted recently in a He^3+He^4 scattering calculation,[7] is also present here, although to a lesser extent. As has been mentioned previously, this effect was first discovered by Gammel and Thaler[40] in their phenomenological analysis of the $\alpha+p$ scattering data using local potentials. Thus, it is indeed gratifying

[39] This hard-core radius is also somewhat energy-dependent, changing from 2.21 F at 2 MeV to 2.09 F at 12 MeV.

[40] J. L. Gammel and R. M. Thaler, Phys. Rev. **109**, 2041 (1958); see also C. C. Giamati, V. A. Madsen, and R. M. Thaler, Phys. Rev. Letters **11**, 163 (1963).

to see that such an effect emerges from a more basic calculation using totally antisymmetrized wave function and a nucleon-nucleon potential.

Figure 13 shows the energy dependence of the potential $\tilde{V}_1$ and a comparison between $\tilde{V}_1$ and the direct potential V_D. Here we see that the energy dependence is quite slight, as in the $\alpha+\alpha$ case. On the other hand, the difference between $\tilde{V}_1$ and V_D is rather large, indicating that in this particular case a proper consideration of the antisymmetrization effect is essential. Further, we note from this figure that there is a peculiar feature in $\tilde{V}_1$ occurring near $r=4.5$ F. At the present moment, we are not sure why this feature arises, although we tend to think that it is connected with the fact that in the $\alpha+N$ case, one of the clusters involved is not a composite particle.

V. CONCLUSION

In this investigation, the resonating-group method is used to study the $\alpha+\alpha$ and $\alpha+N$ systems in the energy region below the reaction thresholds. With a nucleon-nucleon potential of a near-Serber exchange mixture, it is found that good agreement with the empirical phase-shift data can be obtained in both cases. Using the resultant radial scattering functions, effective potentials $\tilde{V}_l$ between the clusters are constructed, which yield the same phase shifts as those obtained from the resonating-group calculations.

Except for minor details, the effective potentials constructed here are quite similar to the phenomenological potentials obtained by Ali and Bodmer[2] and by Darriulat *et al.*[3] in the $\alpha+\alpha$ case and by Swan and Pearce[1] in the $\alpha+N$ case. Essentially, they consist of a hard core with a radius which is weakly energy-dependent but strongly l-dependent, and an attractive long-range part which has only a slight dependence on the relative orbital angular momentum and the energy in both even- and odd-l states. In addition, it is noted that the long-range part has an odd-even feature wherein the potentials in the odd-l states are different from those in the even-l states.

It is our opinion that not too much physical meaning should be attached to the hard core in the effective potential. In our way of construction, the hard core is placed at the outermost node of the radial scattering function in the region where the clusters overlap strongly. In this latter region, the $\alpha+\alpha$ and $\alpha+N$ interactions have a very intricate nature. Thus, in our viewpoint, the introduction of a hard core is mostly an artifice which avoids the construction of a potential inside this region.

Also, we wish to comment briefly on how to construct a potential in a case, such as $O^{16}+O^{16}$ scattering, where a straightforward application of the resonating-group method is impractical. Based on the experience learned here, we propose an l-dependent potential of the form

$$V_l(r)=\infty \qquad r\leq r_{l0}$$
$$=c_l V_D(r), \qquad r>r_{l0}$$

where the hard-core radii r_{l0} can be determined from the positions of the nodes in the oscillator-shell-model wave functions, as outlined in Sec. IV A, and the direct potential V_D can be obtained by a simple folding procedure[41] using assumed matter distributions for the clusters. The quantities c_l, which can take on two values, based on whether l is even or odd, are introduced to take into account the effect of the Pauli principle and the odd-even feature mentioned above; these quantities will be treated as adjustable parameters in this potential. At present, we are planning to use this type of potential in a number of relatively simple cases, such as $p+O^{16}$ scattering, and so on. It is hoped that the results of such an investigation will help us in gaining a better understanding of the features of the effective potential and in extending the local-potential approach to more complicated problems.

At this point, it is perhaps appropriate to mention that the present investigation is our initial attempt to study the features of the effective interactions between composite particles.[42] The construction of these effective interactions is based on the results of a resonating-group calculation, which, although in reasonable agreement with the empirical data, do contain some limitations. For example, the finding that the quantity y has to be somewhat dependent on the relative orbital angular momentum of the clusters is clearly an undesirable feature. To remedy this, it will probably be necessary to calculate with a more realistic nucleon-nucleon potential, e.g., one which has different ranges in the triplet and singlet spin states.[43]

APPENDIX: EXPRESSIONS FOR $V_D(r)$, $V_C(r)$, AND $k_l(r, r')$

$\alpha+\alpha$ System

The direct potential $V_D(r)$ is given by

$$V_D(r)=-V_0(16w-4m+8b-8h)[2\alpha/(2\alpha+3\kappa)]^{3/2} \times\exp[-2\alpha\kappa r^2/(2\alpha+3\kappa)]. \quad \text{(A1)}$$

[41] G. W. Greenlees, G. J. Pyle, and Y. C. Tang, Phys. Rev. **171**, 1115 (1968).

[42] There have been a number of calculations in the $\alpha+\alpha$ case; these have been carefully reviewed by S. A. Afzal, A. A. Z. Ahmad, and S. Ali, Rev. Mod. Phys. **41**, 247 (1969). In particular, we should mention the work of I. Shimodaya, R. Tamagaki, and H. Tanaka, Progr. Theoret. Phys. (Kyoto) **27**, 793 (1962), who have made a study similar to ours. In this latter work, effective interactions between the α clusters were obtained by using, for the functions f_{l_1} trial functions which are consistent with the Pauli principle [see also L. D. Pearlstein, Y. C. Tang, and K. Wildermuth, Nucl. Phys. **18**, 23 (1960)]. This is a procedure different from the one employed here; however, the features of the $\alpha+\alpha$ interactions found by these authors are quite similar to those obtained in this investigation.

[43] I. Reichstein and Y. C. Tang, Nucl. Phys. (to be published).

The Coulomb potential has the form

$$V_C(r) = (zz'e^2/r)\Phi[(\tfrac{2}{3}\alpha)^{1/2}r], \tag{A2}$$

with z and z' being the atomic numbers of the two clusters and

$$\Phi(u) = \frac{2}{\sqrt{\pi}}\int_0^u \exp(-t^2)\,dt. \tag{A3}$$

The kernel $k_l(r, r')$ is written as

$$k_l(r, r') = -(\hbar^2/2M)\mathfrak{I} - V_0\mathfrak{V} + E'\mathcal{E}, \tag{A4}$$

where M is the nucleon mass and E' is given by

$$E' = E + 2E_\alpha, \tag{A5}$$

with

$$E_\alpha = (\hbar^2/2M)\tfrac{9}{2}\alpha - 6(w+m)V_0[\alpha/(\alpha+2\kappa)]^{3/2} + e^2(2\alpha/\pi)^{1/2}. \tag{A6}$$

The quantities $\mathfrak{I}$, $\mathfrak{V}$, and $\mathcal{E}$ are defined as follows:

$$\mathfrak{I} = (8\alpha/3\pi)^{3/2}\exp[-(5/3)\alpha(r^2+r'^2)]$$
$$\times\{62\alpha S_l[-(8/3)\alpha] - (224/9)\alpha^2(r^2+r'^2)S_l[-(8/3)\alpha] + (416/9)\alpha^2 rr' T_l[-(8/3)\alpha]\}$$
$$+ (2\alpha/\pi)^{3/2}\exp[-\alpha(r^2+r'^2)](4\pi rr'\delta_{l0})[6\alpha^2(r^2+r'^2) - (81/2)\alpha], \tag{A7}$$

$$\mathfrak{V} = (8\alpha/3\pi)^{3/2}\Big\{(-24w-24m)[\alpha/(\alpha+2\kappa)]^{3/2}S_l[-(8/3)\alpha]\exp[-(5/3)\alpha(r^2+r'^2)]$$
$$+(-4w+16m-8b+8h)S_l[-(8/3)\alpha-8\kappa]\exp\{-[(5/3)\alpha+4\kappa](r^2+r'^2)\}$$
$$+(-36w+24m-24b+24h)\left(\frac{3\alpha}{3\alpha+4\kappa}\right)^{3/2}S_l\left(-\frac{8\alpha^2+8\alpha\kappa}{3\alpha+4\kappa}\right)\exp\left(-\frac{5\alpha^2+8\alpha\kappa}{3\alpha+4\kappa}(r^2+r'^2)\right)$$
$$+(-24w-24m)\left(\frac{3\alpha}{3\alpha+4\kappa}\right)^{3/2}S_l\left(-\frac{8\alpha^2+16\alpha\kappa}{3\alpha+4\kappa}\right)$$
$$\times\left[\exp\left(-\frac{5\alpha^2+8\alpha\kappa}{3\alpha+4\kappa}r^2 - \frac{5\alpha^2+12\alpha\kappa}{3\alpha+4\kappa}r'^2\right) + \exp\left(-\frac{5\alpha^2+12\alpha\kappa}{3\alpha+4\kappa}r^2 - \frac{5\alpha^2+8\alpha\kappa}{3\alpha+4\kappa}r'^2\right)\right]\Big\}$$
$$+\left(\frac{2\alpha}{\pi}\right)^{3/2}\Big\{(12w+12m)\left(\frac{\alpha}{\alpha+2\kappa}\right)^{3/2}(4\pi rr'\delta_{l0})\exp[-\alpha(r^2+r'^2)] + (24w-36m+24b-24h)$$
$$\times\left(\frac{\alpha}{\alpha+\kappa}\right)^{3/2}S_l\left(-\frac{2\alpha\kappa}{\alpha+\kappa}\right)\exp\left(-\frac{\alpha^2+2\alpha\kappa}{\alpha+\kappa}(r^2+r'^2)\right) + (24w+24m)\left(\frac{2\alpha}{2\alpha+3\kappa}\right)^{3/2}(4\pi rr'\delta_{l0})$$
$$\times\left[\exp\left(-\alpha r^2 - \frac{2\alpha^2+5\alpha\kappa}{2\alpha+3\kappa}r'^2\right) + \exp\left(-\frac{2\alpha^2+5\alpha\kappa}{2\alpha+3\kappa}r^2 - \alpha r'^2\right)\right]\Big\}, \tag{A8}$$

$$\mathcal{E} = 4(8\alpha/3\pi)^{3/2}S_l[-(8/3)\alpha]\exp[-\tfrac{5}{3}\alpha(r^2+r'^2)] - 3(2\alpha/\pi)^{3/2}(4\pi rr'\delta_{l0})\exp[-\alpha(r^2+r'^2)], \tag{A9}$$

with

$$S_l(\lambda) = (4\pi/\lambda)\mathcal{J}_{l+1/2}(\lambda rr') \tag{A10}$$

and

$$T_l(\lambda) = (4\pi/\lambda)[\mathcal{J}_{l+3/2}(\lambda rr') - (l/\lambda rr')\mathcal{J}_{l+1/2}(\lambda rr')], \tag{A11}$$

where $\mathcal{J}(x)$ is a hyperbolic spherical Bessel function.

$\alpha+N$ System

The expressions for V_D, V_C, $\mathfrak{I}$, $\mathfrak{V}$, and $\mathcal{E}$ are as follows:

$$V_D(r) = -V_0(4w-m+2b-2h)[4\alpha/(4\alpha+3\kappa)]^{3/2}\exp\{-[4\alpha\kappa/(4\alpha+3\kappa)]r^2\}, \tag{A12}$$

$$V_C(r) = (zz'e^2/r)\Phi[(\tfrac{4}{3}\alpha)^{1/2}r], \tag{A13}$$

$$\mathfrak{I} = (\tfrac{4}{5})^3(4\alpha/3\pi)^{3/2}\exp[-(34/75)\alpha(r^2+r'^2)]$$
$$\times\{(47/5)\alpha S_l[(32/75)\alpha] - (1216/1125)\alpha^2(r^2+r'^2)S_l[(32/75)\alpha] - (1568/1125)\alpha^2 rr' T_l[(32/75)\alpha]\}, \tag{A14}$$

$$\mathcal{U}=(\tfrac{4}{5})^3(4\alpha/3\pi)^{3/2}\Big\{(-w+4m-2b+2h)\,S_l[(32/75)\alpha-(32/25)\kappa]\exp\{-[(34\alpha+48\kappa)/75](r^2+r'^2)\}$$
$$+(-3w-3m)[\alpha/(\alpha+2\kappa)]^{3/2}S_l[(32/75)\alpha]\exp[-(34/75)\alpha(r^2+r'^2)]$$
$$+(-3w-3m)\left(\frac{3\alpha}{3\alpha+2\kappa}\right)^{3/2}S_l\left(\frac{32\alpha^2+64\alpha\kappa}{75\alpha+50\kappa}\right)$$
$$\times\left[\exp\left(-\frac{34\alpha^2+28\alpha\kappa}{75\alpha+50\kappa}r^2-\frac{34\alpha^2+108\alpha\kappa}{75\alpha+50\kappa}r'^2\right)+\exp\left(-\frac{34\alpha^2+108\alpha\kappa}{75\alpha+50\kappa}r^2-\frac{34\alpha^2+28\alpha\kappa}{75\alpha+50\kappa}r'^2\right)\right]\Big\}, \quad \text{(A15)}$$

$$\mathcal{E}=(\tfrac{4}{5})^3(4\alpha/3\pi)^{3/2}S_l[(32/75)\alpha]\exp[-(34/75)\alpha(r^2+r'^2)]. \quad \text{(A16)}$$

Also, the quantity E' in Eq. (A4) is given by

$$E'=E+E_\alpha. \quad \text{(A17)}$$

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Multiple-Scattering Analysis on a Soluble Neutron-Deuteron Model*

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The multiple-scattering series for elastic scattering is investigated numerically for a model of the neutron-deuteron system, at neutron laboratory energies of 14.1, 50, and 100 MeV. The model is that of Aaron, Amado, and Yam, with spin-dependent, s-wave, separable, two-body interactions. It is found that the doublet $L=0$ series converges only slowly even at 100 MeV, and that it strongly diverges at 14.1 MeV. On the other hand, the convergence is rapid for both doublet and quartet partial waves beyond $L=2$, and for these the single-scattering plus Born-pickup terms provide an accurate approximation. Differential cross sections and partial-wave amplitudes are given for various orders of multiple scattering, and for a unitary version of the first-order approximation, and are compared with the exact results.

1. INTRODUCTION

APPROXIMATIONS based on truncated multiple-scattering series have often been used to study the scattering of elementary particles by deuterons. Of these, the most widely used is the impulse approximation,[1] which in its usual application may be regarded as the first-order contribution to the multiple-scattering series. Some attempts have been made to calculate second-[2,3] and higher-order[3] terms, but with realistic potentials even the first-order term requires approximation, and drastic simplifications must be made to calculate higher-order terms. With separable potentials, however, exact calculation to all orders becomes possible, and valuable insight into the multiple-scattering series may be gained in this way. Such studies have been made previously for the K^-—d,[4] K^+—d,[5] and Λ—d [5] systems. In this paper, we consider the n-d system, which differs in the important respect that rearrangement collisions are possible. Our main aim is to study the convergence of the multiple-scattering series of Faddeev-type[6] for this system. We hope that the results will serve as a guide to the usefulness of the multiple-scattering series for more realistic potentials.

The multiple-scattering series is derived in Sec. 2, and in Sec. 3 we briefly describe the model (the separable-potential model of Aaron, Amado, and Yam[7]). The methods used in calculating the multiple-scattering series and in analyzing the convergence are described

* Research supported by the Australian Research Grants Committee.

[1] G. F. Chew, Phys. Rev. **80**, 196 (1950); G. F. Chew and G. C. Wick, *ibid.* **85**, 636 (1952); J. Ashkin and G. C. Wick, *ibid.* **85**, 686 (1952); G. F. Chew and M. L. Goldberger, *ibid.* **87**, 778 (1952); H. Kottler and K. L. Kowalski, *ibid.* **138**, B619 (1965). The last paper contains many references to earlier calculations with the impulse approximation.

[2] A. Everett, Phys. Rev. **126**, 831 (1962); A. K. Bhatia and J. Sucher, *ibid.* **132**, 855 (1963).

[3] N. M. Queen, Nucl. Phys. **55**, 177 (1964); **66**, 673 (1965).

[4] J. H. Hetherington and L. H. Schick, Phys. Rev. **137**, B935 (1965).

[5] J. H. Hetherington and L. H. Schick, Phys. Rev. **138**, B1411 (1965); **139**, B1164 (1965).

[6] L. D. Faddeev, Zh. Eksperim. i Teor. Fiz. **39**, 1459 (1960) [English transl.: Soviet Phys.—JETP **12**, 1014 (1961)].

[7] R. Aaron, R. D. Amado, and Y. Y. Yam, Phys. Rev. **140**, B1291 (1965).