trajectory is involved. To accommodate this fact, a multichannel scheme can be formulated in a straightforward manner. However, in order to avoid the additional complications of the multichannel problem, we have studied the hypothetical model of a single self-bootstrapping Regge pole. We have solved our equation approximately both at small and large negative t values. It is found that the zero-energy intercept $\alpha(0)$ cannot be 1 if the internal Regge coupling is *strictly* factorizable. If additional structure for this coupling is allowed, the intercept $\alpha(0)$ could in principle be unity. Further applications of our equation will be discussed elsewhere.

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Veneziano Amplitudes and the $\pi\omega\rho\sigma A_1$ System*

A. ZEE†

Department of Physics, Harvard University, Cambridge, Massachusetts 02138

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We write Veneziano amplitudes for all reactions involving the $\pi\omega\rho\sigma A_1$ system, paying particular attention to kinematical constraints arising from spin. The minimal Veneziano amplitude for $A_1 \to 3\pi$ is shown to be dependent on two parameters, which may be related to the $A_1\rho\pi$ vertex. A discussion is given on how examination of the Dalitz plot may resolve the theoretical confusion surrounding the A_1.

RECENTLY, Veneziano[1] has written down an amplitude that satisfies all the postulates of S-matrix theory except unitarity. A series of successful applications[2,3] has generated much optimism that the Veneziano amplitude may be a decent first approximation to the real world.[4] Here we attempt to apply the Veneziano-Lovelace analysis to the $\pi\rho A_1$ system. We consider all possible reactions involving π, ω, ρ, σ (the daughter of ρ), A_1, and the "heavy pion" τ (the daughter of A_1), such as $\pi\rho \to \pi\rho$, $\pi A \to \pi A$, $\pi\pi \to \pi A$, $\pi\sigma \to \pi\sigma$, $\pi\tau \to \pi\tau$, $\pi\tau \to \sigma\sigma$, etc. Consistency may be achieved by incorporating a sufficient number of "Veneziano terms," and relations between coupling constants are obtained. Unfortunately, these relations generally involve the daughters.

It must be emphasized from the outset that the isolated Veneziano model lacks predictive power because of the possibility of adding higher terms. We seek to use a minimum number of terms to construct an amplitude consistent with the correct Regge behavior in all channels, the Adler consistency condition, current algebra (where applicable), the appearance of poles with appropriate residues, the absence of isospin $\geqslant 2$ exotic resonances, and, in cases with spin, the constraints imposed by angular momentum conservation. Apart from simplicity, there is nothing against "nonminimal" terms; we speak of them often in this paper.

To avoid unnecessary complications, we take $m_\pi = 0$ and degenerate trajectories $\rho(s) = \omega(s) = (s+m^2)/2m^2$, $\pi(s) = A(s) = s/2m^2$, where the name of a particle denotes the trajectory to which it belongs ($m = m_\rho$).

We employ the notation

$$\chi(m,\Delta,n) = \frac{\Gamma(m-\chi(s))\Gamma(n-\rho(t))}{\Gamma(\Delta-\chi(s)-\rho(t))}, \qquad (1)$$

which behaves asymptotically as $s^{\rho(t)+m-\Delta}$ and $t^{\chi(s)+n-\Delta}$. The residue of the leading pole $\chi(s) = m$ $[\rho(t) = n]$ is a polynomial of order $m+n-\Delta$ in t (in s).

I. $\pi^-(q_1)\pi^+(k) \to \pi^-(q_2)A_1^+(\varepsilon,\varrho)$

With $M = H(s,t,u)\epsilon(q_1+q_2) + K(s,t,u)\epsilon k$, minimal structure is

$$H = h_1\rho(1,2,2) + h_2\rho(1,2,1), \qquad (2)$$

$$K = k_1\rho(1,1,1) + k_2\rho(1,2,1) + k_3\rho(2,2,1) + k_4\rho(1,2,2). \qquad (3)$$

For example, the h_1 term is necessary both for $t \to \infty$ before and for the correct residue on the s-channel ρ pole, while the h_2 term is necessary to accommodate the t-channel ρ pole. In K, the k_1 term alone appears sufficient at first sight. However, crossing symmetry implies that $k_1 = 0$ and relates k_2, k_3, and k_4 to the two truly independent parameters h_1 and h_2, which upon going on the ρ pole are related to the two $A_1\rho\pi$ coupling constants[5,6] by

$$ga/2m^2 = h_2 = -k_2$$

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† Parker predoctoral fellow.

[1] G. Veneziano, Nuovo Cimento **57A**, 190 (1968).

[2] C. Lovelace, Phys. Letters **28B**, 264 (1968).

[3] M. Ademollo, G. Veneziano, and S. Weinberg, Phys. Rev. Letters **22**, 83 (1969).

[4] G. Veneziano, in Proceedings of the Sixth Coral Gables Conference on Symmetry Principles at High Energy, 1969 (unpublished); S. Weinberg, Comments Nucl. Particle Phys. (to be published).

[5] Our coupling constants are summarized by

$$\begin{aligned}\mathcal{L} = {} & g\epsilon^{abc}\rho_\mu{}^a\pi^b\partial^\mu\pi^c + g_\omega\epsilon_{\mu\nu\lambda\sigma}\partial^\sigma\rho_a{}^\mu\omega^\nu\partial^\lambda\pi_c\delta^{ac} \\ & + \epsilon^{abc}(aA_\mu{}^a\rho^{\mu b}\pi^c - bA_\mu{}^a\rho_\nu{}^b\partial^\mu\partial^\nu\pi^c) + \lambda_{\sigma\pi\pi}\sigma\pi^a\pi^a + g_{A\sigma\pi}A_\mu{}^a\partial^\mu\pi^a\sigma \\ & + \lambda_{\sigma\tau\tau}\sigma\tau^a\tau^a + \lambda_{\sigma\tau\pi}\sigma\pi^a\tau^a + (\lambda_{\sigma\sigma\sigma}/3!)\sigma\sigma\sigma + g_{\rho\tau\tau}\epsilon^{abc}\rho_\mu{}^a\tau^b\partial^\mu\tau^c \\ & + g_{\rho\tau\pi}\epsilon^{abc}\rho_\mu{}^a\tau^b\partial^\mu\pi^c - \tfrac{1}{2}\epsilon^{abc}g_{\rho AA}A_\mu{}^a\partial^\nu A^{\mu b}\rho_\nu{}^c \\ & + (\text{two other } \rho AA \text{ couplings}).\end{aligned}$$

[6] The relation of our coupling constants to those of Gilman and Harari is $a = \frac{1}{8}g_t m^2$ and $b = \frac{3}{4}g_T + g_L$. For $m_\pi = 0$, $\Gamma(A \to \rho\pi) = \frac{1}{16}\pi \times (a^2/\sqrt{2}m)\{1 + (1/24)[1 - 3bm^2/a + \frac{1}{4}b^2m^4/a^2]\}$. Note that a controls the width unless bm^2/a is very large; also note that the finite m_π correction factor to the width is as large as 1.6.

and (4)

$$\tfrac{1}{2}gb=h_1=k_4=-\tfrac{1}{2}k_3.$$

Of the three Adler[7] points, ($t=0$, $s=m_A^2$) and ($s=0$, $t=m_A^2$), related by crossing, are satisfied because of the mass relations,[3] while ($s=0$, $t=0$) is satisfied by crossing symmetry. By going on the σ pole one obtains[8]

$$2a+bm^2+4\lambda_{\sigma\pi\pi}g_{A\sigma\pi}/g=0. \tag{5}$$

Comparison with experiment would be futile even if $\lambda_{\sigma\pi\pi}$ and $g_{A\sigma\pi}$ could be determined (see below), since the A_1 decay involves interference between the $\rho\pi$ and $\sigma\pi$ modes. Rather, we insist that the spectrum of theoretical values[9–14] for (a,b) must be used as input to a dynamical calculation of $A_1\to 3\pi$. Indeed, the present Veneziano amplitude provides an example of such, and it should be of great interest to attempt to unravel the confused experimental and theoretical situation.[15] For instance, the Dalitz plot for $A_1^+\to\pi_1^+\pi_2^+\pi^-$ summed over the polarization of the A_1 may be written as

$$\int P(a,b,s,t)\,|\rho(1,2,1)|^2 ds\,dt, \tag{6}$$

with $s=(p_1^++p^-)^2$ and $t=(p_2^++p^-)^2$. Here, $\rho(1,2,1)$ gives the expected features of a ρ-σ band at $s\approx m^2$ and at $t\approx m^2$, and current-algebraic zeros at ($s=t=0$), ($s=m_A^2$, $t=0$), and ($s=0$, $t=m_A^2$), i.e., the three corners of the triangular (for $m_\pi=0$) Dalitz region. Indeed, the Veneziano amplitude as a more detailed model goes beyond current algebra and predicts a depletion of events along the line (one edge of Dalitz region) $s+t=2m^2$. [With the "poetic license" already issued by Lovelace,[2] we understand that the trajectory appearing here in $\rho(1,2,1)$ possesses a suitably adjusted imaginary part to give the ρ band a finite width.] The function $P(a,b,s,t)/a^2$, depending on the ratio of b to a, influences the shape of the Dalitz plot. To illustrate: For $b=0$ (we have taken $m_\pi=0$), $P(a,0,s,t)\propto(s+t)^2$, while for $a=0$,

$$P(0,b,s,t)\propto m^4(s+t)^2-4m^2st(s+t)+4s^2t^2.$$

Consider, for example, the ρ band, $P(a,0,s,t\approx m^2)\propto(s+m^2)^2$ and $P(0,b,s,t\approx m^2)\propto(s-m^2)^2$, implying rather different accumulation of events. We stress that the values for a and b so obtained are the numbers to be compared with the previous theoretical predictions. A detailed numerical fit to what scarce (confused) data are available to determine a and b will be presented elsewhere.

II. $\pi^a\sigma\to\pi^b\sigma$

Writing $iM=\gamma\pi(0,1,1)$ and examining the A_1, τ, π pole in the s channel and the σ pole in the t channel, we easily obtain

$$g_{A\sigma\pi}^2=\frac{4\lambda_{\sigma\pi\pi}^2}{m^2}=\frac{2\lambda_{\sigma\pi\pi}\lambda_{\sigma\sigma\sigma}}{m^2}=\frac{8}{3}\frac{\lambda_{\sigma\tau\pi}^2}{m^2}. \tag{7}$$

Now an analysis[2] of $\pi\pi\to\pi\pi$ has already determined $\lambda_{\sigma\pi\pi}=\frac{1}{2}gm$ (corresponding to $\sim$500 MeV for the σ width, and apparently in agreement with the result from sum-rule saturation[11,16]). If one disregards the large width of σ, one also finds

$$\Gamma(A_1\to\sigma\pi)=(1/16\sqrt{2})\,\Gamma(\rho\to\pi\pi). \tag{8}$$

III. $\pi^-\tau^+\to\pi^-\tau^+$

With $iM=\beta\rho(1,1,1)$, we obtain

$$\tfrac{1}{4}g_{\rho\tau\pi}^2=\lambda_{\sigma\tau\pi}^2/m^2=gg_{\rho\tau\tau}=\tfrac{4}{5}(\lambda_{\sigma\tau\tau}\lambda_{\sigma\pi\pi}/m^2). \tag{9}$$

Applying the threshold current-algebraic constraint at $t=0$ yields

$$gg_\tau=m^2/2\pi f_\pi^2, \tag{10}$$

which implies that

$$g^2=(m^2/2f_\pi^2)(8/3\pi). \tag{11}$$

(The result of assigning a universal ρ-coupling value to g_τ differs by a factor of about 2.) It should be recalled that from $\pi\pi$ scattering we obtain

$$g^2=(m^2/2f_\pi^2)(2/\pi)\quad\text{and}\quad 2\approx 8/3. \tag{12}$$

If one should insist on true consistency, an extra term could be added to the present amplitude, say, $\beta'\rho(1,2,1)$, which would lose the (undesirable) prediction.

IV. $\pi^-(p_1)\rho^+(\epsilon_1,q_1)\to\pi^-(p_2)\rho^+(\epsilon_2,q_2)$

The invariant amplitude is[17]

$$iM=A(s,t,u)\epsilon_1P\epsilon_2P+B(s,t,u)(\epsilon_1P\epsilon_2Q+\epsilon_2P\epsilon_1Q)+C(s,t,u)\epsilon_1Q\epsilon_2Q+D(s,t,u)\epsilon_1\epsilon_2. \tag{13}$$

The residue of t-channel poles corresponding to $J^P=J^{(-)J}$ is a polynomial in s of order $J-2$ in A, $J-1$ in B, and J in C and D. The residue of s-channel poles corresponding to $J^P=J^{(-)J}$ is a polynomial in t of order J in A, B, and C and of order $J-1$ in D; and,

[7] S. Adler, Phys. Rev. **137**, B1022 (1965).

[8] We have learned that Fayyazuddin and Riazuddin [Phys. Letters **28B**, 561 (1969)] have independently analyzed $\pi\pi\to\pi A$. However, they neglected to include the σ and consequently fixed the ratio a/b.

[9] H. J. Schnitzer and S. Weinberg, Phys. Rev. **164**, 1828 (1967).

[10] S. G. Brown and G. B. West, Phys. Rev. Letters **19**, 812 (1967).

[11] F. J. Gilman and H. Harari, Phys. Rev. Letters **18**, 1150 (1967); **19**, 142(E) (1967); Phys. Rev. **165**, 1803 (1968).

[12] P. H. Frampton, Nuovo Cimento **48A**, 703 (1967); P. H. Frampton and J. C. Taylor, *ibid.* **49A**, 152 (1967).

[13] M. Ademollo *et al.*, Nuovo Cimento **51A**, 227 (1967).

[14] A. Zee, Phys. Rev. **173**, 1788 (1968).

[15] S. Weinberg, in *Proceedings of the Fourteenth International Conference on High-Energy Physics, Vienna, 1968* (CERN, Geneva, 1968), p. 253. Among the recent experimental work is J. Ballam *et al.*, Phys. Rev. Letters **21**, 934 (1968), which indicates that $|g_T/g_L|^2=0.16\pm0.08$ or $a\ll bm^2$. (A recently discovered numerical error in the SLAC analysis of J. Ballam *et al.* changes the value of $|g_T/g_L|$ to 0.64 ± 0.25.)

[16] S. Weinberg, Phys. Rev. **177**, 2604 (1969).

corresponding to $J^P=J(-)^{J+1}$, a polynomial in t of order J in A, B, C, and D. Of course, these angular momentum statements are also reflected in the asymptotic Regge behavior.[17]

Here the ω (we take $m_\omega=m_\rho$ for simplicity), because of its natural parity, behaves rather differently from the A_1 (and generally forms a disjoint problem until we look at the t-channel ρ pole). Now, we have

$$\operatorname{Im}D=\pi\delta(s-m_\rho^2)\tfrac{1}{2}g_\omega^2m_\rho^2t+\pi\delta(s-m_A^2)a^2 +(t\text{-channel poles}), \quad (14)$$

and immediately we need at least two Veneziano terms to ensure the disappearance of the ω pole at $t=0$ (note that the daughter of the ω uncouples from $\pi\rho$ scattering by parity). This is merely a consequence of angular momentum conservation in the forward direction for helicity $\lambda=\pm1$, and of $m_\omega=m_\rho$. Furthermore, angular momentum conservation in the forward direction for helicity $\lambda=0$ requires all natural-parity mesons in the s channel to uncouple[18]; i.e., in the combination $A(s,t=0)(s-m^2)^2-m^2D(s,t=0)$ there is no trace of the entire ω trajectory. The minimal structure consistent with these kinematical constraints, current algebra, and the prescribed Regge behavior turns out to be[19]

$$A=a_1\pi(0,2,2)+a_2\pi(1,3,3)+a_3\omega(1,3,3) +a_4\omega(1,3,2) \quad (15)$$

and

$$D=d_1\pi(1,2,1)+d_4\pi(2,3,1)+d_5\pi(1,3,2)+d_2\omega(1,1,1) +d_6\omega(1,2,1)+d_7\omega(1,2,2)+d_8\omega(1,3,2) +d_9\omega(1,3,3), \quad (16)$$

where

$$g_\omega^2=8a_3=-16a_4=(4/m^2)d_2=(4/m^2)(2d_8+3d_9) =-(4/3m^2)(d_7+d_9)$$

and

$$d_6+\tfrac{1}{2}d_7=0. \quad (17)$$

From A, one gets the relation $b^2=4g^2/m^2$ and another relation involving $g_{\rho\pi\pi}^2$ (for what these are worth). B and C, if given in minimal structure, also yield various (uninteresting) relations, which are in general inconsistent, indicating that nonminimal terms must be added. One may also contemplate the f and A_2 poles to obtain relations among coupling constants involving these particles. However, in view of the necessity of nonminimal terms, we would view such attempts with suspicion.

[17] We follow the notation of, for example, V. De Alfaro, in *Proceedings of the VI Internationale Universitätswochen für Kernphysik-Schladmung, 1967*, edited by P. Urban (Springer-Verlag, Vienna, 1968).

[18] See, for example, S. Weinberg, Ref. 16.

[19] After this work was completed, we learned that P. G. O. Freund and E. Schonberg [Phys. Letters **28B**, 600 (1969)] had also remarked on the condition for helicity $\lambda=\pm1$. However, they did not take into account the condition for helicity $\lambda=0$.

Current algebra, which concerns only D, does not yield any condition, since there are too many terms. (Although minimality does not require both d_4 and d_5, there is no reason to prefer one over the other. The ω terms uncouple at the current-algebraic threshold.) We may recall that Gilman and Harari[11] and Frampton and Taylor[12] have obtained one current-algebraic relation involving a, g, and f_π^2 by saturating a once-subtracted dispersion relation for D with A_1 and ω. Our Veneziano amplitude D satisfies the same low-energy current-algebraic constraint and high-energy behavior as theirs, and thus serves as a model to illustrate the (possible) importance of high-lying states.

V. $\pi^a\omega\to\pi^a\omega$

Here the entire natural-parity sequence must disappear for $t=0$ and helicity $\lambda=0$. The minimal expression needed to enforce this is analogous to that in Sec. IV, but requires $\omega(1,4,3)$ and $\omega(1,4,4)$ since the ρ pole cannot appear in the t channel.

VI. $\pi^-(p_1)A_1^+(\varepsilon_1,q_1)\to\pi^-(p_2)A_1^+(\varepsilon_2,q_2)$

There are fewer low-lying poles here than in $\pi\rho$ scattering. Nevertheless, most of the relations involve the three ρAA and the two σAA coupling constants, and are uninteresting. One exception is $(a^2+abm^2-\frac{1}{4}b^2m^4)+g_{A\sigma\pi}^2m^2=0$, which follows from the minimal form $\rho(1,1,1)$ for C.[20] The current-algebraic restriction on the forward-scattering amplitude concerns the D amplitude and gives

$$a^2+gg_{\rho AA}m_\rho^2=2m_\rho^4/\pi f_\pi^2, \quad (18)$$

which might be of interest were $g_{\rho AA}$ determinable by other means.

VII. CONCLUSION

In conclusion, because of the abundance of daughters, the study of scattering involving the $\pi\rho A$ system in the Veneziano framework does not prove to be particularly fruitful in terms of predictions. We have shown how some kinematical constraints may be incorporated. We also like to view the Veneziano amplitude as a viable alternative to saturation with a few low-lying poles, as was done, for example, in the work of Gilman and Harari. Finally, the minimal Veneziano amplitude for $A_1\to3\pi$ is dependent on two parameters related to the $A\rho\pi$ vertex.

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[20] If one combines this relation with those obtained in Secs. I and II, one finds $\Gamma(A_1\to\rho\pi)/\Gamma(\rho\to\pi\pi)\approx0.4$, 0.5, or 5.