Proton-Proton Bremsstrahlung Calculations*

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Revised calculations of the cross section for the process $P+P\rightarrow P+P+\gamma$ are presented, using several models and various approximations. These are compared to the latest calculations of other authors. For the Hamada-Johnston potential, they are found to be in good agreement with the data, and with only a small discrepancy among authors. However, strong discrepancies are found with previously published values for the Tabakin and Bryan-Scott potentials and for the Feshbach-Yennie approximation. It is estimated that inclusion of the Coulombic interaction of the protons lowers most of the cross sections by 5–10%.

I. INTRODUCTION

THERE have been a number of calculations of the proton-proton bremsstrahlung (PPB) process $P+P\rightarrow P+P+\gamma$ using a variety of approximations and proton-proton potentials. This paper presents the results of new calculations and also comparisons of the various calculations to each other and to experiment.

All PPB discussions begin with the distorted-wave Born-approximation type of equation[1] for the photon-production amplitude

$$T_{fi}=\langle\psi_f^{(-)}\mid V^{\text{em}}\mid\psi_i^{(+)}\rangle,$$

where the $\psi^{(\pm)}$ are the exact wave functions obtained using the strong nuclear force alone. This formula is exact to all Born orders in the strong interaction, but is only first order in the electromagnetic interaction V^{em}. The problem becomes more tractable if one iterates the wave functions to obtain

$$T_{fi}=V_{fn}^{\text{em}}G_n^{(+)}t_{ni}+\tilde{t}_{fn}G_n^{(+)}V_{ni}^{\text{em}}+\sum_m \tilde{t}_{fn}G_n^{(+)}V_{nm}^{\text{em}}G_m^{(+)}t_{mi}.$$

The amplitude $\tilde{t}_{fn}$ is the time-reversed amplitude corresponding to the usual two-nucleon amplitude $t_{nf}=\langle\phi_n\mid V_N\mid\psi_f\rangle$ for the scattering by the nuclear potential V_N. The first two terms in the series for T are called the "single-scattering" terms, while the last term is referred to as that due to "double scattering." The words "single" and "double" refer to the number of times the exact nuclear t amplitude appears in the production amplitude T, rather than to the number of times the nucleon potential V_N appears (which would be the strong-force Born order). The intermediate states n in the single-scattering terms above are kinematically unique, making their contributions much easier to calculate than those from the double-scattering term. Most calculations are made with the approximation of omitting the latter term. The omission has been justified on general arguments[2] and on the numerical calculations of Brown.[3]

In order to carry out numerical calculations, one makes partial-wave projections of the off-energy-shell two-nucleon amplitudes, as has been described elsewhere.[4] The off-shell partial-wave amplitudes turn out to be a simple generalization of the usual on-shell formula. Specifically,

$$t_{ni,L}=e^{i\delta_L(k_i)}\Delta_L(k_n,k_i),$$

where the real quantity

$$\Delta_L(k_n,k_i)=-m_N\int_0^\infty j_L(k_nr)V_N(r)U_L(k_i,r)rdr$$

is called the "quasiphase." Obviously, it is just the sine of the phase shift for on-shell energies:

$$\Delta_L(k,k)=\sin\delta_L(k).$$

Most experiments have been performed in the "Harvard geometry,"[5] in which the momenta of the protons and photon are all in the same plane, and the exit protons are on opposite sides of the incident beam at equal angles to it. Then for a specific geometry, one need only specify the polar exit angles of one proton and the photon, as shown in Fig. 1. The differential cross section is $d\sigma/d\Omega_1 d\Omega_2 d\theta_\gamma$, and one defines an integrated cross section $d\sigma/d\Omega_1 d\Omega_2$ for the purpose of comparison to many of the experiments.

It should be mentioned that if one uses transverse gauge for the electromagnetic interaction, then the

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[1] M. Sobel and A. Cromer, Phys. Rev. 132, 2698 (1963).

[2] P. Signell, in *Proceedings of the International Conference on Light Nuclei, Few Body Problems, and Nuclear Forces, Brela, Yugoslavia, 1967* (Gordon and Breach Science Publishers, Inc., New York, 1969).

[3] V. Brown, Phys. Letters 25B, 506 (1967); Phys. Rev. 177, 1498 (1969); and (private communication).

[4] M. I. Sobel, Phys. Rev. 138, B1517 (1965).

[5] This geometrical arrangement is due to B. Gottschalk; it was first published in Ref. 1.

TABLE I. The best values calculated by various authors for the integrated cross section, coplanar Harvard geometry, incident-proton energy of 61.7 MeV, proton exit angles of 30°. The value attributed to Felsner, line 9, is an estimate obtained by extrapolation of a line on a graph.

Line	Authors	Potential	$d\sigma/d\Omega_1 d\Omega_2$ (μb/sr^2)	Remarks	Ref.
1	Brown	Hamada-Johnston	2.98	$J\leq 4$	3
2	Drechsel-Maximon	Hamada-Johnston	2.89	$J\leq 4$	10, 11
3	Marker-Signell	Hamada-Johnston	2.90	$J\leq 5$	here
4	Brown	Bryan-Scott II	2.63	$J\leq 4$	3
5	Marker-Signell	Bryan-Scott II	3.24	$J\leq 5$	here
6	Pearce-Gale-Duck	Tabakin	2.4	$\epsilon_2=0$, OPE for $L>2$	8
7	Marker-Signell	Tabakin	2.83	HJ for $L>2$	here
8	Nyman	Single-energy approx.	2.52		14
9	Felsner	Two-energy approx.	1.5 (est.)	extrap. on graph	13
10	Marker-Signell	Two-energy approx.	3.2		here

invariant production amplitude is most easily evaluated in the over-all center-of-mass (c.m.) frame of reference. After taking the nucleon-spin trace of the square of the amplitude, the appropriate phase-space factor can be added, evaluated in the laboratory frame for comparison to experiment. In principle, the calculation could be done directly in the laboratory frame, but then the difficult double-scattering term can not be neglected.[2]

II. THEORETICAL CALCULATIONS

Recently, there have been PPB calculations by Brown,[3] using the Hamada-Johnston[6] (HJ) and Bryan-Scott II[7] (BS-II) nuclear potentials, by Pearce, Gale, and Duck[8] (PGD) using the Tabakin[9] potential, by Drechsel and Maximon[10,11] using the Hamada-Johnston and Reid[12] potentials, and by the present authors using the Hamada-Johnston and Tabakin potentials. In addition, Felsner,[13] Nyman,[14] PGD,[8] and the present authors[2,15–17] have made *approximate* calculations in which the off-energy-shell nucleon-nucleon amplitudes were replaced by on-energy-shell (elastic) ones.

A. Hamada-Johnston Potential

There is now quite close agreement among the various predicted values for the Hamada-Johnston potential, lines 1–3 of Table I. Previous differences among the several authors were due to truncation of the two-nucleon partial-wave series at too low a value of angular momentum[18] and to an error in the transformation from the c.m. to lab frames.[19] At least part of the remaining difference between Brown's result and the others is due to a use of relativistic energy differences for the Green's function or propagator G (see Introduction). The other authors used an invariant form suggested by perturbation and dispersion theory. The reason for the rest of the small discrepancy is not known.

None of these calculations was gauge-invariant, in the sense that no gauge terms were added for the explicit momentum dependence in the angular-momentum operators. However, corrections due to this source have generally been regarded as negligible, because of an early estimation at a single kinematical point by Sobel and Cromer.[1]

In Fig. 2, we compare the latest differential-cross-section predictions of Brown,[18] Drechsel and Maximon,[11] and ourselves, for the HJ potential at 158 MeV, $\theta=35°$. All of these calculations used all partial waves through $J=4$ and included no Coulomb corrections. Only the dashed curve of Brown included double-scattering effects. Except at the very ends, the agreement is seen to be excellent. For the ends, note that the slope of the cross section must be zero at $\theta_\gamma=0°$ and 180°, since the Harvard geometry is symmetric about those angles. This seems not to have been noted previously.

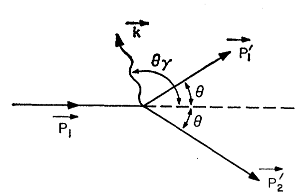

FIG. 1. Kinematical variables used to described proton-proton bremsstrahlung for the so-called Harvard geometry.

[6] T. Hamada and I. D. Johnston, Nucl. Phys. **34**, 382 (1962).
[7] R. A. Bryan and B. Scott, Phys. Rev. **177**, 1435 (1969).
[8] W. A. Pearce, W. A. Gale, and I. M. Duck, Nucl. Phys. **B3**, 241 (1967).
[9] F. Tabakin, Ann. Phys. (N.Y.) **30**, 51 (1964).
[10] D. Drechsel and L. C. Maximon, Phys. Letters **26**, 477 (1968).
[11] D. Drechsel and L. C. Maximon, Ann. Phys. (N.Y.) **49**, 403 (1968).
[12] R. V. Reid, Ann. Phys. (N.Y.) **50**, 411 (1968).
[13] G. Felsner, Phys. Letters **25B**, 290 (1967).
[14] E. M. Nyman, Phys. Letters **25B**, 135 (1967); Phys. Rev. **170**, 1628 (1968).
[15] P. Signell and D. Marker, Phys. Letters **26B**, 559 (1968).
[16] P. Signell and D. Marker, Phys. Letters **28B**, 79 (1968).
[17] P. Signell, in *Advances in Nuclear Physics*, edited by M. Baranger and E. Vogt (Plenum Press, Inc., New York, 1969), Vol. 2.

[18] V. Brown, Phys. Rev. **177**, 1498 (1969).
[19] D. Drechsel (private communication). The calculations of Dreschsel and Maximon reported in Ref. 10 contained an extra factor of $W_{\text{C.M.}}/W_{\text{lab}}$ in the differential cross sections, where W is the photon energy. Since that ratio of the photon energies is roughly antisymmetric about $\theta_\gamma=90°$ (see Refs. 2 or 17), the integrated cross sections were quite close to the correct values, though the differential cross sections from which they were computed were not.

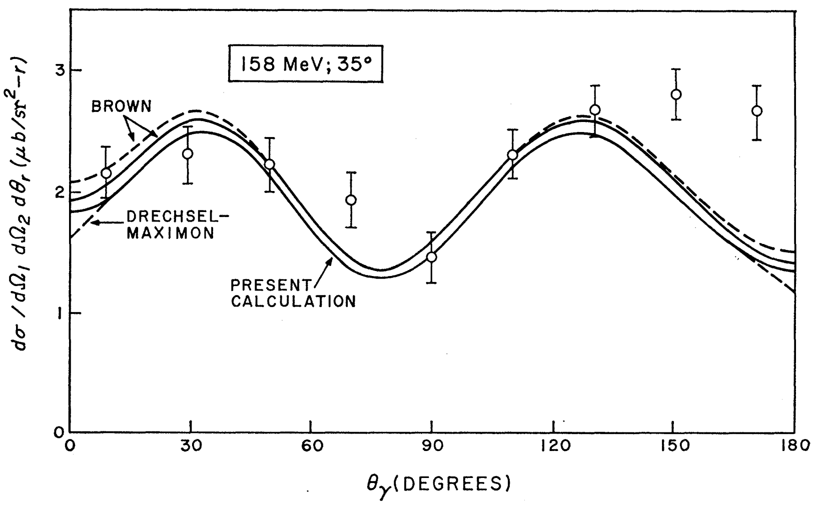

FIG. 2. Photon angular distribution at 158 MeV, $\theta=35°$ in the Harvard geometry, produced by the latest calculations of several authors (see text). The data are those of Gottschalk *et al.*, Ref. 29.

B. Bryan-Scott II Potential

The Bryan-Scott II potential[7] is a moment-dependent one, and Brown[3] has found it to give a considerably lower PPB cross section, line 4 of Table I, than that found for the hard-core HJ potential, lines 1–3. However, Brown was unable[20] to reproduce the elastic-scattering phases found by Bryan and Scott, whereas our calculation, line 5 of Table I, *did* reproduce them.[21] One should also be aware that the various elastic (on-energy-shell) phases for the BS-II potential are up to 3° larger than those of the HJ potential at the nucleon-nucleon energies involved in PPB at 61.7 MeV, 30°. However, the HJ phases are a considerably better fit to the values[22] of the single-energy phase-shift analysis at 25 and 50 MeV than are the phases of the BS-II potential. In addition, there are gauge corrections to be made to the BS-II calculation before comparison to the predictions of other models; no such terms were added by Brown (or us) for the momentum-squared operator in the potential. Although such extra gauge terms may be somewhat small, it is clear from Low's theorem[23] that they must be calculated, or at least estimated, if one is to have an assurance of obtaining the correct model dependence of the bremsstrahlung cross section.

As in the case of the HJ potential above, no gauge terms were added for angular-momentum operators in the potential.

C. Tabakin Potential

The cross section quoted by Pearce, Gale, and Duck[8] (PGD) for the nonlocal separable Tabakin potential,[9] line 6 of Table I, is quite different from our new value for that potential, line 7.

Some of this difference is explainable. First, Tabakin incorrectly assumed that the Blatt-Biedenharn (BB) coupling parameter ϵ_2, and all partial waves with $L>2$, contributed negligibly to the nucleon-nucleon amplitude in the energy range of interest for the nuclear-force problem. PGD[8] added to Tabakin's model a one-pion-exchange (OPE) amplitude for the higher partial waves, but kept ϵ_2 at Tabakin's value of zero. Lines 2 and 3 of Table II indicate that if the one-pion-exchange BB ϵ_2 is included, it increases the PPB cross section by over 50% at 61.7 MeV, 30°. This indicates that it can not be neglected if one wishes to consistently use OPE for the states not specified by Tabakin. Second, the OPE amplitude introduced by PGD differs substantially from the usual one, both on- and off-energy shell. This is illustrated in Fig. 3.

A more direct comparison between our calculation and PGD's can be made by using only the original Tabakin potential, with $\epsilon_2=0$ (no angular-momentum coupling) and with no contributions from states with $L>2$. The results for this case should be identical to the corresponding ones of PGD, yet comparison of lines 4 and 5 and of 6 and 7 still shows a strong disagreement. Although the two calculations were the same in principle, there *was* a practical difference, in that PGD calculated the partial-wave amplitudes directly from the potential parameters, while we had an (unnecessary) intermediate step of quasiphase evaluation. For a check of this part of our calculation, we compared all of our computed *on-shell* quasiphase values against the phase shifts quoted by Tabakin[9]: there was precise agreement. It would seem unlikely that our off-shell values could be wrong when our on-shell ones were right, since they were both computed by the use of the same computer program statement. The other half of our computer program, which calculated the PPB cross sections from

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[20] V. Brown, B. Scott, and R. Bryan (private communications).

[21] P. Signell and M. Ulrickson (private communication). V. Brown's HJ potential phases were in agreement with those of the authors of this reference, in contrast to the situation with the BS-II phases.

[22] M. Sher, P. Signell, and M. Miller (to be published). See also M. H. MacGregor *et al.*, Phys. Rev. **173**, 1272 (1968).

[23] F. E. Low, Phys. Rev. **110**, 974 (1958); L. Heller, *ibid.* **174**, 1580 (1968).

TABLE II. Examination of approximations used by several authors. The integrated cross sections are for 61.7 MeV, 30°, as in Table I. MS refers to the present calculation, PGD to Pearce, Gale, and Duck, and T to Tabakin.

Line	Authors	Potential	Approx.	$d\sigma/d\Omega_1 d\Omega_2$ (μb/sr^2)	Ref.
1	PGD	T+OPE	$\epsilon_2=0$	2.4	8
2	MS	T+OPE	$\epsilon_2=0$	2.20	here
3	MS	T+OPE	OPE ϵ_2	3.40	here
4	PGD	T	$L\leq 2$, $\epsilon_2=0$	1.75	8
5	MS	T	$L\leq 2$, $\epsilon_2=0$	2.23	here
6	PGD	T	$L\leq 2$, $\epsilon_2=0$, two-energy on-shell	2.08	8
7	MS	T	$L\leq 2$, $\epsilon_2=0$, two-energy on-shell	2.53	here
8	MS	T+OPE	OPE for $L>2$	3.40	here
9	MS	T+HJ	HJ for $L>2$	2.83	here
10	MS	HJ		2.90	here
11	MS	T+HJ	HJ for $L>2$	2.83	here
12	MS	HJ		2.90	here
13	MS	HJ	two-energy on-shell	3.21	here
14	MS	T+HJ	two-energy on-shell	2.98	here
15	Nyman	HJ	single-energy on-shell	2.50	14
16	MS	HJ		2.90	here

the quasiphases, was the same one which produced the agreement with the latest calculations of Brown and of Drechsel and Maximon when using the HJ quasiphases. In addition, a state-by-state comparison was made to HJ single-scattering contributions calculated by Brown,[24] with close (but not precise) agreement. Finally, the PPB cross section was computed very close to the energy shell and was found to be in good agreement with the value obtained from the elastic-scattering cross section via Low's theorem.[23]

Another possible source of difference was PGD's direct addition of PPB T-matrix elements computed with respect to several different nucleon-spin quantization axes.[25] This approximation would be expected to be a good one at very low energies where the directions of the quantization axis they used, given by the c.m. proton direction coming into the strong interaction, are not very different in the various single-scattering diagrams.[2,17] However, at 61.7 MeV, 30°, the orientations of the axes they used varied by less than 7°, so this can account for only a small part of the difference shown. One should also note that the integrated cross section $d\sigma/d\Omega_1 d\Omega_2$ was here obtained by numerical integration of the differential cross section $d\sigma/d\Omega_1 d\Omega_2 d\theta_\gamma$ over the photon-emission angle θ_γ. PGD, however, integrated $d\sigma/d\Omega_1 d\Omega_2 dE_1$ over the energy of one of the final protons E_1. It might be difficult to obtain accurate answers in this way, since $d\sigma/d\Omega_1 d\Omega_2 dE_1$ diverges at the limits of integration at low energies.

Concerning the $L>2$ states for the Tabakin model, the HJ potential is certainly much more realistic than is pure OPE. The effect of using HJ rather than OPE for the higher waves unspecified by Tabakin is shown in lines 8 and 9 of Table II. The change in the cross section is striking and is due in good part to the shift from the very large OPE BB ϵ_2 to the smaller HJ BB ϵ_2. In lines 10 and 11, it is seen that this more realistic Tabakin-model prediction is in much better agreement with the HJ prediction. However, one must be cautious about drawing any conclusions from these numbers, since the Tabakin potential gives a much poorer fit to the elastic data than does the HJ potential.

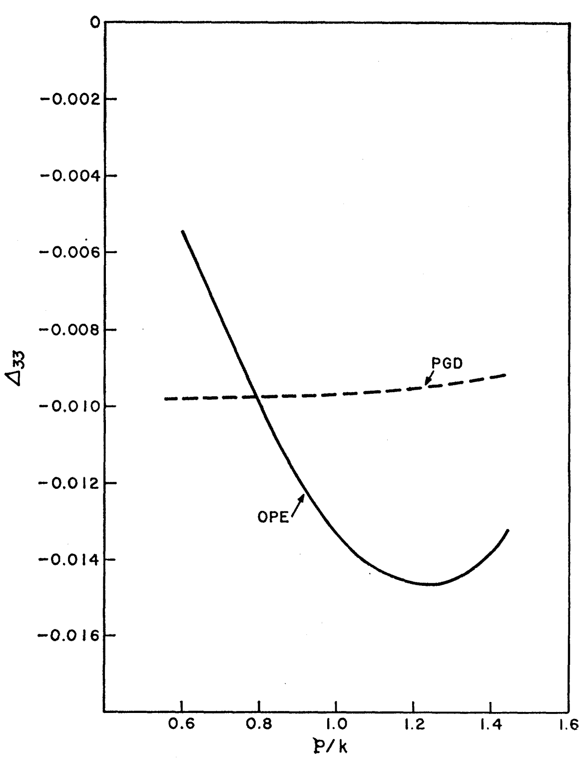

FIG. 3. Comparison of the $^3F_3$-state Pearce-Gale-Duck-OPE quasiphase with the one from the usual OPE formula at 50 MeV. The p/k value is the ratio of the off-shell to on-shell momentum.

[24] V. Brown (private communication).

[25] W. A. Pearce (private communication); see also Ref. 8.

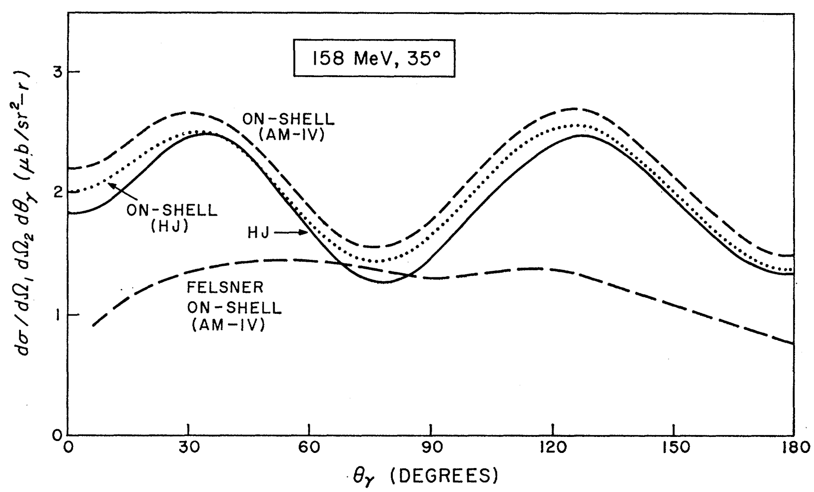

FIG. 4. The line marked HJ is the same as that shown in Fig. 2. The on-shell (HJ) (dotted) curve was calculated in the two-energy on-shell approximation, using HJ phase shifts. The upper (dashed) curve used the same approximation, but with the multienergy-analysis phase shifts given by Arndt and MacGregor. This is to be compared with Felsner's calculation, the lower (dashed) curve, apparently using the same approximation and the same phase shifts (see text).

D. Model-Independent Calculations

There have been two types of "model-independent" calculations, the Feshbach-Yennie[26] type of two-energy *approximation* used by Felsner,[13] by Pearce, Gale, and Duck,[8] and by the present authors,[2,15–17] and the Low[23] type of one-energy *approximation* used by Nyman[14] and the present authors.[16,17] In both of these approximations, only the observed elastic nucleon-nucleon scattering amplitudes are used, thus obviating the need for a model of the nucleon-nucleon interaction. The PPB cross sections so derived are usually described as being model-independent and so they are, but it is not possible to interpret them as being *the* model-independent parts of the cross sections.[27] This is obvious if only because the two model-independent approximations generally give such different answers.[16]

The two-energy approximation can be defined as approximating each off-energy-shell nucleon-nucleon quasiphase Δ_l by its on-energy-shell value:

$$\Delta_l(k', k) \cong \Delta_l(k, k) = \sin\delta_l(k),$$

where $\delta_l(k)$ is the usual elastic-scattering phase shift. The c.m. momentum of either nucleon, k, is to be evaluated for the pair of nucleons which are on their mass shells when entering or leaving the nucleon-nucleon interaction. Thus, one is to use the *initial*-state

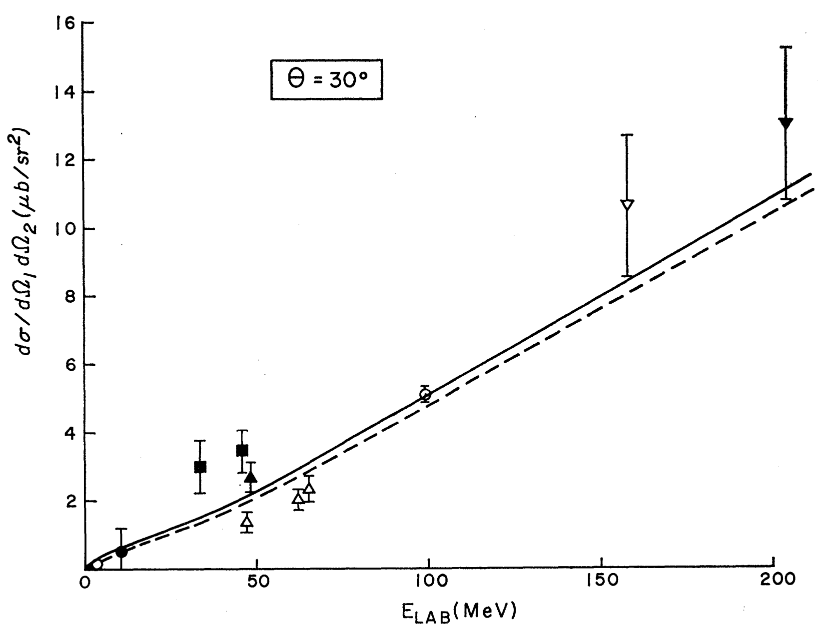

FIG. 5. Integrated cross section for a Harvard-geometry proton exit angle of 30°, as a function of incident-proton energy. The solid line is the single-scattering prediction of the Hamada-Johnston potential without Coulomb correction; the dashed line is the prediction with on-shell Coulomb correction. The experimental points are those listed in Table III and are from Refs. a(○), c(●), d(■), e, g, and h(△), f(▲), i(⊙), j(▽), and k(▼) of Table III.

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[26] H. Feshbach and D. R. Yennie, Nucl. Phys. **37**, 150 (1962).
[27] The terminology used in a number of publications has been a source of confusion on this point.

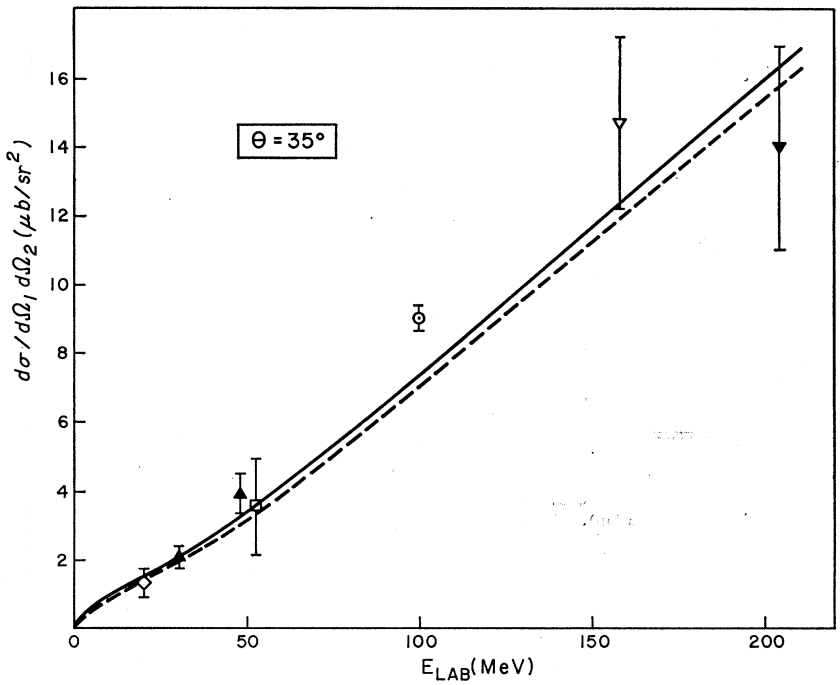

FIG. 6. The integrated cross section for a Harvard-geometry angle of 35°, as a function of incident-proton energy. The solid line is the single-scattering prediction of the Hamada-Johnston potential without Coulomb correction; the dashed line is the prediction with on-shell Coulomb correction. The experimental points are those listed in Table IV and are from Refs. a(◇), b and c(△), d(□), e(⊙), f(▽), and g(▼).

nucleon-nucleon c.m. energy for those terms in which the photon is emitted after the nucleon-nucleon interaction, and the *final*-state nucleon-nucleon c.m. energy when the photon emission occurs before the nucleon-nucleon interaction. For 61.7 MeV, $\theta=30°$, and $\theta_\gamma=0°$, the two equivalent-lab-frame energies are 61.7 and 18.3 MeV, respectively.

We previously found[15,16] that the two-energy on-shell approximation gave a PPB cross section which was accurate to within 1% at 10 MeV, 30° and at 20 MeV, 35°. Comparison of lines 12 and 13 of Table II shows that the error has climbed to 10% at 61.7 MeV, 30°. This increase is due to the increased absolute distance from the energy shell, which tends to make the approximation less valid, and to strong off-shell effects in the *p* waves, which were unimportant at the lower energies. Felsner's two-energy on-shell value, line 9 of Table I, is in strong disagreement with our corresponding value on line 10. A comparison of the photon angular distributions at 158 MeV, 35° is shown in Fig. 4. The exact and two-energy on-shell HJ curves in the figure show that the approximate curve may be expected to lie slightly above the exact one. Our "on-shell (AM-IV)" curve was made using the same set of on-shell elastic-scattering amplitudes used by Felsner, so there should be exact agreement with his curve. We have no explanation for the large discrepancy, but we note that Felsner's result was also found[16] to be much lower than ours at 10 MeV, 30°.

If two potential models gave identical on-energy-shell matrix elements, then their two-energy on-shell PPB cross sections would be identical. Lines 13 and 14 of Table II show that the HJ and HJ+Tabakin potentials are rather different on shell. If one "corrects" the on-shell Tabakin cross section to the on-shell HJ value

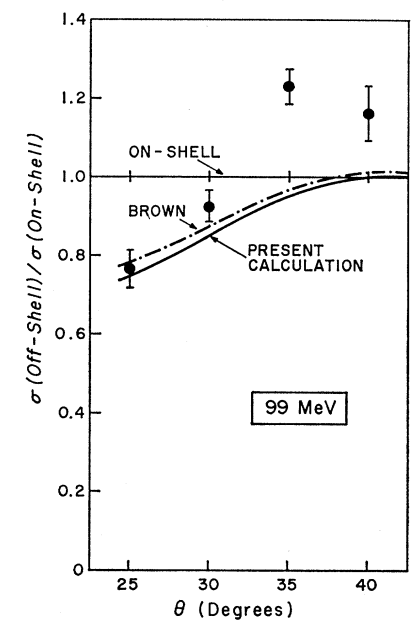

FIG. 7. The ratio of the off-shell Hamada-Johnston model PPB integrated cross section $d\sigma/d\Omega_1 d\Omega_2$ to the on-shell value, as a function of the proton-exit angles θ for an incident-proton energy of 99 MeV. The present-calculation values are the Coulomb-corrected ones of Table V. The Brown values are those of Ref. 3, adjusted for Coulomb effects by use of multiplicative factors obtained by dividing the numbers in line 3 by those in line 1 in Table V. The data are those of Sannes, Trischuk, and Stairs (Ref. e of Table IV).

TABLE III. Measured and calculated values of the PPB integrated cross sections for Harvard geometry, $\theta=30°$. The calculated values are for the Hamada-Johnston potential.

Authors	Ref.	Energy (MeV)	Expt σ (μb/sr^2)	Calc σ (μb/sr^2) No Coulomb	With Coulomb
Silverstein *et al.*	a	3.2	$0.15_{-0.15}^{+0.17}$	0.29	0.10
Joseph *et al.*	b	10	≤ 0.42	0.58	0.44
Crawley *et al.*	c	10.5	$0.5_{-0.5}^{+0.7}$	0.60	0.46
Slaus *et al.*	d	33.5	3.6 ± 1.1	1.47	1.32
Slaus *et al.*	d	46	3.8 ± 0.7	2.06	1.89
Mason *et al.*	e	47.1	1.37 ± 0.29	2.12	1.94
Warner	f	48	2.68 ± 0.45	2.17	1.99
Halbert *et al.*	g	61.7	2.04 ± 0.24	2.90	2.66
Mason *et al.*	h	65	2.34 ± 0.38	3.08	2.82
Sannes *et al.*	i	99	5.14 ± 0.22	5.03	4.75
Gottschalk *et al.*	j	158	10.2 ± 1.7	8.46	8.12
Rothe *et al.*	k	204	$13.\pm2.4$	11.07	10.59

[a] E. A. Silverstein and K. G. Kibler, Phys. Rev. Letters **21**, 922 (1968).
[b] C. Joseph, A. Niiler, V. Valcovic, R. Spiger, S. T. Emerson, T. Canada, J. Sandler, and G. C. Phillips, Bull. Am. Phys. Soc. **13**, 567 (1968); A. Niiler (private communication).
[c] C. M. Crawley, D. L. Powell, and B. V. Narasimha Rao, Phys. Letters **26B**, 576 (1968).
[d] I. Slaus, J. W. Verba, J. R. Richardson, R. F. Carlson, W. J. H. Van Oers, and L. S. August, Phys. Rev. Letters **17**, 536 (1966). Corrected for finite geometry by use of the Drechsel-Maximon (DM) correction factor, W. J. H. Van Oers (private communication); see also Ref. 3.
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[f] R. E. Warner, Can. J. Phys. **44**, 1225 (1966). Corrected for finite geometry by D. Drechsel and R. Warner, Phys. Rev. **181**, 1720 (1969).
[g] M. L. Halbert, D. L. Mason, and L. C. Northcliffe, Phys. Rev. **168**, 1130 (1968); and M. L. Halbert and D. L. Mason (private communica tion). Corrected for finite geometry by use of the DM correction factor, see Ref. e. On the advice of the authors, only the 61.7-MeV cross section obtained from their long large-aperture run is quoted. The relatively small number of events detected in the short, small-aperture run makes the error in the resulting cross section less certain.
[h] D. L. Mason, M. L. Halbert, A. Von der Wonde, and L. C. Northcliffe, Phys. Rev. **179**, 940 (1969).
[i] F. Sannes, J. Trischuk, and D. G. Stairs, Phys. Rev. Letters **21**, 1474 (1968).
[j] B. Gottschalk, W. J. Shlaer, and K. H. Wang, Nucl. Phys. **A94**, 491 (1967); B. Gottschalk and W. J. Shlaer (private communication). Revised to include later measurements.
[k] K. W. Rothe, P. F. M. Koehler, and E. H. Thorndike, Phys. Rev. **157**, 1247 (1967).

by use of an additive or multiplicative factor, and then applies that same factor to the off-shell Tabakin+HJ cross section of line 11, it is seen that the corrected Tabakin+HJ prediction is a small amount *larger* than that of thc HJ potential, line 10. This is in contrast to a naive comparison of lines 3 and 6 of Table I, which would seem to indicate a *smaller* value for the Tabakin potential.

In the single-energy approximation,[16,23] the two-nucleon *off-shell* amplitudes are expanded in Taylor series about mean values of the kinematical variables, and one can then show that in the linear approximation only *on-shell* amplitudes and first derivatives survive. For 61.7 MeV, $\theta=30°$, the (lab-frame energy) expansion point is about 40 MeV, close to the average of the energies cited above for the two-energy approximation.

TABLE IV. Measured and calculated values of the PPB integrated cross sections for Harvard geometry, $\theta=35°$. The calculated values are for the Hamada-Johnston potential. The 52-MeV experiment by Sanada *et al.* was for $\theta=33°$; a theoretically interpolated value for 35° is shown.

Authors	Ref	Energy (MeV)	Expt σ (μb/sr^2)	Calc σ (μb/sr^2) No Coulomb	With Coulomb
Bahnsen-Burman	a	20	1.3 ± 0.4	1.49	1.34
Thompson *et al.*	b	30	2.10 ± 0.28	2.06	1.89
Warner	c	48	3.93 ± 0.57	3.26	3.05
Sanada *et al.*	d	52	(3.6 ± 1.4)		
Sannes *et al.*	e	99	9.01 ± 0.33	7.27	6.93
Gottschalk *et al.*	f	158	14.7 ± 2.5	12.3	11.9
Rothe *et al.*	g	204	14 ± 2.7	16.4	15.8

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TABLE V. Values of the integrated cross section $d\sigma/d\Omega_1 d\Omega_2$ in μb/sr^2 for an incident-proton energy of 99 MeV. The calculated values are for the Hamada-Johnston potential, and the Coulomb corrections are as described previously (in Ref. 16). The data are from Ref. e of Table IV.

Line	On- or off-shell	Coulomb included?	Proton exit angle θ 25°	30°	35°	40°
1	Off-shell	No	4.00	5.03	7.27	16.9
2	On-shell approx.	No	5.21	5.84	7.66	16.9
3	Off-shell	Yes	3.71	4.75	6.93	16.2
4	On-shell approx.	Yes	4.93	5.56	7.32	16.2
5	Experiment	Yes	3.77±0.23	5.14±0.22	9.01±0.33	18.83±1.15

We have previously noted[16] that this procedure, called the "soft-photon approximation" by Nyman, might be expected to be in error by an amount which is roughly independent of the beam energy for low energies, but of course strongly dependent on the proton-exit angle. We found that the exact PPB cross section was 20% higher than Nyman's approximate one at 10 MeV, 30°. Comparison of lines 15 and 16 of Table II shows that the value of the HJ potential model at 61.7 MeV, 30°, appears to be about 16% higher than Nyman's approximation to it. However, Nyman's calculation was completely covariant, in contrast to the potential calculations. Although it seems unlikely at the energies under consideration, it is conceivable that the potential-model calculations need to be corrected for relativistic effects, especially in the magnetic-moment term. The *nucleon-nucleon* parts of the PPB potential-model amplitudes are already relativistic, in the sense that the better potential models yield elastic-scattering amplitudes which are in essential agreement with those used in the covariant calculations.

III. COMPARISON TO EXPERIMENT

Before comparison can be made between predictions for the coplanar Harvard geometry and experimental values, corrections must usually be applied to the latter to correct for the finite (noncoplanar) geometry of the counters. The earliest corrections[28] were made by simply integrating the essentially constant phase-space factor over the counter geometry, under the assumption of negligible variation in the matrix elements over this region. A measurement of out-of-plane cross sections by Gottschalk, Schlaer, and Wang[29] was at one time interpreted as suggesting a linear fall-off of the cross section with the out-of-plane angle Φ to a value of zero at the kinematical limit Φ_m. Then PGD[8] computed out-of-plane cross sections, and their numerical results were later fitted by a parabolic formula[30]:

$$\sigma(\Phi)=\sigma(0)\times[1-(\Phi/\Phi_m)^2].$$

Subsequently, however, Drechsel and Maximon[10] found the calculated cross section to approach a finite value at the kinematical limit. Since the phase-space factor stays finite as the limit is approached,[31] and since there is no reason for the matrix elements to all be zero, the behavior found by Drechsel and Maximon seems plausible. Most of the large-counter data has now been corrected for finite geometry by the use of out-of-plane curves produced by these latter authors.

The calculated angular distribution at 158 MeV, 30° is compared to the data in Fig. 2. Our integrated cross sections are compared to the experimental ones in Figs. 5–7, which include curves with on-shell Coulomb corrections. Although the accuracy of these corrections is unknown, it perhaps gives an idea of what one might obtain from a more accurate calculation. The Coulombic correction is of course much larger at very low energies. Nevertheless, the spin-triplet Coulomb amplitude increases with energy as the nucleon-nucleon scattering angle moves away from 90°.[17]

Tables III–V show numerical values corresponding to the figures. Note that Fig. 7 shows only the ratios

TABLE VI. Ratios of on-shell-approximation values to off-shell values for some representative quasiphases and for integrated cross sections at 61.7 meV, 30°. For example, the off-shell effects in the BS-II and HJ quasiphases are as large as 34% (BS-II $^3P_1$), but the model splitting of those effects is at most 6% in the quasiphases ($^3P_0$) and only 1% in the integrated PPB cross sections.

Potential (see text)	Type	Quasiphases ($p/k=0.55$) $^1S_0$	$^3P_0$	$^3P_1$	PPB cross sections
BS-II	p^2-dependent	0.81	1.25	1.34	1.10
HJ	hard core	0.80	1.17	1.33	1.11
Tabakin ($L\leq 2$)+HJ ($L>2$)	nonlocal, sep.	0.72	1.00	1.47	1.05

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of Coulomb-corrected off-shell to on-shell HJ cross sections for an incident-proton energy of 99 MeV and various proton-exit angles θ. It is seen that the off-shell cross section approaches the on-shell one as θ approaches $\approx 45°$, where the elastic limit is reached. Brown's calculation is slightly higher, but this seems to be due to other causes besides the omission of double scattering by the present authors (see Fig. 2). The bump in the curve in Fig. 5 near 10 MeV is due to the high peak in the nuclear $^1S_0$-state interaction near that energy. The ^3P-state magnetic-moment part of the PPB amplitude is negligible at 10 MeV, but becomes dominant by 200 MeV, as has been previously noted.[2,8]

In general, the agreement between calculated and measured values is good, although the recent Oak Ridge measurements[32] at 47 and 62 MeV indicate a somewhat lower value than expected. The general trend of the 99-MeV data, shown in Fig. 7, is indeed downward as one moves away from the energy shell ($\theta \approx 45°$). Nevertheless, there is a very serious discrepancy at 35°.

Table VI displays the model splitting of the quasi-phases and PPB cross section for three potentials of different types, according to our calculations.

Note added in proof: Bryan and Scott (Ref. 7) indicate the notation BS-III for the potential here denoted BS-II.

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Configuration Mixing of Major Shells in Mass-16 and Mass-17 Nuclei

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The odd-parity ($T=\frac{1}{2}$) levels of O^{17} are investigated within the framework of the spherical shell model in order to determine the importance of configuration mixing of the ($2s$, $1d$) and $1p$ major shells. Particles in the $2s_{1/2}$ and $1d_{5/2}$ subshells and holes in the $1p_{1/2}$ subshell are considered. It is shown that a configuration space which includes both 2-particle–1-hole states and 4-particle–3-hole states is required to achieve good agreement between calculated and experimental energies in the range of excitation from 3.06 to 8.88 MeV. As confirmed by previous work, calculations based on a truncated 2-particle–1-hole space predict excitation energies that are higher by about 2 MeV than the observed values. In addition, the effects of $(2s_{1/2}, 1d_{5/2})$-$1p_{1/2}$ mixing on the odd- and even-parity ($T=1$) states of N^{16} above its four lowest levels are evaluated. The consequences of this mixing are a lowering of calculated energies by an average of 2–3 MeV and many changes in the predicted sequence of levels. The significance of these results in relation to the neutron elastic scattering cross section for N^{15} is discussed, and a need for more experimental data on the spins and parities of N^{16} levels is indicated.

1. INTRODUCTION

THE region of nuclei around mass number 16 has proven to be relatively fruitful for applying the spherical shell model to predict or correlate the energies of low-lying nuclear levels. Yet there have appeared in the literature a number of conspicuous examples of the apparent failure of this model to interpret the energies of such states. Included among these cases are the following discrepancies:

(1) Several of the odd-parity ($T=0$) states of O^{16} are predicted to have energies too high by more than 1 MeV on the basis of a $(2s, 1d)^1$-$1p^{-1}$ configuration space. This error persists regardless of whether the residual interaction is described by effective matrix elements,[1] by a finite-range potential with fitted exchange mixture,[2] or by realistic forces.[3]

(2) The second (0, 1$^+$) state of F^{18} is actually lower by several MeV than the predictions based on $(2s, 1d)^2$

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