lations were then performed using Cooper's theory for the frozen-lattice model. Good qualitative agreement between theory and experiment resulted. Quantitative agreement is obtained if the values $P_2S=-18$ K/atom and $D^{\gamma}=1.7$ K/atom are chosen for the twofold anisotropy[14] and magneto-elastic constants.

[14] This value appears to be in satisfactory agreement with the neutron diffraction results of H. Bjerrum Møller, J. C. Gylden Houmann, and A. R. Mackintosh [Phys. Rev. Letters **19**, 312 (1967)], for P_2S.

In conclusion, we believe that this paper reconciles the earlier discrepancy between experiment and theory for ferromagnetic resonance in Tb at high microwave frequencies.

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Rigorous Boson Formulation for Calculating Time-Dependent Thermal Properties of Localized Spin Systems*

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We outline a method for rigorously calculating the time-dependent as well as the static thermal properties of localized spin systems from those of an ordinary many-boson system. Our method retains the advantages of both the Holstein–Primakoff and the Dyson–Maleev transformations without having their main disadvantages. The spin operators and the boson Hamiltonian are all finite series in the boson creation and annihilation operators. Our boson Hamiltonian is Hermitian. This method establishes a rigorous correspondence between the thermal properties of the spin-½ isotropic Heisenberg model and those of a hard-core boson system with only two-body interactions.

1. INTRODUCTION

THE methods available for the calculation of the thermodynamic properties of spin systems do not, as yet, seem to be as advanced as the methods[1,2] already developed for treating the thermodynamic properties of many-body systems of bosons or fermions. The reason for this is that the spin-operator commutation rules are more complicated than the boson or fermion commutation rules. In two famous papers in 1956,[3] Dyson used the idea of establishing a correspondence between a given spin system and a boson system, which then could be used to calculate thermodynamic properties of spin systems if the corresponding properties of the boson system could be calculated. He then proceeded to use this correspondence to calculate some of the static (nontime-dependent) properties of the Heisenberg ferromagnet.

Dyson's work was later extended in an effort to calculate the spin Green's functions which would, of course, give one a means of calculating the time-dependent as well as the static thermodynamic properties of spin systems.[4,5]

This approach, however, is only one of several that have been attempted. The usual starting point with these theories is to establish a transformation from the spin operators to a set of boson operators, the two most often used being the Dyson–Maleev[6] and the Holstein–Primakoff[7] transformations. The Dyson–Maleev transformation leads to a "boson system" described by a non-Hermitian "Hamiltonian." The Holstein–Primakoff transformation leads to a boson system with a Hermitian Hamiltonian which is an infinite series in the boson operators. The objections to using these transformations are obvious.

The object of this paper is to present a theory whereby the time-dependent as well as the static properties of a spin system could, in principle, be calculated by means of the application of ordinary many-body boson theory to a system of bosons described by a finite-series Hermitian Hamiltonian. The procedure we have in mind is the following: In order to calculate a given thermodynamic property of the spin system one must, in general, calculate some kind of time-dependent correla-

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tion function. Such functions usually are not calculated directly. It is more convenient to rely on some indirect method such as that provided by the Green's function theory.[1,2] If we can show how one can rigorously calculate a given spin Green's function from some corresponding boson Green's function; then in this way, we can calculate any thermodynamic property of the spin system. The known techniques of boson many-body theory can be used to calculate the relevant boson Green's function.

This paper will concern itself primarily with establishing a rigorous correspondence between the spin and ordinary boson Green's functions. In Secs. 1 and 2 we present the properties of the spin and boson operators, respectively. Section 3 is concerned with establishing a rigorous correspondence between a given spin-correlation function and an ordinary boson correlation function, and gives the method for the construction of the relevant boson system. The correspondence between the spin and boson Green's functions is given in Sec. 4. Section 5 is concerned with the spin image operators which are necessary for the relevant boson calculation, and Sec. 6 deals with a specific example, the spin-$\frac{1}{2}$ Heisenberg ferrogmagnet.

2. SPIN SYSTEMS AND THEIR PROPERTIES

The model we wish to consider is a finite crystal lattice with peridoic boundary conditions. The total number of lattice points is N. The set of lattice vectors is then defined to be $\mathbf{n}_1, \mathbf{n}_2, \cdots, \mathbf{n}_N$.

At each lattice point there is a spin of magnitude S represented by the operator $\mathbf{S}_j$, or simply S_j. The generalization to more complicated spin structures, such as ferrimagnets, is straightforward. The Cartesian components of this operator satisfy the commutation rules:

$$[S_j^x, S_{j'}^y] = iS_j^z \delta_{j,j'}, \quad (1)$$

where $\delta_{j,j'}$ is the Kronecker δ function.

The spin ladder operators $S_j^{\pm}$, are defined by

$$S_j^{\pm} = S_j^x \pm iS_j^y, \quad (2)$$

and satisfy the commutation rules

$$[S_j^+, S_{j'}^-] = 2S_j^z \delta_{j,j'}, \quad (3)$$

$$[S_j^{\pm}, S_{j'}^z] = \mp S_j^{\pm} \delta_{j,j'}, \quad (4)$$

and the identity

$$(S_j^z)^2 - S_j^z \equiv S(S+1) - S_j^+ S_j^-. \quad (5)$$

The major difficulty in theoretically treating spin systems is represented by Eq. (3), which shows that the commutator of S^+ and S^- is not a c number.

The operator S_j^z can be expanded in a series of normal products of the spin ladder operators. That is,

$$S_j^z = \sum_{\nu=1}^{2S} a_\nu(S)(S_j^+)^n(S_j^-)^n,$$

$$a_0(S) = -S, \quad a_1(S) = 1/2S, \quad (6)$$

$$a_\nu(S) = \frac{2}{\nu(2S+1-\nu)} \sum_{\mu=1}^{\nu-1} \mu a_\mu(S) a_{\nu-\mu}(S), \quad 2 \leq \nu \leq 2S. \quad (7)$$

We can now define a complete and orthonormal set of states $\{|p\rangle\}$. Let

$$|p\rangle = \prod_j F^{-1/2}(p_j)(2S)^{-1/2p_j}(p_j!)^{-1/2}(S_j^+)^{p_j}|0\rangle, \quad (8)$$

$$|p\rangle = |p_1, \cdots, p_N\rangle, \quad (9)$$

$$p_j = 0,1,2, \cdots, 2S, \quad (10)$$

$$F(p_j) = 1(1-1/2S)\cdots[1-(p_j-1)/2S]. \quad (11)$$

The state $|0\rangle$ is definee by

$$S_j^-|0\rangle = 0 \quad \text{for all } j. \quad (12)$$

The set $\{|p\rangle\}$ is then the set $\{|p_1 \cdots p_N\rangle\}$, where p_j ranges over all of its possible values, and each distinct ordered N tuple is taken only once. The number of states in the set $\{|p\rangle\}$ is therefore $(2S+1)^N$. The set $\{|p\rangle\}$ also forms a basis for the Hilbert space of the spin system. The orthonormality condition

$$\langle u|v\rangle = \delta_{u_1,v_1} \cdots \delta_{u_N,v_N} = \delta_{u,v} \quad (13)$$

can be derived by using Eqs. (3), (4), and (8).

The matrix elements of the operators S^+, S^-, and S^z with respect to this set of states are

$$\langle p'|S_j^+|p\rangle = (2S)^{1/2}(1+p_j)^{1/2}[1-(p_j/2S)]^{1/2} \times \delta_{p_1,p_1'} \cdots \delta_{p_j+1,p_j'} \cdots \delta_{p_N,p_N'}, \quad (14)$$

$$\langle p'|S_j^-|p\rangle = (2S)^{1/2}(p_j)^{1/2}[1-(p_j-1)/2S]^{1/2} \times \delta_{p_1,p_1'} \cdots \delta_{p_j-1,p_j'} \cdots \delta_{p_N,p_N'}, \quad (15)$$

$$\langle p'|S_j^z|p\rangle = (-S+p_j)\delta_{p,p'}. \quad (16)$$

Equations (14)–(16) can be derived by using Eqs. (3), (4), and (8).

In order to calculate the thermodynamic properties of a spin system described by the Hamiltonian H, we must in general calculate time-dependent correlation functions. It is well known that these functions can be analytically continued into a strip of the complex time plane,[2] giving

$$F_{AB}(t-t') = \sum_n \langle n|A(t)B(t')e^{-\beta H}|n\rangle / \sum_n \langle n|e^{-\beta H}|n\rangle, \quad (17)$$

$$\beta = 1/k_B T, \quad \Omega(t) = e^{iHt}\Omega(0)e^{-iHt}, \quad \Omega = A,B, \quad (18)$$

where k_B is Boltzmann's const. and T is the temperature. The imaginary part of the time difference $t-t'$ is restricted by

$$-\beta \leq \text{Im}\,(t-t') \leq 0, \tag{19}$$

and the set $\{|n\rangle\}$ represents any complete set of states. Throughout this paper, the variable t is to be considered as complex unless otherwise stated.

Since any arbitrary operator in a spin system can be expressed as a function of the operators $S^{\pm}$ and S^z only, and since the set $\{|p\rangle\}$ is complete, we may expand (17) in terms of sums of products of the matrix elements $\langle p|S^\alpha|p'\rangle$, $(\alpha=+,-,z)$. Then by the use of Eqs. (14)–(16), we can in prinicple calculate the correlation function F_{AB}.

3. BOSON OPERATORS ON A LATTICE AND THEIR PROPERTIES

The model we consider here is the same as that used in Sec. 2. However, instead of considering the spin operators, we will consider the pure boson creation and destruction operators b_j and $b_j^\dagger$, which for our discrete system satisfy the commutation rules

$$[b_j,b_{j'}^\dagger]=\delta_{j,j'}, \tag{20}$$

As before, we can define a complete and orthornormal set of states $\{|B)\}$, where

$$|B)=\prod_{j=1}^{N}(B_j!)^{-1/2}(b_j^\dagger)^{B_j}|0), \tag{21}$$

$$|B)=|B_1,\cdots,B_N), \tag{22}$$

$$B_j=0,1,2,\cdots \text{ (any positive integer)}, \tag{23}$$

$$b_j|0)=0 \quad \text{for all } j. \tag{24}$$

In contrast to the set of states $\{|p\rangle\}$, the set $\{|B)\}$ is an infinite set. It is clear, however, by comparing Eqs. (21) and (8) that there exists a subset of $\{|B)\}$ which is isomorphic to the set $\{|p\rangle\}$. This set will be called the physical subset, and it is clearly the set

$$\{|p)\}=\{|p_1,\cdots,p_N)\}p_j=0,1,\cdots,2S. \tag{25}$$

The vector space formed from the basis set $\{|p)\}$ will be called the physical subspace. We also define the unphysical subset $\{|u)\}$ as the set which contains those state vectors which belong to $\{|B)\}$, but not to $\{|p)\}$.

4. SPIN CORRELATION FUNCTIONS FROM BOSON STATISTICS

As was pointed out in the discussion following Eq. (17), any spin correlation function can be expressed in terms of sums of products of the matrix elements $\langle p'|S^\alpha|p\rangle$, and therefore if we define a set of image operators $\tilde{S}_j^\alpha$ in the boson space by the equations

$$\langle p'|S_j^\alpha|p\rangle=(p'|\tilde{S}_j^\alpha|p)f^{-1}(p')f(p), \quad \alpha=+,-,z, \tag{26}$$

where f is any arbitrary function for which f and f^{-1} exist for all $(p_1,\cdots,p_N)$, then by the simple replacement given by (26) the spin correlation function (17) can be written in terms of the matrix elements $(p'|\tilde{S}^\alpha|p)$ only. That is, the result is independent of f.

If we now make the additional requirement that either

$$(u|\tilde{S}_j^\alpha|p)=0 \quad \text{for all } j, \tag{27}$$

or

$$(p|\tilde{S}_j^\alpha|u)=0 \quad \text{for all } j, \tag{28}$$

so that

$$\sum_{p''}(p|\tilde{S}_j^\alpha|p'')(p''|\tilde{S}_{j'}^{\alpha'}|p') = \sum_B (p|\tilde{S}_j^\alpha|B)(B|\tilde{S}_{j'}^{\alpha'}|p')=(p|\tilde{S}_j^\alpha\tilde{S}_{j'}^{\alpha'}|p') \tag{29}$$

then the complicated expression we have for $F_{AB}(t)$ in terms of the $(p'|\tilde{S}^\alpha|p)$ matrix elements will collapse giving

$$F_{AB}(t-t')=\sum_p (p|\tilde{A}(t)\tilde{B}(t')e^{-\beta H}|p)/\sum_p (p|e^{-\beta H}|p), \tag{30}$$

where, for any spin operator Ω, the image operator $\tilde{\Omega}$ is obtained from

$$\tilde{\Omega}(\tilde{S}^\alpha)=\Omega(S^\alpha\to\tilde{S}^\alpha). \tag{31}$$

That is, we express the spin operator Ω in terms of S^α and then replace S^α by $\tilde{S}^\alpha$ to obtain $\tilde{\Omega}$.

It should be noted here that the sums in (30) are only over the physical subspace. This restriction can be removed by introducing a projection operator P with the properties

$$P|p)=|p), \quad P|u)=0. \tag{32}$$

Such an operator can be constructed in the following manner. First define

$$P_j(0)=\eta e^{-b_j^\dagger b_j}=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(b_j^\dagger)^n b_j^n, \tag{33}$$

where η is the normal product operator. It can easily be established from the definitions (33) and (21) that

$$\begin{aligned}P_j(0)|B)&=|B) \quad \text{if } B_j=0,\\ &=0 \quad \text{if } B_j\neq 0,\end{aligned} \tag{34}$$

and therefore

$$\begin{aligned}P_j(m)|B)&\\ =(1/m!)(b_j^\dagger)^m P_j(0)b_j^m|B)&=|B) \quad \text{if } B_j=m,\\ &=0 \quad \text{if } B_j\neq m.\end{aligned} \tag{35}$$

Thus, the operator

$$P_j=\sum_{n=0}^{2S}P_j(n) \tag{36}$$

will project out of the states $\{|B)\}$ only those states for which $B_j \leq 2S$. That is,

$$P_j|B) = |B) \quad \text{if } B_j \leq 2S, \qquad = 0 \quad \text{if } B_j > 2S. \tag{37}$$

Hence, a projection operator which will project out the physical subset from the full set $\{|B)\}$ is

$$P = \prod_{j=1}^{N} P_j. \tag{38}$$

Then,

$$F_{AB}(t-t') = \sum_n (n|\tilde{A}(t)\tilde{B}(t')e^{-\beta\tilde{H}}P|n) / \sum_n (n|e^{-\beta\tilde{H}}P|n), \tag{39}$$

where $\{|n)\}$ is any complete set of boson states. It should be noted that because of the condition (27) [or (28)] the position of P in (39) is not important.

The right-hand side of (39) could be interpreted as a correlation function calculated with respect to a boson space with a metric operator P. However, since P is not the identity operator in the full boson space, we cannot directly apply the usual techniques of many-body boson theory to calculate F_{AB}.

In order to circumvent this problem, we will now construct a boson system and then show how the thermodynamic properties of this can be related to those of the spin system.

The first step is to choose a set of image spin operators $\tilde{S}^\alpha$ to work with. We can then construct the corresponding image operators $\tilde{\Omega}$ for any spin operator Ω, as pointed out above. The Hamiltonian for this boson system is, however, not taken to be the image Hamiltonian $\tilde{H}$ of the spin Hamiltonian H. Instead, we define the Hamiltonian $\hat{H}$ for the boson system by

$$\hat{H} = \bar{H} + V(S), \tag{40}$$

where

$$V(S) = \frac{v_0}{(2S+1)!} \sum_j \Big[\prod_{\mu=-S}^{S} (S_j^z - \mu) \Big] = \frac{v_0}{(2S+1)!} \sum_j (b_j^\dagger)^{2S+1} b_j^{2S+1}, \tag{41}$$

and where $\bar{H}$ is chosen to be the simplest operator which satisfies the following conditions:

(a) $\bar{H}$ is Hermitian, (42a)

(b) $(p'|\bar{H}|p) = (p'|\tilde{H}|p)$ for any $|p),(p'|$; $(u|\bar{H}|p) = (p|\bar{H}|u) = 0$, (42b)

(c) $\hat{H} = \bar{H} + V(S)$ has a lower bound on its energy eigenvalue spectrum. (42c)

From (42b), one obtains

$$[\bar{H}, P] = 0, \tag{43}$$

and therefore the normalized eigenstates of $\hat{H}$ can be written in the form

$$|\psi_p) = \sum_p A_p|p), \quad \sum_p |A_p|^2 = 1, \qquad |\psi_u) = \sum_u A_u|u), \quad \sum_u |A_u|^2 = 1. \tag{44}$$

That is, the energy eigenstates of $\hat{H}$ can be classified as physical or unphysical. Since, according to (41),

$$V(S)|p) = 0, \quad V(S)|u) = \frac{v_0}{(2S+1)!} \sum_j \prod_{\mu=-S}^{S} (u_j - \mu)|u), \tag{45}$$

the energy eigenvalues of all of the $|\psi_u)$ states will diverge in the limit $v_0 \to \infty$.

The procedure for calculating any arbitrary F_{AB} is now simple. We first calculate the image correlation function $\hat{F}_{AB}$, defined by

$$\hat{F}_{AB}(t-t') = (\hat{A}(t)\hat{B}(t'))_{\hat{H}} = \sum_n (n|\hat{A}(t)\hat{B}(t')e^{-\beta\hat{H}}|n) / \sum_n (n|e^{-\beta\hat{H}}|n), \tag{46}$$

$$\hat{\Omega}(t) = e^{i\hat{H}t}\tilde{\Omega}e^{-i\hat{H}t}, \quad \Omega = A, B,$$

where $\{|n)\}$ represents any complete set of boson states in the full boson space. Then from (27) or (28), (42), and (45), it is straightforward to show that

$$\lim_{v_0\to\infty} \hat{F}_{AB}(t-t') = \sum_n (p|\tilde{A}(t)\tilde{B}(t')e^{-\beta\tilde{H}}|p) / \sum_p (p|e^{-\beta\tilde{H}}|p) = F_{AB}(t-t'), \tag{47}$$

where we have used Eq. (30) to establish the last part of this equation. This result shows how F_{AB} can be obtained from $\hat{F}_{AB}$. We can use ordinary boson many-body theory to calculate the correlation function $\hat{F}_{AB}$ for finite v_0 and then take the limit $v_0 \to \infty$ in order to obtain F_{AB}. This limit corresponds to a boson system with a hard-core interaction.

It should be pointed out here that the requirement that $\bar{H}$ be Hermitian is necessary if one wants to be able to use ordinary many-body theory to calculate $\hat{F}_{AB}$.

5. CALCULATION OF SPIN GREEN'S FUNCTIONS

As mentioned before, in order to calculate the thermodynamic properties of the spin system, one must, in general, calculate some appropriate time-dependent correlation function. In Sec. 4, it was shown how one could calculate such correlation functions by performing a corresponding calculation in a properly constructed boson system and taking an appropriate limit.

To calculate the correlation function $\langle A(t)B(t')\rangle$ one first calculates the appropriate Green's function

$$G_{AB}(t-t') = -i\langle T[A(t)B(t')]\rangle_H \quad 0\leqslant it,it'\leqslant\beta. \quad (48)$$

The imaginary time-ordering operator T is chosen to be the one for the time ordering of boson operators,

$$\begin{aligned} T[A(t)B(t')] &= A(t)B(t') \quad \text{if } it>it' \\ &= B(t')A(t) \quad \text{if } it<it'. \end{aligned} \quad (49)$$

Thus,

$$\begin{aligned} G_{AB}(t-t') &= -i\langle A(t)B(t')\rangle_H \quad it>it', \\ &= -i\langle B(t')A(t)\rangle_H \\ &\qquad it<it', \quad 0\leqslant it,it'\leqslant\beta. \end{aligned} \quad (50)$$

The quantities on the right-hand side of Eq. (50) are nothing more than time-dependent correlation functions. Thus, we have immediately, from Eq. (47), the result

$$G_{AB}(t-t') = \lim_{v_0\to\infty} \hat{G}_{AB}(t-t'), \quad (51)$$

where

$$\hat{G}_{AB}(t-t') = -i(T\hat{A}(t)\hat{B}(t'))_{\hat{H}}. \quad (52)$$

This result allows us to rigorously calculate the spin Green's functions by the known techniques of boson many-body theory.

6. SPIN IMAGE OPERATORS

The spin image operators $\tilde{S}^\alpha$ are defined by Eqs. (26) and (27) [or (28)]. These equations do not uniquely define a set of image operators, nor do they indicate a unique form for the expansion in terms of the boson operators.

We choose to expand the $\tilde{S}^\alpha$ in a series of normal products of the boson operators, which from (26) must be of the form

$$\begin{aligned} S_j^+ &= (2S)^{1/2} b_j^\dagger \Omega_j^+, \\ S_j^- &= (2S)^{1/2} \Omega_j^- b_j, \\ S_j^z &= \Omega^z, \\ \Omega^\alpha &= \sum_\nu D_\nu^\alpha(S)(b_j^\dagger)^\nu b_j^\nu, \quad \alpha=+,-,z. \end{aligned} \quad (53)$$

The corresponding expansions in terms of the number operator can be determined with the help of the identity

$$(b_j^\dagger)^\nu b_j^\nu = \prod_{\mu=1}^{\nu} [b_j^\dagger b_j - (\nu-\mu)]. \quad (54)$$

If the D_ν^α are real and

$$D_\nu^+ = D_\nu^-, \quad (55)$$

then

$$\tilde{S}_j^+ = (\tilde{S}_j^-)^\dagger, \quad (56)$$

and the Hermitian property

$$S_j^+ = (S_j^-)^\dagger \quad (57)$$

is preserved by the transformation $S^\alpha \to \tilde{S}^\alpha$. If, on the other hand,

$$D_\nu^+ \neq D_\nu^-, \quad (58)$$

then (56) is not satisfied and the Hermitian property (57) is not preserved. We will consider below the two simplest expansions corresponding, respectively, to (55) and (58).

If we want (56) to be satisfied by the image operators, then from Eq. (26) we must have

$$f(p) = \text{const} \quad \text{for all } p. \quad (59)$$

Then

$$\begin{aligned} (p'|\tilde{S}_j^\alpha|p) &= \langle p'|S_j^\alpha|p\rangle, \quad \alpha=+,-,z, \quad \text{for all } j, \\ (u|\tilde{S}_j^\alpha|p) &= (p|S_j^\alpha|u) = 0 \quad\quad \text{for all } j. \end{aligned} \quad (60)$$

It is shown in Appendix A that the simplest set of operators which will satisfy the conditions in Eq. (60) is

$$\begin{aligned} \tilde{S}_j^+ &= (2S)^{1/2} b_j^\dagger \sum_{\nu=0}^{2S} H_\nu^+(S)(b_j^\dagger)^\nu b_j^\nu, \\ S_j^- &= (\tilde{S}_j^+)^\dagger \\ \tilde{S}_j^z &= -S + b_j^\dagger b_j, \end{aligned} \quad (61)$$

where

$$H_\nu^+(S) = \sum_{\mu=0}^{\nu} \frac{(-1)^{\mu+\nu}}{\mu!(\nu-\mu)!}\left(1-\frac{\mu}{2S}\right)^{1/2}. \quad (62)$$

It is noted here that any transformation will do which has the form

$$\begin{aligned} \tilde{S}^+ &= (2S)^{1/2} b_j^\dagger \sum_{\nu=0}^{\infty} B_\nu(S)(b_j^\dagger)^\nu b_j^\nu, \\ B_\nu(S) &= H_\nu^+(S), \quad \nu\leq 2S, \end{aligned} \quad (63)$$

regardless of what B_ν is for $\nu>2S$. For example, as is shown in Appendix A, the Holstein–Primakoff transformation is given by

$$\tilde{S}_j^+ = (2S)^{1/2} b_j^\dagger \sum_{\nu=0}^{\infty} H_\nu^+(S)(b_j^\dagger)^\nu b_j^\nu, \quad (64)$$

and this transformation is just a more complicated set of operators which satisfy the conditions (60). Thus, the image operators in (61) are just a "truncated" version of the Holstein–Primakoff operators written in normal product form.

The simplest set of operators which satisfy (58) can be determined by choosing

$$f(p) = \prod_j [F(p_j)]^x, \quad x=\tfrac{1}{2} \text{ or } -\tfrac{1}{2}, \quad (65)$$

where $F(p_j)$ is defined by Eq. (11). Then from (14), (15), and (26),

$$(p'|\tilde{S}_j^+|p)=(2S)^{1/2}(p_j+1)^{1/2}(1-p_j/2S)^{1/2+x}\times\delta_{p_1,p_1'}\cdots\delta_{p_j+1,p_j'}\cdots\delta_{p_N,p_N'},$$
$$(p'|\tilde{S}_j^-|p)=(2S)^{1/2}(p_j)^{1/2}(1-(P_j-1)/2S)^{1/2-x}\times\delta_{p_1,p_1'}\cdots\delta_{p_j-1,p_j'}\cdots\delta_{p_N,p_N'},\quad(66)$$
$$p('|\tilde{S}_j^z|p)=-(S+p_j)\delta_{p,p'}.$$

Then for $x=\frac{1}{2}$, the obvious choice is

$$\tilde{S}_D^+=(2S)^{1/2}b_j^\dagger(1-b_j^\dagger b_j/2S),$$
$$\tilde{S}_D^-=(2S)^{1/2}b_j,\quad(67)$$
$$\tilde{S}_D^z=-S+b_j^\dagger b_j,$$

and for $x=-\frac{1}{2}$

$$\tilde{S}_{CD}^+=(2S)^{1/2}b_j^\dagger,$$
$$\tilde{S}_{CD}^-=(2S)^{1/2}(1-b_j^\dagger b_j/2S)b_j,\quad(68)$$
$$\tilde{S}_{CD}^z=-S+b_j^\dagger b_j.$$

Clearly, the correspondences (67) and (68) satisfy either (27) or (28). Notice that the image operators in (67) are just the Dyson–Maleev operators and the operators in (68) are the Hermitian conjugates of the Dyson–Maleev operators.

Thus, (61) gives the simplest expressions for the image operators which preserve the Hermitian property (57), while (67) and (68) give an even simpler set of image operators, which, however, do not preserve the Hermitian property.

APPLICATION OF THEORY

As an example of how this theory is to be applied, we now consider one of the simplest of all spin systems, the spin-$\frac{1}{2}$ Heisenberg ferromagnet for cubic lattices. The Hamiltonian for this system is

$$H=g\mu h\sum_j S_j^z-\sum_j\sum_\rho J_\rho(S_j^+S_{j+\rho}^-+S_j^zS_{j+\rho}^z),\quad(69)$$

where the first term represents the Zeeman energy due to the external field h, and the second term represents the exchange interaction between the spins. The J_ρ are the exchange constants, and the sum on ρ represents a sum over all the neighbors of j.

We will set this problem up in two ways. The first will be in terms of the image operators (61), which preserve the Hermitian property (57), and the second in terms of the Dyson-Maleev operators which do not preserve this property.

First we consider the operators $\tilde{S}_j^\pm$ and $\tilde{S}_j^z$ given by Eqs. (61) with $S=\frac{1}{2}$:

$$\tilde{S}_j^+=b_j^\dagger-b_j^\dagger b_j^\dagger b_j,$$
$$\tilde{S}_j^-=b_j-b_j^\dagger b_jb_j,\quad(70)$$
$$\tilde{S}_j^z=-\tfrac{1}{2}+b_j^\dagger b_j.$$

In order to obtain a suitable Hamiltonian $\hat{H}$ we must first determine a suitable $\bar{H}$ which satisfies the conditions (42). As a first guess we might try $\tilde{H}$, where from (31)

$$\tilde{H}=\tilde{H}_0+\tilde{H}_2+\tilde{H}_3,$$
$$\tilde{H}_0=E_0+\sum_{j,\rho}J_\rho(b_j^\dagger b_j-b_j^\dagger b_{j+\rho})+g\mu h\sum_j b_j^\dagger b_j,$$
$$E_0=\text{const},\quad(71)$$
$$\tilde{H}_2=\sum_{j,\rho}J_\rho(b_{j+\rho}^\dagger b_j^\dagger b_jb_j+b_j^\dagger b_j^\dagger b_jb_{j+\rho}-b_j^\dagger b_{j+\rho}^\dagger b_{j+\rho}b_j),$$
$$\tilde{H}_3=-\sum_{j,\rho}J_\rho b_j^\dagger b_j^\dagger b_{j+\rho}^\dagger b_{j+\rho}b_{j+\rho}b_j.$$

However, it is trivial to show by direct substitution that

$$(p'|\tilde{H}|p)=(p'|\tilde{H}_0+\tilde{H}_2+\tilde{H}_3|p)=(p'|\tilde{H}_0+\tilde{H}_2|p),\quad(72)$$
$$(p|\tilde{H}_0+\tilde{H}_2|u)=(u|\tilde{H}_0+\tilde{H}_2|p)=0,$$

and thus we can take

$$\bar{H}=\tilde{H}_0+\tilde{H}_2=\bar{H}^\dagger,\quad(73)$$

since all of the requirements (42) are met. Then, from (40),

$$\hat{H}=\tilde{H}_0+\tilde{H}_2+V(\tfrac{1}{2}).\quad(74)$$

In terms of the space Fourier transforms of b_j, $b_j^\dagger$, and J_ρ, defined by

$$b_k=\frac{1}{(N)^{1/2}}\sum_j b_je^{i\mathbf{k}\cdot\mathbf{j}},$$
$$J_k=\sum_\rho J_\rho e^{i\mathbf{k}\cdot\rho},\quad(75)$$

we have

$$\hat{H}=\hat{H}_0+\hat{H}_2,\quad(76)$$

where

$$\hat{H}_0=E_0+\sum_k E_k^0b_k^\dagger b_k=\tilde{H}_0,\quad(77)$$

$$\hat{H}_2=\frac{1}{2N}\sum_{k,k',K}V_K(k,k')\times b_{1/2K+k}^\dagger b_{1/2K-k}^\dagger b_{1/2K+k'}b_{1/2K-k'},\quad(78)$$

$$E_k^0=g\mu h+J(0)-J(k),\quad(79)$$

$$V_K(k,k')=v_0+J(\tfrac{1}{2}K+k)+J(\tfrac{1}{2}K-k)+J(\tfrac{1}{2}K+k')+J(\tfrac{1}{2}K-k')-J(k+k')-J(k-k').\quad(80)$$

This Hamiltonian describes a system of bosons interacting by means of the two-body interaction $\hat{H}_2$.

We can now calculate any Green's function, and thus any time-dependent correlation function, for the $S=\frac{1}{2}$ Heisenberg ferromagnet, by first calculating the corresponding Green's function with respect to this system of interacting bosons and then taking the hard-core limit $v_0 \to \infty$.

As an example, suppose we want to calculate the spin Green's function

$$G_1(1;1') = -i\langle TS^-(1)S^+(1')\rangle_H. \tag{81}$$

Then from Eqs. (51) and (52),

$$G_1(1;1') = \lim_{v_0\to\infty} [-i(T\hat{S}^-(1)\hat{S}^+(1')_{\hat{H}}], \tag{82}$$

where 1 refers to (r_1,t_1).

Substitution of the expansions (70) into (82) gives the result

$$G_1(1;1') = \lim_{v_0\to\infty} [\Gamma_1(1;1') - i\Gamma_2(1,1';1'^+1'^+) \\ -i\Gamma_2(1,1;1^+,1') - \Gamma_3(1,1,1';1^+,1'^+,1'^+)], \tag{83}$$

where the n-body boson Green's functions Γ_n are defined as

$$\Gamma_n(1,2,\cdots,n;1',2',\cdots,n') \\ = (-i)^n[Tb(1)\cdots b(n)b^\dagger(1')\cdots b^\dagger(n')]_{\hat{H}}, \tag{84}$$

and where 1^+ simply means to evaluate (r_1,t_1) at $(r_1,t_1+\epsilon)$, with $\epsilon>0$ and infinitesimal, and then let $\epsilon\to 0$ after the time ordering has been performed.

However,

$$\lim_{v_0\to\infty} [i\Gamma_2(1,1;1^+,1') + \Gamma_3(1,1,1';1^+1'^+1'^+)] \\ = \lim_{v_0\to\infty} \{-i[Tb^\dagger(1)b(1)b(1)\tilde{S}^+(1')]_{\hat{H}}\} = 0, \tag{85}$$

since $\tilde{S}^+$ does not connect the physical and unphysical subspaces, and

$$b^2(1)|p) \equiv 0, \quad S=\tfrac{1}{2}. \tag{86}$$

Thus, we have the rigorous result

$$G_1(1,1') = \lim_{v_0\to\infty} [\Gamma_1(1,1') - i\Gamma_2(1,1';1'^+1'^+)]. \tag{87}$$

Secondly, in order to set the problem up in terms of the Dyson–Maleev operators given by (67), we must obtain a Hermitian operator $\bar{H}_D$ which satisfies (42b); that is,

$$(p'|\bar{H}_D|p) = (p'|\tilde{H}_D|p), \tag{88}$$

where from (31)

$$\tilde{H}_D = \tilde{H}_0 + \tilde{H}_{2D}, \\ \tilde{H}_{2D} = \sum_{j,\rho} J_\rho (b_j^\dagger b_j^\dagger b_j b_{j+\rho} - b_j^\dagger b_{j+\rho}^\dagger b_{j+\rho} b_j), \tag{89}$$

and where $\tilde{H}_0$ is given in (71). $\tilde{H}_D$ is clearly not Hermitian.

We can now construct a suitable $\bar{H}_D$ by simply adding to $\tilde{H}_D$ the term necessary to make it Hermitian; that is,

$$\bar{H}_D = \tilde{H}_D + \sum_{j,\rho} J_\rho b_j^\dagger b_{j+\rho}^\dagger b_j b_j = \bar{H}_D^\dagger, \tag{90}$$

since

$$(p'|\bar{H}_D|p) = (p'|\tilde{H}_D|p), \quad (p|\bar{H}_D|u) = (u|\bar{H}_D|p) = 0, \tag{91}$$

then

$$\hat{H}_D = \bar{H}_D + V(\tfrac{1}{2}). \tag{92}$$

However, from (92), (71), and (74), we have

$$\hat{H}_D \equiv \hat{H}. \tag{93}$$

Thus, for $S=\frac{1}{2}$, the simplest Hamiltonians for the relevant boson system for these two cases are identical.

In order to calculate $G_1(1;1')$ given by (81), we now form $\hat{G}_1$ in accordance with (52) in terms of the Dyson–Maleev operators, giving immediately the result

$$G_1(1;1') = \lim_{v_0\to\infty} [\Gamma_1(1;1') - i\Gamma_2(1,1';1'^+1'^+)], \tag{94}$$

which is identical with the result (87), since the boson Green's functions are calculated with respect to the same Hamiltonian $\hat{H}$.

Equation (94) or (87) can be simplified even further if we restrict t_1 and t_1' by

$$0 < it, it' < \beta, \tag{95}$$

where we have only excluded the end points of the interval, that is, the points 0 and β. With it and it' restricted by (95), it is easy to prove that the limit $v_0\to\infty$ exists for both Γ_1 and Γ_2 appearing in (94). In fact, by retracing the steps used to establish (47), it is straightforward to prove that

$$\lim_{v_0\to\infty} \Gamma_2(1,1';1'^+,1'^+) = 0 \quad 0 < it, it' < \beta. \tag{96}$$

Thus,

$$G_1(1;1') = \lim_{v_0\to\infty} \Gamma_1(1;1'), \quad 0 < it, it' < \beta. \tag{97}$$

This function can then be analytically continued in the usual manner to obtain $G(1;1')$ for real times. We therefore have the result that the spin Green's function $G_1(1;1')$, for $S=\frac{1}{2}$, can be calculated directly from the ordinary one-particle boson Green's function $\Gamma_1(1;1')$.

Thus, from either starting point we arrive at the result that in order to calculate $G_1(1;1')$ we need only calculate the one-body Green's functions for a system of bosons described by the Hamiltonian $\hat{H}$, which contains at most two-body interactions.

At $T=0$ the result is trivial, since G_1 is then the Green's function for a system of noninteracting bosons described by the Hamiltonian $\hat{H}_0$.

In order to extend the calculation to finite temperatures, one must use some method of calculation of Γ, such as the T-matrix approximation, which will give meaningful results for hard-core boson problems. It is clear that simple approximations such as the Hartree-Fock approximation cannot be used here because of the hard-core limit.

It has been shown for $S=\frac{1}{2}$ that within the framework of the T-matrix approximation this theory will reproduce all of the low-temperature results obtained by Dyson.[8] We obtain terms in the calculation of the space-time Fourier transform of $\hat{G}_1(1;1')$, $\hat{G}_1(p,z)$, which for finite v_0 give an exponentially small contribution from the nonphysical states similar to that found by Dyson in the partition function calculation. However, in the limit $v_0\rightarrow\infty$, which must be taken, these terms disappear.

Several authors have attempted to calculate the spin Green's function by means of the Hamiltonian $\tilde{H}_D$.[4,5] These calculations are all based on the neglect of the effects of the nonphysical states. Calculations based on $\tilde{H}_D$ are suspect for several reasons. In the first place, as Dyson points out, $\tilde{H}_D$ has no lower bound on its energy spectrum, and hence one is faced with a convergence problem as regards the thermodynamic properties of the system. Secondly, in order to prove that for the partition function the effects of the nonphysical states were exponentially small at low temperatures, Dyson had to modify $\tilde{H}_D$, making it in effect an infinite series Hamiltonian in the boson operators. Finally, it is not clear to us how one can apply the techniques of many-body theory to a system described by a non-Hermitian Hamiltonian unless the calculation is modified by introducing a metric operator which in turn introduces further complications.[9] We have, in fact, been able to show[8] that a calculation of the space-time transform $G(p,z)$ based on $\tilde{H}_D$ and on the procedures outlined in Refs. 4 or 5 leads apparently to an expression for $G_1(p,z)$ which is different in form from the result we get from this theory.

In contrast to this, the theory presented above provides a rigorous connection between the spin-$\frac{1}{2}$ Heisenberg ferromagnet and a system of bosons interacting via a Hermitian two-body interaction.

We should also point out that the Hamiltonian (74) is very similar to the one Shaw obtained as an approximation to a more complicated Hamiltonian which contained n-body interactions of all orders.[10] We have, however, shown that $\hat{H}$ can be used to obtain rigorous results for the spin-$\frac{1}{2}$ Heisenberg ferromagnet, a feature which is lacking in Shaw's work.

8. SUMMARY AND CONCLUSIONS

We have outlined a way of calculating the thermodynamic properties of spin systems by using boson many-body theory. Each thermodynamic property of the spin system is calculated by first calculating a corresponding property in an ordinary many-boson system and then subjecting the result to a limiting procedure. For any S, this limit amounts to a $(2S+1)$-body hard-core limit which, by making the $(2S+1)$-body interaction energy diverge, prohibits any more than $2S$ bosons from occupying the same lattice site; for $S=\frac{1}{2}$ we only need a hard-core boson system with two-body forces.

We have found that we have some choice as to the boson "image system." There were two such "simplest" choices: As long as we insisted on preserving the Hermitian property (i.e., on letting the Hermitian adjoint of the boson image of each spin operator serve as the image of that spin operator's Hermitian adjoint), the simplest transformation turned out to be a "truncated" version of the Holstein–Primakoff transformation. If we do not require that the Hermitian property be preserved, the Dyson–Maleev transformation or its Hermitian conjugate gives even simpler spin image operators.

For $S=\frac{1}{2}$, both turned out to lead to the same "simplest" boson Hamiltonian. The correspondence between a given Green's function for the spin system and its image in the boson system depends on which set of image operators is used. However, in the limit $v_0\rightarrow\infty$ both image Green's functions lead to the same result, namely, the spin Green's function. We explicitly demonstrated this in the calculation of $G_1(1;1')$.

For higher spin, the situation is more complicated. In the first place, it is not clear to us how to obtain a suitable $\hat{H}$ which satisfies conditions (42) for the Dyson–Maleev transformation. In the case of the transformation given by (61) and (62), which we found was a "truncated" Holstein–Primakoff transformation written in normal product form, we obtain a boson system with n-body forces ($n>2$). If we introduce a low-density two-body approximation, it is possible to obtain the correct low-temperature behavior of the ferromagnet for arbitrary spin (to all orders in $1/2S$). This work will be presented in a subsequent paper.

APPENDIX A

In this Appendix we want to show that the image operators $\tilde{S}^{\pm}$ given by (61) and (62) are the simplest set which satisfy Eq. (60). Since $\tilde{S}^{-}=(\tilde{S}^{+})^{\dagger}$, we only need to show $\tilde{S}^{+}$ satisfies (60). The choice of $\tilde{S}^{z}$ is obvious from Eq. (16).

Consider now the general expression given by (53). Obviously in such an expression, all of the summands $(b^{\dagger})^{\nu+1}b^{\nu}$ with $\nu>2S$ do not contribute to (60), so it is simplest to set them equal to zero.

Hence from (14), (53), and (60)

$$(p'|\sum_{\nu=0}^{2S}D_\nu^{+}(S)(b_j^{\dagger})^{\nu+1}b_j^{\nu}|p)=(p_j+1)^{1/2}\left(1-\frac{p_j}{2S}\right)^{1/2} \times\delta_{p_1,p_1'}\cdots\delta_{p_j+1,p_j'}\cdots\delta_{p_N,p_N'},\quad p_j=0,1,\cdots,2S. \quad \text{(A1)}$$

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[8] J. F. Cooke and H. H. Hahn (to be published).
[9] R. E. Mills and R. P. Kenan, Ann. Phys. (N. Y.) 37, 104 (1966).
[10] W. M. Shaw, Ph.D. thesis, University of Washington (unpublished).

Since

$$(b_j^\dagger)^\nu b_j^\nu|p)=\frac{p_j!}{(p_j-\nu)!}|p),\quad \nu\leq p_j$$
$$=0 \qquad \nu>p_j, \tag{A2}$$

we get

$$(p'|(b_j^\dagger)^{\nu+1}b_j^\nu|p)=(p_j+1)^{1/2}[p_j!/(p_j-\nu)!]\times\delta_{p_1,p_1}{'}\cdots\delta_{p_j+1,p_j}{'}\cdots\delta_{p_N,p_N}{'},\quad \nu\leq p_j,$$
$$=0 \quad \nu>p_j. \tag{A3}$$

Thus, from (A1)

$$\sum_{\nu=0}^{p_j} D_\nu^+(S)\frac{p_j!}{(p_j-\nu)!}=\left(1-\frac{p_j}{2S}\right)^{1/2},\quad p_j=0,\cdots,2S, \tag{A4}$$

or

$$D_0^+=1,$$

$$D_\nu^+(S)=\frac{[1-(\nu/2S)]^{1/2}}{\nu!}-\sum_{\mu=0}^{\nu-1}\frac{D_\mu^+(S)}{(\nu-\mu)!},\quad 1\leq\nu\leq 2S. \tag{A5}$$

We suggest the unique solution

$$D_\nu^+(S)=\sum_{\mu=0}^{\nu}\frac{(-1)^{\nu+\mu}}{\mu!(\nu-\mu)!}[(1-(\mu/2S)]^{1/2}\equiv H_\nu^+(S). \tag{A6}$$

It is straightforward to show by direct substitution that (A6) satisfies (A4). It is simpler, however, to prove this by induction. Clearly, from (A6), $D_0^+(S)=1$. It is now assumed that (A6) is valid for all $\nu\leq p_j$, and we want to prove that (A6) is correct for $\nu=p_j+1$. For simplicity, we will drop the j subscript.

In order to do this we only need to prove the identity

$$R\equiv\sum_{\nu=0}^{p}\frac{H_\nu^+(S)}{(p+1-\nu)!}=\sum_{\mu=0}^{p}\frac{(-1)^{p+\mu}}{\mu!(p+1-\mu)!}\left(1-\frac{\mu}{2S}\right)^{1/2}. \tag{A7}$$

From (A6)

$$R=\sum_{\nu=0}^{p}\sum_{\mu=0}^{\nu}\frac{(-1)^{\nu+\mu}}{\mu!(\nu-\mu)!}\frac{[1-(\mu/2S)]^{1/2}}{(p+1-\nu)!}. \tag{A8}$$

Since for any function $f(\mu,\nu)$

$$\sum_{\nu=0}^{p}\sum_{\mu=0}^{\nu}f(\mu,\nu)=\sum_{\mu=0}^{p}\sum_{\nu=\mu}^{p}f(\mu,\nu), \tag{A9}$$

we have

$$R=\sum_{\mu=0}^{p}\frac{(-1)^\mu}{\mu!}\left(1-\frac{\mu}{2S}\right)^{1/2}\left[\sum_{\nu=\mu}^{p}\frac{(-1)^\nu}{(\nu-\mu)!(p+1-\nu)!}\right]. \tag{A10}$$

Let

$$\nu=\mu+\Delta,\quad \Delta\geq 0, \tag{A11}$$

and consider

$$E_\mu\equiv\sum_{\nu=\mu}^{p}\frac{(-1)^\nu}{(\nu-\mu)!(p+1-\nu)!}$$
$$=\sum_{\Delta=0}^{p-\mu}\frac{(-1)^{\mu+\Delta}}{\Delta!(p+1-\mu-\Delta)!},\quad \mu\leq p \tag{A12}$$
$$=\frac{(-1)^\mu}{(p+1-\mu)!}\sum_{\Delta=0}^{p-\mu}\binom{p+1-\mu}{\Delta}(-1)^\Delta,$$

$$\binom{m}{n}=\frac{m!}{n!(m-n)!} \tag{A13}$$

$$=\frac{(-1)^\mu}{(p+1-\mu)!}\left\{-(-1)^{p+1-\mu}+\sum_{\Delta=0}^{p+1-\mu}\binom{p+1-\mu}{\Delta}(-1)^\Delta\right\}=\frac{(-1)^p}{(p+1-\mu)!}. \tag{A14}$$

Substitution of (A14) into (A10) establishes (A7).

Notice that since we did not use the fact that $p\leq 2S$ in the proof of (A7), it is true that (A4),

$$\sum_{\nu=0}^{B_j}H_\nu^+(S)\frac{B_j!}{(B_j-\nu)!}=\left(1-\frac{B_j}{2S}\right)^{1/2}, \tag{A15}$$

is true for any integer, B_j, where $H_\nu^+(S)$ is given by (A6). Then, clearly,

$$(2S)^{1/2}\sum_{\nu=0}^{\infty}H_\nu^+(S)(b_j^\dagger)^{\nu+1}b_j^\nu|B)=(2S)^{1/2}(B_j+1)^{1/2}\times\left(1-\frac{B_j}{2S}\right)^{1/2}|B_1\cdots,B_j+1,\cdots,B_N). \tag{A16}$$

That is,

$$\tilde{S}_{\mathrm{HP}}{}^+=(2S)^{1/2}\sum_{\nu=0}^{\infty}H_\nu^+(S)(b_j^\dagger)^{\nu+1}b_j^\nu$$
$$=(2S)^{1/2}b_j^\dagger\left(1-\frac{b_j^\dagger b_j}{2S}\right)^{1/2} \tag{A17}$$

is the Holstein–Primakoff transformation written in normal product form.