# Elastic Surface Waves in Crystals

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(Received 21 November 1968)

Elastic surface waves in a general semi-infinite, anisotropic medium are discussed in terms of a six-dimensional vector formalism. The six-dimensional state vectors have the physical significance that their first three components constitute the displacement of and their last three components the force on the surface of the medium. For a semi-infinite medium with no sources of energy in its interior, a definite relation exists between force and particle velocity at the surface. This relation defines an impedance matrix for the semi-infinite medium which is a function of frequency, wave vector, and material parameters. The impedance matrix exhibits interesting symmetry properties and provides us with some generally valid relations for surface waves. In particular, formulas for energy and power relations attain attractive forms especially suitable for numerical computation. Finally, some characteristic properties of surface waves along free surfaces are discussed, including undamped and damped ("leaky") surface waves.

## I. INTRODUCTION

THE elastic surface waves propagating along the free surface of a solid consist of combinations of two or three nonuniform plane waves. In a Cartesian coordinate system where the $x$ axis is the direction of propagation in the surface and where the $y$ axis is the direction normal to the surface, the displacement at an arbitrary point may be written

$$\boldsymbol{\xi}=\sum_{i=1}^{3} c_i\boldsymbol{\xi}_i e^{-j(\omega t-qx+n_iy)}. \tag{1.1}$$

Here $\boldsymbol{\xi}_i$ is the normalized displacement of the $i$th component wave. The imaginary part of $n_i$ ensures that the component waves decrease exponentially with increasing distance from the surface. To each displacement vector $\boldsymbol{\xi}_i$ there corresponds a force $\mathbf{v}_i$ per unit area of the surface. Since there can be no force on a free surface the boundary condition is

$$\mathbf{v}=\sum_{i=1}^{3} c_i\mathbf{v}_i=0. \tag{1.2}$$

The equation requires the force vectors $\mathbf{v}_i$ to be linearly dependent. This requirement is satisfied only for particular values of $\omega/q$. Thus, (1.2) determines the phase velocities of the surface waves. The solution of (1.2) also gives the relative values of the amplitude factors $c_i$, i.e., the amplitudes of the component waves.

In an isotropic material, first discussed by Lord Rayleigh,[1] the surface wave (Rayleigh wave) is a combination of two component waves each having zero displacement in the $z$ direction, whereas the nonparticipating wave is purely transverse with displacement and force in the $z$ direction. The transverse wave numbers $n_i$ are in this case all purely imaginary. In anisotropic materials these results may be altered in various ways. In general the wave numbers $n_i$ become complex (generalized Rayleigh waves), the surface wave contains three rather than two component waves, and the displacement in the $z$ direction is nonzero. Examples of such waves have been published.[2–13] A few authors[6,8,12–14] have considered the general case of triclinic symmetry, which is the subject of the present paper.

The solution of (1.2) is, in general, very complicated and analytic solutions can be found only in cases of high crystalline symmetry. Therefore, the surface wave solutions in crystals must be obtained numerically. In this paper we shall present a formalism which has been found particularly useful in numerical work.

Assume that a force $\mathbf{v}$ is applied to the surface. This force causes a displacement $\boldsymbol{\xi}$ of the surface which can be obtained from Eq. (1.1) and the first of Eqs. (1.2) by elimination of the amplitude coefficients $c_i$. We shall write the relation between force and displacement as

$$\mathbf{v}=j\omega\mathbf{Z}\boldsymbol{\xi}, \tag{1.3}$$

where the $3\times3$ matrix $\mathbf{Z}$ is determined by the displacement vectors $\boldsymbol{\xi}_i$ and force vectors $\mathbf{v}_i$. It follows from the definition that $\mathbf{Z}$ may be interpreted as a mechanical impedance matrix for the semi-infinite medium. The surface waves along the free surface occur for those values of $\omega/q$ for which $\mathbf{Z}$ is singular. On the basis of orthogonality relations for the displacement vectors $\boldsymbol{\xi}_i$ and force vectors $\mathbf{v}_i$ we shall find some very useful relations for $\mathbf{Z}$. In particular we shall find simple expressions for power flow and energy densities which are suitable for numerical work. It will also be shown that

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the numerical search for surface wave solutions may be simplified by using the impedance matrix.

Finally, some properties of the "leaky" surface waves which have been observed recently[15,16] will be considered. Numerical investigations have shown that these waves exist in several materials.[17]

## II. VECTOR FORM OF THE EQUATIONS OF MOTION

The equations of motion will be referred to a Cartesian coordinate system where we take the $xz$ plane to be a plane boundary of the crystal with the positive $y$ axis pointing into the crystal. Without loss of generality we may assume that the tensor of elasticity of the medium is referred to the same coordinate system.

Denoting the displacement by the vector $\hat{\xi}_l$ and the strain tensor by $\hat{\xi}_{lk}$ we have

$$\hat{\xi}_{lk}=\tfrac{1}{2}(\partial\hat{\xi}_l/\partial x_k+\partial\hat{\xi}_k/\partial x_l), \tag{2.1}$$

where the subscripts $l$ and $k$ refer to any one of the coordinates $(x,y,z)=(x_1,x_2,x_3)$. The stress tensor may then be written

$$\hat{\sigma}_{mn}=\lambda_{mnlk}\partial\hat{\xi}_l/\partial x_k, \tag{2.2}$$

where we have used the general symmetry properties of the elastic tensor, viz.,

$$\lambda_{mnlk}=\lambda_{nmlk}=\lambda_{nmkl}=\lambda_{klmn}. \tag{2.3}$$

In addition to Eq. (2.2) we need the equation of motion

$$s\partial^2\hat{\xi}_l/\partial t^2=\partial\hat{\sigma}_{lk}/\partial x_k, \tag{2.4}$$

where $s$ is the density of the medium.

We shall find it convenient to work with the vector

$$\hat{\nu}_l=\lambda_{l2mn}\partial\hat{\xi}_m/\partial x_n, \tag{2.5}$$

which is the force per unit area on a plane parallel with the $xz$ plane. Equations (2.2), (2.4), and (2.5) will now be Fourier transformed with respect to $x$, $z$, and $t$. That is, we multiply by $e^{j(\omega t-q_xx-q_zz)}$ and integrate over all $x$, $z$, and $t$. By this means we obtain equations in $\omega$, $q_x$, $q_z$, and $y$ which take the form

$$\sigma_{lk}=(jq_x\lambda_{lkm1}+jq_z\lambda_{lkm3})\xi_m+\lambda_{lkm2}d\xi_m/dy, \tag{2.6}$$

$$-s\omega^2\xi_m=jq_x\sigma_{m1}+jq_z\sigma_{m3}+d\nu_m/dy, \tag{2.7}$$

$$\nu_l=j(q_x\lambda_{l2m1}+q_z\lambda_{l2m3})\xi_m+\lambda_{l2m2}d\xi_m/dy. \tag{2.8}$$

Here, $\sigma_{ik}$, $\xi_l$, and $\nu_l$ are the Fourier transforms of $\hat{\sigma}_{ik}$, $\hat{\xi}_l$, and $\hat{\nu}_l$, respectively. In (2.7), $\sigma_{m1}$ and $\sigma_{m3}$ are the forces per unit area on planes parallel with the $yz$ plane and the $xy$ plane, respectively.

In order to introduce a matrix formalism we define the column vectors

$$\boldsymbol{\xi}=\begin{pmatrix}\xi_x\\ \xi_y\\ \xi_z\end{pmatrix} \quad\text{and}\quad \boldsymbol{\nu}=\begin{pmatrix}\nu_x\\ \nu_y\\ \nu_z\end{pmatrix} \tag{2.9}$$

and the 3×3 matrices

$$\mathbf{A}=(\lambda_{l1m1}),\quad \mathbf{B}=(\lambda_{l2m2}),\quad \mathbf{C}=(\lambda_{l3m3}),$$
$$\mathbf{D}=(\lambda_{l1m3}+\lambda_{l3m1}),\quad\text{and}\quad \mathbf{L}=(q_x\lambda_{l2m1}+q_z\lambda_{l2m3}), \tag{2.10}$$

where $\lambda_{l1m1}$ denotes the element in row number $l$ and column number $m$ of $\mathbf{A}$, and similarly for the other matrices. It is worth noting that the symmetry properties of $\lambda_{lkmn}$ require $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ to be symmetric matrices.

After elimination of $\sigma_{m1}$ and $\sigma_{m3}$ by means of (2.6) Eq. (2.7) takes the form

$$d\boldsymbol{\nu}/dy+(s\omega^2\mathbf{I}-q_x^2\mathbf{A}-q_xq_z\mathbf{D}-q_z^2\mathbf{C}+\mathbf{L}^T\mathbf{B}^{-1}\mathbf{L})\boldsymbol{\xi}+j\mathbf{L}^T\mathbf{B}^{-1}\boldsymbol{\nu}=0. \tag{2.11}$$

Here, $\mathbf{I}$ denotes the three-dimensional unit matrix, and $\mathbf{L}^T$ is the transpose of the matrix $\mathbf{L}$. Likewise, Eq. (2.8) may be written

$$d\boldsymbol{\xi}/dy+j\mathbf{B}^{-1}\mathbf{L}\boldsymbol{\xi}-\mathbf{B}^{-1}\boldsymbol{\nu}=0. \tag{2.12}$$

If we introduce the six-dimensional vector

$$\boldsymbol{\zeta}=\begin{pmatrix}\boldsymbol{\xi}\\ \boldsymbol{\nu}\end{pmatrix}, \tag{2.13}$$

we may combine (2.11) and (2.13) in a six-dimensional vector equation

$$d\boldsymbol{\zeta}/dy+j\mathbf{N}\boldsymbol{\zeta}=0, \tag{2.14}$$

where the 6×6 matrix $\mathbf{N}$ is defined in terms of 3×3 matrix blocks as

$$\mathbf{N}=\begin{pmatrix}\mathbf{B}^{-1}\mathbf{L}, & j\mathbf{B}^{-1}\\ j\mathbf{S}, & \mathbf{L}^T\mathbf{B}^{-1}\end{pmatrix} \tag{2.15}$$

with $\mathbf{S}$ given by

$$\mathbf{S}=q_x^2\mathbf{A}+q_zq_x\mathbf{D}+q_z^2\mathbf{C}-\mathbf{L}^T\mathbf{B}^{-1}\mathbf{L}-s\omega^2\mathbf{I}. \tag{2.16}$$

Equation (2.14) may be regarded as a six-dimensional form of the equations of motion of the medium. The vector $\boldsymbol{\zeta}$ has a simple physical significance. Its first three components are the components of the displacement vector and the last three components are the components of the force per unit area on a plane normal to the $y$ axis. The matrix $\mathbf{N}$ is composed of terms which are homogeneous polynomials in $\omega$, $q_x$, and $q_z$. The off-diagonal blocks are symmetric while the diagonal blocks are nonsymmetric. We notice that $\omega$ appears only in the diagonal terms of the matrix block $\mathbf{S}$.

If it is assumed that a volume force density acts on the volume element, a slightly generalized version of (2.14) is obtained, viz.,

$$d\boldsymbol{\zeta}/dy+j\mathbf{N}\boldsymbol{\zeta}=\begin{pmatrix}\mathbf{0}\\ \mathbf{f}\end{pmatrix}, \tag{2.17}$$

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where $\mathbf{0}$ is the three-dimensional zero vector and $\mathbf{f}$ is the Fourier transform with respect to $x$, $z$, and $t$ of the volume force density.

In this paper we shall consider primarily free waves in the crystal, i.e., solutions of (2.14). A general solution may be expressed in terms of eigenvectors of $\mathbf{N}$. Let the equation

$$(\mathbf{N}-n\mathbf{I})\boldsymbol{\zeta}=0 \qquad (2.18)$$

have for solutions the eigenvalues

$$n_1,\, n_2,\, \cdots,\, n_6,$$

and eigenvectors

$$\boldsymbol{\zeta}_1,\, \boldsymbol{\zeta}_2,\, \cdots,\, \boldsymbol{\zeta}_6. \qquad (2.19)$$

Then a general solution may be written

$$\boldsymbol{\zeta}=\sum_1^6 c_m e^{-jn_m y}\boldsymbol{\zeta}_m. \qquad (2.20)$$

Since $\mathbf{N}$ is a function of $q_x$, $q_z$, and $\omega$ this will be the case for $n_m$ and $\boldsymbol{\zeta}_m$ as well. The coefficients $c_m$ are determined by the boundary conditions. In general they will also be functions of $q_x$, $q_z$, and $\omega$. However, before we consider the boundary value problem, it is evidently important to study the eigenvalues and eigenvectors $n_m$ and $\boldsymbol{\zeta}_m$. This subject will be considered in the Sec. III.

## III. EIGENVALUES AND EIGENVECTORS OF N; IMPEDANCE MATRIX OF THE SEMI-INFINITE MEDIUM

Clearly the eigenvalue $n_m$ is the $y$ component of the wave vector of a plane wave. The displacement and force vectors of this plane wave are given respectively by the first three and the last three components of the eigenvector $\boldsymbol{\zeta}_m$. When we eliminate $\mathbf{v}$ from the eigenvalue equation (2.18), we obtain

$$[n^2\mathbf{B}-n(\mathbf{L}+\mathbf{L}^T)+q_x^2\mathbf{A}+q_xq_z\mathbf{D}+q_z^2\mathbf{C}-s\omega^2\mathbf{I}]\boldsymbol{\xi}=0. \qquad (3.1)$$

Some properties of the eigenvalues and eigenvectors are deduced most easily from the form (3.1) while others are conveniently obtained from the six-dimensional matrix equation. We shall work, therefore, with both forms alternatively. Consider first the three-dimensional vector equation (3.1). The eigenvalues $n_m$ are found by requiring the determinant of (3.1) to be zero.

$$|n^2\mathbf{B}-n(\mathbf{L}+\mathbf{L}^T)+q_x^2\mathbf{A}+q_xq_z\mathbf{D}+q_z^2\mathbf{C}-s\omega^2\mathbf{I}|=0. \qquad (3.2)$$

Since each term is a homogeneous polynomial of degree 2 in $q_x$, $n$, $q_z$, and $\omega$, it follows that the determinantal equation will be of degree 6 in these variables.

For a given $\omega$ this equation describes three closed surfaces in the real $(q_x,n,q_z)$ space. These are the slowness surfaces.[2] It follows from the equation that the surfaces are point symmetric with respect to the origin.

Assume now that we keep $q_x$ and $q_z$ real and fixed and consider the solutions for $n$ as functions of $\omega$ when $\omega$ is real. Since the equation is a sextic equation in $n$ with real coefficients the following possibilities exist: (1) Six complex solutions in three conjugate pairs. This case occurs for sufficiently small values of $\omega$. (2) Six real solutions. This case occurs for sufficiently large values of $\omega$. For intermediate values of $\omega$ we obtain either (3) two real solutions and two complex conjugate pairs, or (4) four real solutions and one complex conjugate pair. Synge[6] has shown that in each case the solutions may be divided into two parts in the following way. Since the complex solutions occur in conjugate pairs one half of these represent waves which increase exponentially with increasing $y$, the others decrease exponentially. One half of the solutions with real eigenvalues, represent plane waves with energy transport in the positive $y$ direction; the other half, in the negative $y$ direction. A little care must be taken when two or more of the eigenvalues coincide. Waves which otherwise have different directions of energy flow then degenerate into waves of identical phase velocities and zero-energy flow in the $y$ direction.

The classification of solutions outlined above is important for the selection of what we may call "physically acceptable solutions" for the semi-infinite medium.

Let us now return to the six-dimensional eigenvalue equation (2.18). The matrix $\mathbf{N}$ exhibits symmetry properties that impose orthogonality relations between the eigenvectors. These relations are conveniently expressed by means of the auxiliary matrices

$$\mathbf{T}=\begin{pmatrix}\mathbf{0} & \mathbf{I}\\ \mathbf{I} & \mathbf{0}\end{pmatrix} \quad\text{and}\quad \mathbf{P}=\begin{pmatrix}\mathbf{0} & -\mathbf{I}\\ \mathbf{I} & \mathbf{0}\end{pmatrix}, \qquad (3.3)$$

where $\mathbf{0}$ and $\mathbf{I}$ are three-dimensional zero and unit matrices, respectively. When the wave vector $\mathbf{q}=(q_x,q_z)$ and frequency $\omega$ are complex it is found that $\mathbf{N}$ satisfies the following relations:

$$\mathbf{TNT}=\mathbf{N}^T, \qquad (3.4)$$

$$\mathbf{PN}(\omega,\mathbf{q})\mathbf{P}^T=\mathbf{N}^\dagger(\omega^*,\mathbf{q}^*), \qquad (3.5)$$

$$\mathbf{N}(\omega,\mathbf{q})=-\mathbf{N}^*(\omega^*,-\mathbf{q}^*). \qquad (3.6)$$

Here $\mathbf{N}^\dagger$ denotes the Hermitian conjugate of $\mathbf{N}$. Since $\mathbf{T}=\mathbf{T}^{-1}$ and $\mathbf{P}^T=\mathbf{P}^{-1}$, the transformations (3.4) and (3.5) are similarity transformations which preserve the eigenvalues of $\mathbf{N}$.

The relation (3.4) leads to important orthogonality properties for the eigenvectors of $\mathbf{N}$. If $n_k$ is an eigenvalue of $\mathbf{N}$ and $\boldsymbol{\zeta}_k$ the corresponding eigenvector, they satisfy

$$\mathbf{N}\boldsymbol{\zeta}_k-n_k\boldsymbol{\zeta}_k=0. \qquad (3.7)$$

Multiplying by $\boldsymbol{\zeta}_l^T\mathbf{T}$ on the left and using the relation $\mathbf{TT}=\mathbf{I}$, we obtain

$$(n_l-n_k)\boldsymbol{\zeta}_l^T\mathbf{T}\boldsymbol{\zeta}_k=0. \qquad (3.8)$$

That is, eigenvectors corresponding to different eigenvalues are orthogonal, with the matrix $\mathbf{T}$ as a metric.

The relations (3.5) and (3.6) may be used to relate the eigenvalues and eigenvectors at $(\omega,\mathbf{q})$ to those at

$(\omega^*,\mathbf{q}^*)$ and $(\omega^*, -\mathbf{q}^*)$, respectively. Multiplying Eq. (3.7) on the left by $\mathbf{P}$ and using the relation $\mathbf{PP}^T=\mathbf{1}$ and (3.5) we obtain

$$(\mathbf{N}^\dagger(\omega^*,q^*)-n_k\mathbf{I})\mathbf{P}\boldsymbol{\zeta}_k=0. \tag{3.9}$$

Multiplying (3.9) on the left by $\mathbf{T}$ and taking the complex conjugate we find

$$(\mathbf{N}(\omega^*,q^*)-n_k^*\mathbf{I})\mathbf{TP}\boldsymbol{\zeta}_k^*=0. \tag{3.10}$$

This shows that $n_k^*$ is an eigenvalue of $\mathbf{N}(\omega^*,\mathbf{q}^*)$ and that the corresponding eigenvector is $\mathbf{TP}\boldsymbol{\zeta}_k^*$. Notice that when $\boldsymbol{\zeta}_k$ is written in terms of displacement $\boldsymbol{\xi}_k$ and force $\mathbf{v}_k$ as in Eq. (2.14), we obtain

$$\mathbf{TP}\boldsymbol{\zeta}_k^*=\begin{pmatrix}\boldsymbol{\xi}_k^*\\ -\mathbf{v}_k^*\end{pmatrix}. \tag{3.11}$$

It is an important consequence of (3.10) that for real values of $\omega$ and $\mathbf{q}$ the vectors $\boldsymbol{\zeta}_k$ and $\mathbf{TP}\boldsymbol{\zeta}_k^*$ are both eigenvectors of $\mathbf{N}$ corresponding to eigenvalues $n_k$ and $n_k^*$, respectively. When $\omega$ and $\mathbf{q}$ are real the eigenvectors $\boldsymbol{\zeta}$ also satisfy an orthogonality relation with respect to $\mathbf{P}$, viz.

$$(n_l^*-n_k)\boldsymbol{\zeta}_l^\dagger\mathbf{P}\boldsymbol{\zeta}_k=0. \tag{3.12}$$

This follows from Eq. (3.9) by multiplication on the left by $\boldsymbol{\zeta}_l^\dagger$. In terms of the three-dimensional subvectors $\boldsymbol{\xi}$ and $\mathbf{v}$ this relation may be written

$$(n_l^*-n_k)(\mathbf{v}_l^\dagger\boldsymbol{\xi}_k-\boldsymbol{\xi}_l^\dagger\mathbf{v}_k)=0. \tag{3.13}$$

When $n_k$ is a complex eigenvalue this equation states that the quantity $\mathbf{v}_k^\dagger\boldsymbol{\xi}_k$ must be real. On the other hand, if $n_k$ is a real eigenvalue Eq. (3.1) shows that $\boldsymbol{\xi}_k$ may be chosen real. From the eigenvalue equation (2.19) it then follows that $\mathbf{v}_k$ will be purely imaginary. It should be emphasized that in this case the relations (3.8) and (3.12) are equivalent.

To summarize: $\mathbf{v}_k^\dagger\boldsymbol{\xi}_k$ is real whenever $n_k$ is complex and pure imaginary whenever $n_k$ is real.

### Impedance Matrix

Let us consider waves which are excited by a surface force acting on the surface $y=0$. The free waves will then appear as a limiting case. The boundary conditions on the surface may be satisfied by a linear combination of three eigensolutions. This means that the sum in (2.20) is reduced to three terms, the remaining three coefficients $c_m$ being zero. The choice of three from a total of six eigensolutions may be done in 20 different ways. This choice must be determined from additional requirements imposed on the solutions. We shall denote the three eigenvectors satisfying these requirements as the *physically acceptable eigenvectors* and assume that they are for given $\omega$ and $\mathbf{q}$: $\boldsymbol{\zeta}_1$, $\boldsymbol{\zeta}_2$, and $\boldsymbol{\zeta}_3$. The remaining eigenvectors are the *physically unacceptable eigenvectors.*

Equation (2.20) may then be written

$$\boldsymbol{\xi}=\sum_1^3 c_m e^{-jn_m y}\boldsymbol{\xi}_m\,,\qquad \mathbf{v}=\sum_1^3 c_m e^{-jn_m y}\mathbf{v}_m\,. \tag{3.14}$$

If $\omega$ and $\mathbf{q}$ are real, the acceptable eigenvectors are the ones corresponding either to uniform plane waves carrying power away from the surface or to nonuniform plane waves with amplitudes decreasing exponentially as $y\to\infty$. For complex-valued $\omega$ and $\mathbf{q}$, the principle of selecting the physically acceptable eigenvectors is more complicated, and we leave it aside for the moment.

In order to write (3.14) in a more convenient way let us introduce the third-order matrices

$$\mathbf{X}_1=(\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3)\,,\quad \mathbf{Y}_1=(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3)\,, \tag{3.15}$$

$$\mathbf{X}_2=(\boldsymbol{\xi}_4,\boldsymbol{\xi}_5,\boldsymbol{\xi}_6)\,,\quad \mathbf{Y}_2=(\mathbf{v}_4,\mathbf{v}_5,\mathbf{v}_6)\,, \tag{3.16}$$

the third-order diagonal matrices

$$\mathbf{D}_1=\mathrm{diag}(e^{-jn_1y},e^{-jn_2y},e^{-jn_3y})\,,\qquad \mathbf{D}_2=\mathrm{diag}(e^{-jn_4y},e^{-jn_5y},e^{-jn_6y})\,, \tag{3.17}$$

and the column vector

$$\mathbf{c}_1=\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}. \tag{3.18}$$

Equation (3.14) may then be written

$$\boldsymbol{\xi}=\mathbf{X}_1\mathbf{D}_1\mathbf{c}_1\,,\quad \mathbf{v}=\mathbf{Y}_1\mathbf{D}_1\mathbf{c}_1. \tag{3.19}$$

When $\mathbf{c}_1$ is eliminated from these equations we obtain

$$\mathbf{v}=j\omega\mathbf{Z}_1\boldsymbol{\xi}\,, \tag{3.20}$$

where $\mathbf{Z}_1$ is the impedance matrix of the medium and is defined by

$$\mathbf{Z}_1=-j\omega^{-1}\mathbf{Y}_1\mathbf{X}_1^{-1}. \tag{3.21}$$

It is important to notice that $\mathbf{Z}_1$ is independent of $y$ and is, therefore, the same for any plane parallel with the boundary plane $y=0$. We shall see that this matrix exhibits some interesting properties which are particularly useful in the discussion of surface waves.

Consider first the analytical form of the matrix elements. From Eq. (3.1) it is observed that the elements of $\boldsymbol{\xi}$ are homogeneous polynomials of degree 4 in the variables $\omega$, $q_x$, $q_z$, and $n$. Therefore, the elements of $\mathbf{X}_1$ are homogeneous polynomials of degree 4 in the variables $\omega$, $q_x$, $q_z$, $n_1$, $n_2$, and $n_3$. The elements of $\mathbf{Y}_1$ are polynomials of degree 5 in the same variables. A little reflection shows that the elements of $\mathbf{Z}_1$ are rational functions of the same six variables, the numerator and denominator of each element being homogeneous polynomials of degree 13.

So far $\omega$, $q_x$, $q_z$, $n_1$, $n_2$, and $n_3$ have been regarded as independent variables. However, the eigenvalues are

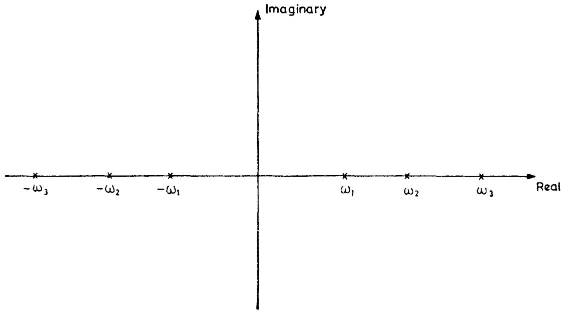


FIG. 1. Branch points of **Z** located on the real frequency axis. **q** is kept real, fixed, and normalized to $|\mathbf{q}|=1$.

actually algebraic functions of $\omega$, $q_x$, and $q_z$, defined by the eigenvalue equation

$$|\mathbf{N}-n\mathbf{I}|=0. \tag{3.22}$$

We may imagine that $n_1$, $n_2$, and $n_3$ are eliminated from the expression for $\mathbf{Z}_1$ by means of (3.22). $\mathbf{Z}_1$ then appears as one branch of a multivalued function **Z** of the three variables $\omega$, $q_x$, and $q_z$. We have seen already that the selection of three eigenvectors out of a total of six may be done in 20 different ways. Thus, **Z** has twenty branches, one of which is $\mathbf{Z}_1$. It may be shown that the matrix retain the property of being a homogeneous function of order zero of the three variables, i.e., for an arbitrary constant $\alpha$ we have

$$\mathbf{Z}(\omega,q_x,q_z)=\mathbf{Z}(\alpha\omega,\alpha q_x,\alpha q_z). \tag{3.23}$$

Furthermore, from Euler's rule for homogeneous polynomials it may be shown that

$$q_x(\partial/\partial q_x)\mathbf{Z}+q_z(\partial/\partial q_z)\mathbf{Z}+\omega(\partial/\partial\omega)\mathbf{Z}=0. \tag{3.24}$$

Let us now choose a constant real value for **q** with $|\mathbf{q}|=1$ and consider $\omega$ as the only free variable. The matrix function $\mathbf{Z}(\omega)$ has a number of poles and branch points. The latter are also branch points of $n=n(\omega)$. They may be defined as those points of the $\omega$ plane for which Eq. (3.22) has double roots. In Ref. 18 (Appendix IV) the algebraic equation satisfied by the branch points is derived, and the number of branch points is shown to be at most 30. Of these at least 6 are located on the real axis (see Fig. 1). These branch points subdivide the real $\omega$ axis into four regions, each corresponding either to 0, 2, 4, or 6 real eigenvalues of $N$, as explained before. The location of the remaining branch points in the complex $\omega$ plane is symmetric with respect to the real and imaginary axes.

The branch constructed from the physically acceptable solutions has been denoted by $\mathbf{Z}_1$. It is related in a particular way to the branch $\mathbf{Z}_2$ constructed from the physically unacceptable solutions. This is seen from the orthogonality relation (3.8) which may be written

$$\mathbf{X}_1^T\mathbf{Y}_2+\mathbf{Y}_1^T\mathbf{X}_2=0, \tag{3.25}$$

where the matrices involved are defined in (3.15) and (3.16). When multiplied on the right by $\mathbf{X}_2^{-1}$ and on the left by $(\mathbf{X}_1^T)^{-1}$ Eq. (3.25) yields

$$\mathbf{Z}_1^T(\omega,\mathbf{q})=-\mathbf{Z}_2(\omega,\mathbf{q}). \tag{3.26}$$

It is obvious that this equation remains valid for a matrix $\mathbf{Z}_1$ constructed from any three solutions provided that $\mathbf{Z}_2$ is constructed from the remaining three.

Let us next consider the effect on $\mathbf{Z}_1$ of reversing the sign of **q**. If $n_k$ and $\boldsymbol{\zeta}_k$ are a set of eigensolutions of **N** at $(\omega,\mathbf{q})$ it follows from (3.6) that $-n_k^*$ and $\boldsymbol{\zeta}_k^*$ are eigensolutions at $(\omega^*, -\mathbf{q}^*)$. Combining this result with the relations (3.4) and (3.5) we observe that $-n_k$ and $\mathbf{TP}\boldsymbol{\zeta}_k$ are solutions at $(\omega, -\mathbf{q})$. For real-valued $\omega$ and **q**, reversion of the sign of the eigenvalues has the effect of transforming the acceptable solutions into unacceptable ones and vice versa. In terms of impedance matrices we then have

$$\mathbf{Z}_1(\omega, -\mathbf{q})=-\mathbf{Z}_2(\omega,\mathbf{q}). \tag{3.27}$$

Combining (3.27) with (3.26) we obtain

$$\mathbf{Z}_1(\omega, -\mathbf{q})=\mathbf{Z}_1^T(\omega,\mathbf{q}), \tag{3.28}$$

i.e., inversion of the sign of **q** transposes the impedance matrix. A little reflection shown that this holds for any branch of $\mathbf{Z}(\omega,\mathbf{q})$. This property of **Z** may be taken to be an expression of reciprocity. If $\omega$ is replaced by $-\omega$ neither the eigenvalues nor the eigenvectors change since $\omega$ always appears in **N** in even powers. If the impedance constructed from the physically acceptable solutions at $(\omega,\mathbf{q})$ is denoted as before by $\mathbf{Z}_1(\omega,\mathbf{q})$, we may conclude that the impedance matrix at $(-\omega, \mathbf{q})$ has a branch $\mathbf{Z}_s$ that satisfies

$$\mathbf{Z}_s(-\omega, \mathbf{q})=-\mathbf{Z}_1(\omega,\mathbf{q}). \tag{3.29}$$

However, this need not be the branch that is relevant for our physical problem, since the change

$$\omega\rightarrow-\omega$$

may transform physically acceptable solutions into unacceptable ones. For example a bulk wave carrying power away from the surface is changed by a reversion of sign into a wave carrying power towards the surface.

When we use the relation (3.11) for eigenvectors with complex conjugate eigenvalues and remember that for real eigenvalues $n_k$ we may chose $\boldsymbol{\zeta}_k$ real and $\mathbf{v}_k$ purely imaginary, a little reflection shows that for real values of $\omega$ and **q** we have

$$\mathbf{Z}_1(-\omega, \mathbf{q})=-\mathbf{Z}_2^*(\omega,\mathbf{q}), \tag{3.30}$$

which according to (3.26) is the same as

$$\mathbf{Z}_1(-\omega, \mathbf{q})=\mathbf{Z}_1^\dagger(\omega,\mathbf{q}). \tag{3.31}$$

This relation is complementary to (3.28) in expressing reciprocity.

[18] K. A. Ingebrigtsen and A. Tonning, Elab Report No. TE-74, Electronics Research Lab., Norwegian Institute of Technology, Trondheim, Norway (unpublished).

In the range of $\omega$ where all eigenvalues are complex

$$-\omega_1 \leq \omega \leq \omega_1 \tag{3.32}$$

(see Fig. 1), the equations above lead to an important relation for the impedance matrix. When $\omega$ is in this range it follows, from the definition of $\mathbf{Z}_1$, that

$$\mathbf{Z}_1(-\omega, \mathbf{q}) = -\mathbf{Z}_1(\omega, \mathbf{q}). \tag{3.33}$$

Combining this equation with (3.31) we obtain

$$\mathbf{Z}_1 = -\mathbf{Z}_1^\dagger. \tag{3.34}$$

The impedance matrix is thus seen to be skew-Hermitian.

The poles and zeros of $|\mathbf{Z}_1(\omega)|$ give rise to a particular type of wave mode called surface waves. Consider first a point $\omega_0$ in the frequency plane for which $|\mathbf{Z}_1| = 0$, i.e., one of the eigenvalues of $\mathbf{Z}_1$ is zero. We can find a nonzero vector $\boldsymbol{\xi}_0$ such that

$$\mathbf{Z}_1(\omega_0)\boldsymbol{\xi}_0 = 0. \tag{3.35}$$

This means that the displacement may be nonzero while the stress is zero $\mathbf{v} = 0$, which is the boundary condition of a free surface.

If $\omega_0$ is a pole of $|\mathbf{Z}_1(\omega)|$ one of the eigenvalues of $\mathbf{Z}_1^{-1}(\omega_0)$ is zero, i.e., a nonzero vector $\mathbf{v}_0$ exists so that

$$\mathbf{Z}_1^{-1}(\omega_0)\mathbf{v}_0 = 0. \tag{3.36}$$

This means that we have a nonzero stress $\mathbf{v}_0$ while the displacement is zero, which is the boundary condition of a clamped surface.

It is the former type of surface wave that has most practical interest and we shall confine our attention to this type. Evidently the null-vector $\boldsymbol{\xi}_0$ of $\mathbf{Z}_1$ is the displacement of the surface. The displacement $\boldsymbol{\xi}$ at a distance $y$ from the surface may now be found from (3.14) or its equivalent (3.19)

$$\boldsymbol{\xi} = \mathbf{X}_1\mathbf{D}_1\mathbf{c}_1. \tag{3.37}$$

Here, $\mathbf{D}_1$ is the diagonal matrix defined by (3.17). For $y = 0$ we obtain

$$\boldsymbol{\xi}_0 = \mathbf{X}_1\mathbf{c}_1, \tag{3.38}$$

which can be inserted into (3.37) to yield

$$\boldsymbol{\xi} = \mathbf{X}_1\mathbf{D}_1\mathbf{X}_1^{-1}\boldsymbol{\xi}_0. \tag{3.39}$$

Let us finally see how the orthogonality relations may be expressed in terms of the impedance matrix.

By means of

$$\mathbf{v}_k = j\omega\mathbf{Z}_1\boldsymbol{\xi}_k, \quad k = 1, 2, 3; \tag{3.40}$$

Eq. (3.13) may be written

$$\boldsymbol{\xi}_l^\dagger(\mathbf{Z}_1 + \mathbf{Z}_1^\dagger)\boldsymbol{\xi}_k = C_k\delta_{lk} \quad l,k = 1, 2, 3, \tag{3.41}$$

where the constant $C_k$, if it is not zero, may be chosen equal to unity by suitable normalization of the vectors, i.e.,

$$\begin{aligned} C_k &= 0 \quad \text{if} \quad n_k \text{ is complex} \\ &= 1 \quad \text{if} \quad n_k \text{ is real.} \end{aligned} \tag{3.42}$$

This relation has an important consequence for the surface-wave solutions. Let any vector $\boldsymbol{\xi}$ be expanded in terms of $\boldsymbol{\xi}_1$, $\boldsymbol{\xi}_2$, and $\boldsymbol{\xi}_3$ as shown in (3.14). From (3.41) we then obtain

$$\boldsymbol{\xi}^\dagger(\mathbf{Z}_1 + \mathbf{Z}_1^\dagger)\boldsymbol{\xi} = \sum_{k,l=1}^{3} c_l c_k^* \boldsymbol{\xi}_k^\dagger(\mathbf{Z}_1 + \mathbf{Z}_1^\dagger)\boldsymbol{\xi}_l^\dagger = \sum{}' |c_k|^2, \tag{3.43}$$

where the last sum is taken over the $k$'s for which $n_k$ is real. However, if $\boldsymbol{\xi}$ is a null vector of $\mathbf{Z}_1$,

$$\mathbf{Z}_1\boldsymbol{\xi} = 0, \tag{3.44}$$

the left-hand side of (3.43) is zero. According to (3.43) this means that the coefficient of expansion $c_k$ is zero if $n_k$ is real. Thus, a null vector of $\mathbf{Z}_1$ for real $\omega$ and $\mathbf{q}$ belongs to the subspace $K$ spanned by the vectors $\boldsymbol{\xi}_k$ corresponding to complex eigenvalues $n_k$. Examples of surface-wave solutions are known for which the frequency satisfies

$$\omega_1 < \omega < \omega_2 \tag{3.45}$$

(see Fig. 1). This means that there are two real and four complex eigenvalues. The result given above shows that the eigenvector corresponding to a real eigenvalue is not part of the surface wave solution. The surface wave for the frequency range given by (3.45) is therefore a two-component wave.

## IV. POWER AND ENERGY RELATIONS

### A. Power Flux

In this section we derive expressions for power flux and energy density. It will be seen that the impedance matrix appears in much the same way as in the case of an electric network. The relations will provide us with generally valid criteria for selecting the physically acceptable solutions.

Consider the instantaneous power passing a unit area parallel with the boundary. This power density is

$$S_y = \mathrm{Re}\{\mathbf{v}^T e^{j\omega t}\}\ \mathrm{Re}\{j\omega\boldsymbol{\xi}e^{j\omega t}\}. \tag{4.1}$$

Averaging over one period we obtain

$$S_y = \tfrac{1}{2}\,\mathrm{Re}\{j\omega\mathbf{v}^\dagger\boldsymbol{\xi}\}. \tag{4.2}$$

This expression may also be written

$$S_y = \tfrac{1}{4}j\omega\boldsymbol{\zeta}^\dagger\mathbf{P}\boldsymbol{\zeta}. \tag{4.3}$$

For real $\omega$ and $\mathbf{q}$, $S_y$ is independent of $y$. This is seen from

$$(\partial/\partial y)S_y = \tfrac{1}{4}j\omega(\partial/\partial y)(\boldsymbol{\zeta}^\dagger\mathbf{P}\boldsymbol{\zeta}) + \tfrac{1}{4}j\omega\boldsymbol{\zeta}^\dagger\mathbf{P}(\partial/\partial y)\boldsymbol{\zeta}. \tag{4.4}$$

When the derivatives are eliminated by means of the differential equation

$$(\partial/\partial y)\boldsymbol{\zeta} = -j\mathbf{N}\boldsymbol{\zeta}, \tag{4.5}$$

and the symmetry properties of $\mathbf{N}$ are used, we obtain

$$(\partial/\partial y)S_y = 0. \tag{4.6}$$

This shows that the power flux in the direction normal to the surface is constant. The result holds only if $\omega$ and $\mathbf{q}$ are real. An alternative formulation of (4.3) is

$$S_y=\tfrac{1}{4}\omega^2\boldsymbol{\xi}^\dagger(\mathbf{Z}+\mathbf{Z}^\dagger)\boldsymbol{\xi}. \tag{4.7}$$

The power flow in the $y$ direction in thus determined by the Hermitian part of the impedance matrix. Since we exclude power sources located in the interior of the medium, or at $y=\infty$, we must require, in order that our solution be physically acceptable, that the expression (4.7) be nonnegative for any acoustic field. This means that the Hermitian part of $\mathbf{Z}$ must be positive semidefinite. Most branches of $\mathbf{Z}$ will not satisfy this requirement and are, therefore, physically unacceptable. A sufficient condition for the requirement to be satisfied is that $\mathbf{Z}$ is constructed from the eigenvectors which represent waves carrying power away from the surface and into the medium, i.e., the branch which was denoted by $\mathbf{Z}_1$ in the Sec. III. In the following we shall consider this branch only, and we shall therefore omit the subscript on $\mathbf{Z}$.

Assuming now that $\mathbf{Z}$ has a positive semidefinite Hermitian part, let us discuss the condition for having zero power flow in the $y$ direction. Since $S_y$ is independent of $y$, it is sufficient to impose

$$S_y=0$$

at the surface. When $\boldsymbol{\xi}_0$ is the displacement of the surface, we have

$$\boldsymbol{\xi}_0^\dagger(\mathbf{Z}+\mathbf{Z}^\dagger)\boldsymbol{\xi}_0=0, \tag{4.8}$$

and, since the matrix is positive semidefinite, this means that $\boldsymbol{\xi}_0$ must belong to the null space of $(\mathbf{Z}+\mathbf{Z}^\dagger)$, i.e.,

$$(\mathbf{Z}+\mathbf{Z}^\dagger)\boldsymbol{\xi}_0=0. \tag{4.9}$$

The dimension of the null space is 0, 1, 2, or 3, depending on the value of $\omega$. As $\omega$ varies the dimension changes abruptly at the critical frequencies $\omega_1$, $\omega_2$, and $\omega_3$ defined in connection with Fig. 1.

To every $\boldsymbol{\xi}_0$ corresponds an impressed surface force $\boldsymbol{\phi}_0$ given by

$$\boldsymbol{\phi}_0=j\omega\mathbf{Z}\boldsymbol{\xi}_0. \tag{4.10}$$

For particular values of $\omega$, $\mathbf{Z}$ is singular and has a null vector satisfying

$$\mathbf{Z}\boldsymbol{\xi}_0=0. \tag{4.11}$$

This solution represents a lossless mode satisfying the boundary condition of a traction-free surface.

Let us next consider power transport in directions parallel with the surface, assuming zero transport in the $y$ direction. If we denote by $\mathbf{v}_x$ the force per unit area normal to the $x$ axis the time average of power flow density in the $x$ direction is

$$S_x=\tfrac{1}{2}\,\mathrm{Re}\{j\omega\mathbf{v}_x{}^\dagger\boldsymbol{\xi}\}. \tag{4.12}$$

In terms of $\boldsymbol{\xi}$, $\mathbf{v}_x$ may be written

$$\mathbf{v}_x=\{jq_x\mathbf{A}+jq_z(\lambda_{l1m3})+(\lambda_{l1m2})\partial/\partial y\}\boldsymbol{\xi}, \tag{4.13}$$

where $(\lambda_{l1m3})$ and $(\lambda_{l1m2})$ are $3\times3$ matrices with row and column subscripts denoted by $l$ and $m$ as in the matrices defined by Eq. (2.10). Using Eq. (4.13) we find, after some manipulations,

$$S_x=\tfrac{1}{4}j\omega\boldsymbol{\zeta}^\dagger(\partial/\partial q_x)(\mathbf{PN})\boldsymbol{\zeta}. \tag{4.14}$$

Likewise

$$S_z=\tfrac{1}{4}j\omega\boldsymbol{\zeta}^\dagger(\partial/\partial q_z)(\mathbf{PN})\boldsymbol{\zeta}. \tag{4.15}$$

For a lossless surface mode, $S_x$ is conveniently expressed in terms of the impedance matrix. From (4.14) we have

$$S_x=\tfrac{1}{4}j\omega\boldsymbol{\zeta}^\dagger(\partial/\partial q_x)(\mathbf{PN}\boldsymbol{\zeta})-\tfrac{1}{4}j\omega\boldsymbol{\zeta}^\dagger\mathbf{PN}(\partial/\partial q_x)\boldsymbol{\zeta}. \tag{4.16}$$

Making use of Eq. (4.5), the relation

$$\boldsymbol{\zeta}^\dagger\mathbf{PN}=\boldsymbol{\zeta}^\dagger\mathbf{N}^\dagger\mathbf{P}$$

and the symmetry properties of $\mathbf{N}$ for real $\omega$ and $\mathbf{q}$, we find

$$S_x=j\tfrac{1}{4}\omega^2\partial/\partial y[\boldsymbol{\xi}^\dagger(\partial/\partial q_x)\mathbf{Z}\boldsymbol{\xi}+\boldsymbol{\xi}^\dagger(\mathbf{Z}+\mathbf{Z}^\dagger)\partial/\partial q_x\boldsymbol{\xi}]. \tag{4.17}$$

If $\boldsymbol{\xi}\to0$ as $y\to\infty$, we may introduce the power flux integrated with respect to $y$,

$$\bar{S}_x=\int_0^\infty S_x dy=\frac{\omega^2}{4j}\boldsymbol{\xi}_0{}^\dagger\left(\frac{\partial}{\partial q_x}\mathbf{Z}\right)\boldsymbol{\xi}_0, \tag{4.18}$$

where $\boldsymbol{\xi}_0$ is the displacement of the surface and Eq. (4.9) has been used. In the same way we find

$$\bar{S}_z=(\omega^2/4j)\boldsymbol{\xi}_0{}^\dagger(\partial/\partial q_z)\mathbf{Z}\boldsymbol{\xi}_0. \tag{4.19}$$

### B. Energy Density

The time average density of kinetic energy is, with $\omega$ and $\mathbf{q}$ real,

$$w_k=\tfrac{1}{4}\omega^2 s\boldsymbol{\xi}^\dagger\boldsymbol{\xi}. \tag{4.20}$$

This may also be written

$$w_k=(\omega/8j)\boldsymbol{\zeta}^\dagger(\partial/\partial\omega)(\mathbf{PN})\boldsymbol{\zeta}. \tag{4.21}$$

The density of potential energy is $\frac{1}{2}\hat{\sigma}_{ij}\hat{\xi}_{ij}$. For harmonic time variation its time average is

$$w_p=\tfrac{1}{4}\,\mathrm{Re}\left\{jq_x\mathbf{v}_x{}^\dagger\boldsymbol{\xi}+\mathbf{v}^\dagger\frac{\partial}{\partial y}\boldsymbol{\xi}+jq_z\mathbf{v}_z{}^\dagger\boldsymbol{\xi}\right\}. \tag{4.22}$$

The total energy density is the sum of $w_k$ and $w_p$. After some manipulations we find

$$w=w_p+w_k=\tfrac{1}{4}j\boldsymbol{\zeta}^\dagger\mathbf{PN}\boldsymbol{\zeta}+(q_x/\omega)S_x+(q_z/\omega)S_z. \tag{4.23}$$

Another form of the same expression is

$$\begin{aligned}w&=2w_k+(w_p-w_k)\\&=(\omega/4j)\boldsymbol{\zeta}^\dagger(\partial/\partial\omega)(\mathbf{PN})\boldsymbol{\zeta}+(w_p-w_k).\end{aligned} \tag{4.24}$$

The last term is the time average of the Lagrangian density and may be written

$$\begin{aligned}w_p-w_k&=(1/8j)(\boldsymbol{\zeta}^\dagger\mathbf{TN}\boldsymbol{\zeta}-\boldsymbol{\zeta}^\dagger\mathbf{N}^\dagger\mathbf{T}\boldsymbol{\zeta})\\&=-\frac{1}{8}\frac{\partial}{\partial y}(\boldsymbol{\xi}^\dagger\mathbf{v}+\mathbf{v}^\dagger\boldsymbol{\xi}).\end{aligned} \tag{4.25}$$

This expression is zero for a uniform, plane acoustic wave, but, in general, nonzero for nonuniform waves. We shall assume, as before, zero-power flow in the $y$ direction, i.e., that (4.9) is satisfied. The acoustic field then tends to zero for $y\rightarrow\infty$, and we may integrate with respect to $y$ to obtain

$$\bar{w}_p-\bar{w}_k=\int_0^\infty (w_p-w_k)dy=-\tfrac{1}{8}(\xi_0^\dagger v_0+v_0^\dagger\xi_0). \quad (4.26)$$

When the impedance matrix is introduced and Eq. (4.9) is used, this gives

$$\bar{w}_p-\bar{w}_k=-\tfrac{1}{4}j\omega\xi_0^\dagger\mathbf{Z}\xi_0. \quad (4.27)$$

For a lossless surface wave mode $\xi_0$ is a null vector of $\mathbf{Z}$, and for this case (4.27) shows that the kinetic energy per unit area equals the potential energy.

On the basis of (4.24) we may now find the energy density per unit area of the surface

$$\bar{w}=\int_0^\infty wdy=\frac{\omega}{4j}\int_0^\infty dy\boldsymbol{\zeta}^\dagger\frac{\partial}{\partial\omega}(\mathbf{PN})\boldsymbol{\zeta}-\tfrac{1}{4}j\omega\xi_0^\dagger\mathbf{Z}\xi_0. \quad (4.28)$$

Using the differential equation for $\boldsymbol{\zeta}$ in the same way as in the derivation of (4.18), we find that the integrand is a perfect differential in $y$, and obtain

$$\bar{w}=-\tfrac{1}{4}\omega\boldsymbol{\zeta}_0^\dagger\mathbf{P}\frac{\partial}{\partial\omega}\boldsymbol{\zeta}_0-\tfrac{1}{4}j\omega\xi_0^\dagger\mathbf{Z}\xi_0. \quad (4.29)$$

Introducing the impedance matrix and using (4.9), we obtain

$$\bar{w}=j\tfrac{1}{4}\omega^2\xi_0^\dagger\left(\frac{\partial}{\partial\omega}\mathbf{Z}\right)\xi_0. \quad (4.30)$$

Taking account of (3.24), (4.18), and (4.19) we find that this leads to

$$\omega\bar{w}=q_x\bar{S}_x+q_z\bar{S}_z. \quad (4.31)$$

### C. Group Velocity of a Surface Wave

Suppose that for real values of $\omega_0$ and $\mathbf{q}$ we have found a vector $\xi_0$ satisfying

$$\mathbf{Z}(\omega_0)\xi_0=0. \quad (4.32)$$

From (4.7) it is then evident that $S_y=0$, and $\xi_0$ accordingly satisfies (4.9) which together with (4.32) leads to

$$\mathbf{Z}^\dagger(\omega_0)\xi_0=0. \quad (4.33)$$

As we have seen already, $\xi_0$ is the displacement of the surface in a lossless surface wave mode. Suppose now that $q_x$ is changed by a small amount $\delta q_x$ while $q_z$ is kept constant. This will cause changes $\delta\mathbf{Z}$ and $\delta\xi_0$ in impedance matrix and displacement. In fact we have from (4.32)

$$\delta\mathbf{Z}\xi_0+\mathbf{Z}\delta\xi_0=0. \quad (4.34)$$

Multiplying on the left by $\xi_0^\dagger$, we have

$$\xi_0^\dagger\delta\mathbf{Z}\xi_0+\xi_0^\dagger\mathbf{Z}\delta\xi_0=0, \quad (4.35)$$

or, taking account of (4.33),

$$\xi_0^\dagger\delta\mathbf{Z}\xi_0=0. \quad (4.36)$$

For the variation of $\mathbf{Z}$ we have

$$\mathbf{Z}=\frac{\partial}{\partial q_x}\mathbf{Z}dq_x+\frac{\partial}{\partial\omega}\mathbf{Z}\frac{\partial\omega}{\partial q_x}dq_x. \quad (4.37)$$

When this expression is inserted into (4.36) we obtain

$$\xi_0^\dagger\left(\frac{\partial}{\partial q_x}\mathbf{Z}\right)\xi_0+\xi_0^\dagger\left(\frac{\partial}{\partial\omega}\mathbf{Z}\right)\xi_0\frac{\partial\omega}{\partial q_x}=0, \quad (4.38)$$

or, by comparison with (4.18) and (4.30)

$$\partial\omega/\partial q_z=\bar{S}_x/\bar{w}. \quad (4.39)$$

In the same way we find for the $x$ component of the group velocity

$$\partial\omega/\partial q_z=\bar{S}_z/\bar{w}. \quad (4.40)$$

The last two equations show that the velocity of energy propagation of a surface wave is equal to the group velocity.

## V. SURFACE-WAVE SOLUTIONS

In Sec. III we found that a lossless surface wave corresponds to a real zero of the determinant $\mathbf{Z}$ when the latter is regarded as a function of $\omega$. The null vector $\xi_0$ of $\mathbf{Z}$ at this zero is the displacement of the surface.

An important relation for the eigenvalues of $\mathbf{Z}$ may be deduced from Sec. IV. Consider the frequency range for which $|\omega|<|\omega_1|$ (Fig. 1). In this region $\mathbf{Z}$ is skew-Hermitian and from (4.30) it follows that the Hermitian matrix $(\partial/\partial\omega)(j\mathbf{Z})$ is positive semidefinite. From this fact it is easily shown that the eigenvalues of $\mathbf{Z}$ must have a monotonic variation with frequency. Denoting the real eigenvalues of $j\mathbf{Z}$ by $\lambda_i$ we find that

$$(\partial/\partial\omega)\lambda_i\geq 0;\quad i=1, 2, 3. \quad (5.1)$$

In the search for numerical surface-wave solutions it is therefore advantageous to work with the matrix $\mathbf{Z}$ rather than with $\mathbf{Y}_1$ defined by Eq. (3.15). The problem of finding the surface-wave solutions is reduced to finding the zeros of the three real functions $\lambda_i$, which all have monotonic variation with frequency.

Solutions of an entirely different kind are the zeros of $|\mathbf{Z}|$ for complex values of $\omega$. Suppose that

$$\omega_0=|\omega_0|e^{j\alpha} \quad (5.2)$$

is a zero for a real-valued $\mathbf{q}$. From (3.23) we have

$$\mathbf{Z}(\omega_0,q_x,q_z)=\mathbf{Z}(|\omega_0|,q_xe^{-j\alpha},q_ze^{-j\alpha}), \quad (5.3)$$

i.e., the impedance matrix is unchanged if we replace the complex $\omega_0$ by its absolute value and the real $\mathbf{q}$ by $\mathbf{q}e^{-j\alpha}$. It may be shown that this multiplication of the variables by a common factor subjects the eigenvalues

$n_i$ of $\mathbf{N}$ to the change

$$n_i \to n_i e^{-j\alpha}. \tag{5.4}$$

After this transformation has been carried out the solution evidently represents a wave with exponentially decreasing (or increasing) amplitude in the direction of propagation in the surface. Since we have assumed zero dissipation in the medium, the decrease in amplitude must be due to radiation of energy from the surface into the medium. Waves of this kind have been observed experimentally[15,16] and are called *leaky surface waves.* As before the power transport in the $y$ direction is given by Eq. (4.7) having now a nonzero and nonconstant value.

Consider a leaky surface wave propagating along the surface in the $x$ direction. Since its amplitude decreases exponentially with $x$ this also is the case with the power flux $S_x$. Accordingly we have

$$(\partial/\partial x)S_x<0. \tag{5.5}$$

On the other hand, conservation of energy requires

$$\frac{\partial}{\partial x}S_x+\frac{\partial}{\partial y}S_y=0, \tag{5.6}$$

and hence

$$(\partial/\partial y)S_y>0. \tag{5.7}$$

As before the acoustic field amplitude has an exponential variation with $y$, determined by the eigenvalues $n_i$ of $\mathbf{N}$. The above result shows that the amplitude is exponentially increasing with increasing distance from the surface, tending to infinity as $y\to\infty$. This is an unavoidable consequence of the fact that the field in the surface tends to infinity as $x\to-\infty$. In any concrete physical problem where the excitation of the waves must be accounted for, the permitted ranges of variation of $x$ and $y$ are limited. This removes the difficulty of infinite field. However, in our earlier discussion of the surface-wave solutions, the requirement of finite amplitudes at $y=\infty$ has been used in selecting the physically acceptable solutions and the physically acceptable branches of the impedance matrix. In the case of leaky surface waves this requirement is not applicable and should be replaced by a more direct application of the radiation condition. We are thus led to impose the following requirement: The physically acceptable branch of the impedance matrix must satisfy

$$\boldsymbol{\xi}^\dagger(\mathbf{Z}+\mathbf{Z}^\dagger)\boldsymbol{\xi}\geqslant 0 \tag{5.8}$$

for an arbitrary vector $\boldsymbol{\xi}$, i.e., the Hermitian part of $\mathbf{Z}$ must be positive semidefinite.

Most important in practice are leaky waves that are weakly damped, corresponding to zeros $\omega_0$ with an imaginary part which is small compared to the real part. The leaky waves that have been studied so far, experimentally or by computation, are of this kind. In some cases the damping of the waves along the surface has been so small that it has scarcely been measurable. In these cases an eigenvalue $z_i(\omega)$ of $\mathbf{Z}$ computed for a series of real values of $\omega$ shows a sharp minimum in its absolute value at some real frequency $\omega=\Omega_0$. From (5.1) it may be concluded that no minimum can occur in the frequency region $-\omega_1<\omega<\omega_1$, hence $\Omega_0>\omega_1$, i.e., the phase velocity of the leaky wave is larger than the velocity of the slowest bulk wave, with energy transport parallel to the free surface.

If the minimum is small, the zero of $|\mathbf{Z}|$ may be taken to be

$$\omega_0=\Omega_0-[z_i/(\partial z_i/\partial\omega)]. \tag{5.9}$$

Since $\Omega_0$ corresponds to a minimum in $|z_i|$ we have

$$[(\partial/\partial\omega)(z_iz_i^*)]_{\omega=\Omega_0}=0. \tag{5.10}$$

Splitting $z_i$ into its real and imaginary parts,

$$z_i=u+jv, \tag{5.11}$$

we find

$$\Delta\omega=\omega_0-\Omega_0=j[u/(\partial v/\partial\omega)]. \tag{5.12}$$

This shows again that if $z_i$ is purely imaginary for real values of $\omega$, i.e., if $u=0$, then the minimum of $|z_i|$ must be zero.

When $\Delta\omega$ is small we may discuss the acoustic field of the leaky wave in terms of the eigenvalues of $\mathbf{N}$ for $\omega=\Omega_0$. Since $\Omega_0>\omega_1$, $\mathbf{N}$ has at least two real eigenvalues. The displacement vector $\boldsymbol{\xi}$ of the leaky wave may be written

$$\boldsymbol{\xi}=\alpha_1\boldsymbol{\xi}_1+\alpha_2\boldsymbol{\xi}_2, \tag{5.13}$$

where $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$, are combinations of displacement vectors corresponding to complex eigenvalues and real eigenvalues, respectively. Then $\boldsymbol{\xi}_1$ is a null vector for the Hermitian part of the impedance matrix while $\boldsymbol{\xi}_2$ is a null vector of the imaginary part. The acoustic field is therefore composed of two different parts: A nondissipative field which stores the energy in a layer close to the surface and a dissipative part which has a component of energy flow in the $y$ direction. In order to maintain stationary conditions we must accordingly apply in the surface the force

$$\boldsymbol{\phi}=j\Omega_0z_i\boldsymbol{\xi}, \tag{5.14}$$

which excites the dissipative mode $\boldsymbol{\xi}_2$. When we suddenly remove the surface force, the acoustic field will decay with a decay time which is the inverse of $\Delta\omega$. Under stationary conditions the power flow in the $y$ direction is

$$S_y=|\alpha_2|^2\tfrac{1}{4}\Omega_0^2\boldsymbol{\xi}_2^\dagger(\mathbf{Z}+\mathbf{Z}^\dagger)\boldsymbol{\xi}_2=\tfrac{1}{2}\Omega_0^2u\boldsymbol{\xi}^\dagger\boldsymbol{\xi}. \tag{5.15}$$

The energy stored in the surface wave mode is

$$\bar{w}=j|\alpha_1|^2\frac{\Omega_0^2}{4}\boldsymbol{\xi}_1^\dagger\partial\mathbf{Z}/\partial\omega\boldsymbol{\xi}_1\bigg|_{\omega=\Omega_0}. \tag{5.16}$$

The amplitude $\alpha_2$ may be very small compared to $\alpha_1$. If the dissipative part of the wave has an energy velocity $v_{Gy}$ in the $y$ direction, and if the surface-wave part has fields with an effective penetration depth $\delta$ which is of the order of a wavelength, then we find

$$|\alpha_2|^2/|\alpha_1|^2 \simeq \frac{v'}{v_{Gy}} \frac{u}{\omega\partial v/\partial\omega}\bigg|_{\omega=\Omega_0}, \qquad (5.17)$$

where we have neglected higher-order terms in $u/(\omega\partial v/\partial\omega)$. Here $v'$ is a characteristic velocity defined as

$$v' = \Omega_0 \delta .$$

When $|\alpha_2|$ is small compared to $|\alpha_1|$, the energy density in the surface wave is

$$\bar{w} \simeq -\tfrac{1}{4}\omega^2 \partial v/\partial\omega \xi^\dagger \xi|_{\omega=\Omega_0}, \qquad (5.18)$$

which shows that

$$\Delta\omega \simeq -jS_y/2W . \qquad (5.19)$$

Numerical calculations given by Lim[17] have shown that the leaky waves very often have an attenuation per period which is only $10^{-4}-10^{-3}$. In these cases the approximate results given above should be quite good.

## ACKNOWLEDGMENT

The authors acknowledge the financial support received from the Norwegian Council of Scientific Research.

PHYSICAL REVIEW VOLUME 184, NUMBER 3 15 AUGUST 1969

# Low-Temperature Thermal Conductivity of Impure Insulators

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(Received 17 February, 1969)

The low-temperature thermal transport properties of an insulating crystal with an arbitrary concentration of randomly located phonon scattering centers are found for slab geometry. The phonon distribution function, the distribution of temperature, the Kapitza resistance at the boundaries, and the heat flux are evaluated by analytically and numerically solving the complete Boltzmann equation with energy-dependent scattering cross section.

## I. INTRODUCTION

THE problem of heat conduction through an electrically insulating crystal at low temperatures leads to the formulation of the Boltzmann transport equation for phonons. In this equation, the collision operator plays the most significant part because it is responsible for thermal resistance. The collision operator introduces into the equation the effect of the two most important scattering mechanisms: scattering of phonons by stationary obstacles (impurities, defects, boundaries, etc.) and scattering of phonons by phonons. Because of mathematical difficulties it has been an accepted practice not to solve the Boltzmann transport equation as it is, but to treat it with the aid of the *relaxation-time approximation.*

A large body of experimental observations has been successfully explained in terms of this approximation, relying, in particular, on the work of Callaway.[1] At the same time, however, a large number of questions remain unanswered. They motivate the present work.

The first question we raise, concerns the validity of the relaxation-time approximation: If, by suitable choice of the available parameters, the theoretical curves resulting from the relaxation-time approximation may be fitted to the experimental data, does this mean that further conclusions of the theory are correct? This question cannot be answered without working out the more exact theory, since the relaxation-time approximation is not capable of estimating its own accuracy.

As we shall see in the discussion (see, in particular, Sec. VIII the discussion of Fig. 8), some conclusions of the approximate theory are born out; others are contradicted by the results of a more exact theory. Questioning further, let us point out that there are some basic difficulties with the relaxation-time approximation.

According to this approximation, if more than one relaxation (scattering) mechanism is present, the system should approach equilibrium faster, than if any one of the individual mechanisms alone is active: More channels enable quicker relaxation. The speed of relaxation being measured by the inverse relaxation time, this means

$$\frac{1}{\tau} > \max_i \left(\frac{1}{\tau_i}\right), \qquad (1.1)$$

where $\tau$ is the combined relaxation time, and the $\tau_i$ are the individual relaxation times. We contend that in order for the concept of relaxation times to make physical sense, this equation must be valid.

It has been customary to assume additivity of $n$

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[1] J. Callaway, Phys. Rev. **113**, 1046 (1959).