# Simplified Integral Equation for the Multiperipheral Bootstrap*

MARVIN L. GOLDBERGER, CHUNG-I TAN, AND JIUNN-MING WANG
*Palmer Physical Laboratory, Princeton University, Princeton, New Jersey 08540*
(Received 24 March 1969)

An integral equation for the two-body absorptive amplitude based on the multiperipheral model is proposed. The general property and the bootstrap potentiality of the equation are discussed.

IN this paper, we propose an integral equation for the absorptive part $A(s,t)$ of a two-body amplitude based on an approximation to the multiperipheral bootstrap model of Chew and Pignotti.[1] Our equation is similar to the one derived by Chew, Goldberger, and Low[2] (hereafter referred to as the CGL equation), where an auxiliary function $B(s,t)$ is employed. We discuss the general structure of our equation as well as the mechanism through which Regge poles are produced by the "coherence" of moving cuts in the $j$ plane.

The simplest version of the CGL model is defined by the following equations: The absorptive part $A(p,p_0,Q)$ of the elastic two-body amplitude $(p+\frac{1}{2}Q)+(p_0-\frac{1}{2}Q) \to (p-\frac{1}{2}Q)+(p_0+\frac{1}{2}Q)$ is given by

$$A(p,p_0,Q)=\int d^4p'\,\delta[(p-p')^2-\mu^2] \times g(t_+,t_+';t_-,t_-')B(p,p',p_0,Q), \quad (1)$$

where

$$B(p,p',p_0,Q)=B_0(p,p',p_0,Q) + \int d^4p''\,\delta[(p'-p'')^2-\mu^2]g(t_+',t_+'';t_-',t_-'') \times[(p-p'')^2]^{\alpha(t_+')+\alpha(t_-')}B(p',p'',p_0,Q) \quad (2)$$

and

$$B_0(p,p',p_0,Q)=g(t_+'(p_0-\tfrac{1}{2}Q)^2;t_-',(p_0+\tfrac{1}{2}Q)^2) \times[(p+p_0)^2]^{\alpha(t_+')+\alpha(t_-')}\delta((p'+p_0)^2-\mu^2). \quad (3)$$

For simplicity, we have taken all particle masses to be the same $\mu$. We also introduce $Q^2\equiv t$ (the negative squared momentum transfer), $t_\pm=(p\pm\frac{1}{2}Q)^2$, $t_\pm'=(p'\pm\frac{1}{2}Q)^2$, and $t_\pm''=(p''\pm\frac{1}{2}Q)^2$; and we shall also use the conventional variables $s=(p+p_0)^2$, $s'=(p'+p_0)^2$, and $s''(p''+p_0)^2$. The residue-propagators function $g$ need not be specified at the moment, but in the simplest model $g(t_+,t_+';t_-,t_-')=G(t_+,t_+')G^*(t_-,t_-')$.

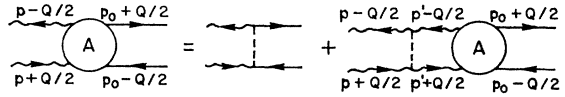


FIG. 1. Diagrammatic representation of the $A$ equation.

The $B$ equation above is quite complicated, and the equation we propose for the absorptive amplitude itself is (see Fig. 1)

$$A(p,p_0,Q) =g(t_+,(p_0-\tfrac{1}{2}Q)^2;t_-,(p_0+\tfrac{1}{2}Q)^2)\delta[(p+p_0)^2-\mu^2] +\mu^2\int d^4p'\,\delta((p-p')^2-\mu^2)\left[\frac{(p+p_0)^2}{(p'+p_0)^2}\right]^{\alpha(t_+')+\alpha(t_-')} \times g(t_+,t_+';t_-,t'_-)A(p',p_0,Q). \quad (4)$$

To "derive" this equation, we note that in the limit of large $s$,

$$(p-p'')^2\cong\mu^2[(p+p_0)^2/(p'+p_0)^2] \times\sinh q'/(\cosh q'+\cos\omega'),$$

where $\cosh q'=\frac{1}{2}(\mu^2-t_+-t_+')/(t_+t_+')^{1/2}$, and $\omega'$ is the angle introduced by Bali, Chew, and Pignotti.[3] We remark that if the function $g$ is large only in the neighborhood of $t_\pm'$, $t_\pm$ near zero, then $\cosh q'\gg1$, and we may disregard the factor $(\sinh q')(\cosh q'+\cos\omega')^{-1}$, since it is essentially unity. If we use the resulting approximation for $(p-p'')^2$ in the $B$ equation and the definition[4] of $A$ in terms of $B$, we obtain what is essentially the once-iterated form of the above $A$ equation.[5]

Our equation for $A$ may be regarded as a different approximation to the multi-Regge model or simply as a large-$s$ approximation of the CGL equation. Their main difference lies in the treatment of the angle $\omega$. In the case of $\omega$-independent residues, they are equivalent to each other in the large-$s$, small-$t$ regime. Although the CGL equation appears to be able to handle the $\omega$ dependence more readily, our $A$ equation seems to have all the essential physical content and to be easier to work with. In order to gain some insight into the structure of these equations, let us study the behavior of the solution at large value of $s$.

* Work supported by the U. S. Air Force Office of Scientific Research under Contract No. AF49(638)-1545.

[1] G. F. Chew and A. Pignotti, Phys. Rev. **176**, 2112 (1968).

[2] G. F. Chew, M. L. Goldberger, and F. Low, Phys. Rev. Letters **22**, 208 (1969); also see L. Caneschi, and A. Pignotti, Phys. Rev. **180**, 1525 (1969); G. F. Chew and C. DeTar, *ibid.* **180**, 1577 (1969).

[3] N. F. Bali, G. F. Chew, and A. Pignotti, Phys. Rev. **163**, 1572 (1967).

[4] We have extended the definition of $A$ to an arbitrary value of $t_+$ and $t_-$ through Eq. (1). This presupposes that the end-vertex function in a multiperipheral chain coincides with the analytic continuation of the internal vertex function.

[5] Under our approximation to the Regge cell $[(p_i-p_{i-2})^2]^{\alpha(t_{i-1})}$, the $A$ equation can also be written directly without any reference to the $B$ equation.

In terms of the conventional variables $s$, $s'$, $s''$, and $t$, the large-$s$ limit of the $A$ equation is[6]

$$A(s,t_+,t_-;t)$$
$$=g(t_+,\mu^2;t_-,\mu^2)\delta(s-\mu^2)+\mu^2\int\frac{dt_+'dt_-'}{[-\Delta(tt_+'t_-')]^{1/2}}$$
$$\times\theta(-\Delta(t,t_+',t_-'))\int_{s_0}^{s}\frac{ds'}{s}\left(\frac{s}{s'}\right)^{\alpha(t_+')+\alpha(t_-')}$$
$$\times g(t_+,t_-',t_-,t_+')A(s',t_+,t_-',t), \quad (5)$$

where

$$\Delta(t,t_+',t_-')=t^2+t_+'^2+t_-'^2-2tt_+'-2tt_-'-2t_+'t_-'.$$

To solve this equation, we make a partial-wave analysis by writing

$$A_l(t_+,t_-;t)=\int_{s_0}^{\infty}ds\,s^{-l-1}A(s,t_+,t_-;t).$$

The large-$s$ behavior of $A$ is determined by the singularity structure of $A_l(t_+,t_-;t)$ in the complex $l$ plane. Forming this (Mellin) transform of the $A$ equation, we find

$$A_l(t_+,t_-;t)$$
$$=(\mu^2)^{-l-1}g(t_+,\mu^2;t\mu^2)+\mu^2\int\frac{dt_+'dt_-'\theta(-\Delta)}{[-\Delta(t_+',t_-',t)]^{1/2}}$$
$$\times\frac{g(t_+,t_+',t_-',t_-')A_l(t_+',t_-',t)}{l-\Lambda(t_+',t_-')}, \quad (6)$$

where $\Lambda(t_+,t_-)\equiv\alpha(t_+)+\alpha(t_-)-1$. If the functions $g$ are sufficiently well behaved, we expect this equation to be of the Fredholm type. The zeros of the Fredholm determinant $D(l,t)$ will give rise to $l$-plane poles which are just the Regge poles referred to in CGL. The eigenvalue condition is simply $D(\alpha(t),t)=0$.

To see this in more detail and to illustrate some other features of the solution, assume that $g(t_+t_+',t_-t_-')=f(t_+,t_-)f(t_+',t_-')$. In this case we may solve for $A_l(t_+,t_-;t)$ explicitly, with the result

$$A_l(t_+,t_-;t)=(\mu^2)^{-l-1}g(t_+,\mu^2;t_-,\mu^2)$$
$$+(\mu^2)^{-l}\int\frac{dt_+'dt_-'\theta(-\Delta)}{[-\Delta(t_+',t_-',t)]^{1/2}}\frac{g(t_+,t_-';t_-,t_-')g(t_-',\mu^2;t_-',\mu^2)}{l-\Lambda(t_+',t_-')}\Bigg/\left(1-\mu^2\int\frac{dt_+'dt_-'\theta(-\Delta)}{[-\Delta(t_+',t_-',t)]^{1/2}}\frac{g(t_+'t_+';t_-'t_-')}{l-\Lambda(t_+',t_-')}\right). \quad (7)$$

The first term above is trivial and irrelevant; its inverse (Mellin) transform simply gives back the one-particle absorptive part $\sim\delta(s-\mu^2)$. The second term is the interesting one. We have written the solution which is exact for our separable $g$ in a form which is also approximately valid for any $g$ which vanishes very rapidly as $|t|$ increases. The Regge poles are given by the solutions of

$$D(l,t)\equiv 1-\mu^2\int\frac{dt_+'dt_-'\theta(-\Delta)}{[-\Delta(t_+',t_-',t)]^{1/2}}\frac{g(t_+',t_+';t_-',t_-')}{l-\Lambda(t_+',t_-')}=0. \quad (8)$$

It is evident that a leading pole is located very close to the maximum value of $\Lambda(t_+',t_-')$ in the range of integration. One finds approximately for this leading pole $\alpha_{\text{out}}(t)$, for small $t$, $\alpha_{\text{out}}(t)=2\alpha(\frac{1}{4}t)-1+g_0^2(t)$, where $\alpha_{\text{out}}(t)$ may or may not be equal to the input $\alpha(t)$, and $g_0^2(t)$ is small if the coupling is weak.

It is evident from Eq. (7) that the numerator function contains branch points in the complex $l$ plane—in particular, the well-known Amati-Fabini-Stanghellini cuts. However, the denominator function contains the same set of branch points and serves to damp the numerator. It has been a long-standing problem that unitarity in the $t$ channel demands[7] that the partial-wave amplitudes be finite at the branch points, whereas models in the past have always led to branch points of logarithmic type. The solution of our equation, however, will not have this defect. In general, both our numerator and denominator functions will have logarithmic branch points in the $l$ plane. Consequently, the resulting amplitude $A(l,t)$ will have only finite branch points with discontinuities vanishing like $1/\ln[l-\alpha_c(t)]$. The summation of terms with increasing order of logarithmic singularity has led not only to a Regge pole; it has also yielded smoothened branch points in the $l$ plane for our partial-wave amplitude.

Starting from a particular form of multiperipheral hypothesis, we have derived an integral equation for the absorptive amplitude $A(s,t;t_+,t_-)$. This function $A(s,t;t_+,t_-)$ coincides with the physical absorptive part when $t_+$ and $t_-$ are set equal to $\mu^2$, and it is defined completely in terms of observable quantities. By solving our equation, we find that $A(s,t;t_+,t_-)$ has Regge behavior, and the output Regge-trajectory function $\alpha(t)$ is independent of $t_+$ and $t_-$. Furthermore, if one is to identify the output trajectory function with the input function, a bootstrap condition is obtained, the consequences of which can readily be studied using our partial-wave equation. In practice, more than one Regge

[6] The upper limit of the $s'$ integration should be $sf(t_\pm,t_\pm',t)$ instead of simply $s$. $f$ is a function which can be calculated explicitly. However, in our problem the presence of $f$ is equivalent to some modification of the internal coupling $g$; therefore, we set $f=1$.

[7] J. B. Bronzan and C. E. Jones, Phys. Rev. **160**, 1494 (1967).

trajectory is involved. To accommodate this fact, a multichannel scheme can be formulated in a straightforward manner. However, in order to avoid the additional complications of the multichannel problem, we have studied the hypothetical model of a single self-bootstrapping Regge pole. We have solved our equation approximately both at small and large negative $t$ values. It is found that the zero-energy intercept $\alpha(0)$ cannot be 1 if the internal Regge coupling is *strictly* factorizable. If additional structure for this coupling is allowed, the intercept $\alpha(0)$ could in principle be unity. Further applications of our equation will be discussed elsewhere.

PHYSICAL REVIEW VOLUME 184, NUMBER 5 25 AUGUST 1969

# Veneziano Amplitudes and the $\pi\omega\rho\sigma A_1$ System*

A. ZEE†
*Department of Physics, Harvard University, Cambridge, Massachusetts 02138*
(Received 21 April 1969)

We write Veneziano amplitudes for all reactions involving the $\pi\omega\rho\sigma A_1$ system, paying particular attention to kinematical constraints arising from spin. The minimal Veneziano amplitude for $A_1 \to 3\pi$ is shown to be dependent on two parameters, which may be related to the $A_1\rho\pi$ vertex. A discussion is given on how examination of the Dalitz plot may resolve the theoretical confusion surrounding the $A_1$.

RECENTLY, Veneziano[1] has written down an amplitude that satisfies all the postulates of $S$-matrix theory except unitarity. A series of successful applications[2,3] has generated much optimism that the Veneziano amplitude may be a decent first approximation to the real world.[4] Here we attempt to apply the Veneziano-Lovelace analysis to the $\pi\rho A_1$ system. We consider all possible reactions involving $\pi$, $\omega$, $\rho$, $\sigma$ (the daughter of $\rho$), $A_1$, and the "heavy pion" $\tau$ (the daughter of $A_1$), such as $\pi\rho \to \pi\rho$, $\pi A \to \pi A$, $\pi\pi \to \pi A$, $\pi\sigma \to \pi\sigma$, $\pi\tau \to \pi\tau$, $\pi\tau \to \sigma\sigma$, etc. Consistency may be achieved by incorporating a sufficient number of "Veneziano terms," and relations between coupling constants are obtained. Unfortunately, these relations generally involve the daughters.

It must be emphasized from the outset that the isolated Veneziano model lacks predictive power because of the possibility of adding higher terms. We seek to use a minimum number of terms to construct an amplitude consistent with the correct Regge behavior in all channels, the Adler consistency condition, current algebra (where applicable), the appearance of poles with appropriate residues, the absence of isospin $\geq 2$ exotic resonances, and, in cases with spin, the constraints imposed by angular momentum conservation. Apart from simplicity, there is nothing against "non-minimal" terms; we speak of them often in this paper.

To avoid unnecessary complications, we take $m_\pi = 0$ and degenerate trajectories $\rho(s) = \omega(s) = (s+m^2)/2m^2$, $\pi(s) = A(s) = s/2m^2$, where the name of a particle denotes the trajectory to which it belongs ($m = m_\rho$).

We employ the notation

$$\chi(m,\Delta,n) = \frac{\Gamma(m-\chi(s))\Gamma(n-\rho(t))}{\Gamma(\Delta-\chi(s)-\rho(t))}, \tag{1}$$

which behaves asymptotically as $s^{\rho(t)+m-\Delta}$ and $t^{\chi(s)+n-\Delta}$. The residue of the leading pole $\chi(s) = m$ $[\rho(t) = n]$ is a polynomial of order $m+n-\Delta$ in $t$ (in $s$).

### I. $\pi^-(q_1)\pi^+(k) \to \pi^-(q_2)A_1^+(\varepsilon,\rho)$

With $M = H(s,t,u)\epsilon(q_1+q_2) + K(s,t,u)\epsilon k$, minimal structure is

$$H = h_1\rho(1,2,2) + h_2\rho(1,2,1), \tag{2}$$

$$K = k_1\rho(1,1,1) + k_2\rho(1,2,1) + k_3\rho(2,2,1) + k_4\rho(1,2,2). \tag{3}$$

For example, the $h_1$ term is necessary both for $t \to \infty$ before and for the correct residue on the $s$-channel $\rho$ pole, while the $h_2$ term is necessary to accommodate the $t$-channel $\rho$ pole. In $K$, the $k_1$ term alone appears sufficient at first sight. However, crossing symmetry implies that $k_1 = 0$ and relates $k_2$, $k_3$, and $k_4$ to the two truly independent parameters $h_1$ and $h_2$, which upon going on the $\rho$ pole are related to the two $A_1\rho\pi$ coupling constants[5,6] by

$$ga/2m^2 = h_2 = -k_2$$

* Research supported in part by the Office of Naval Research, under Contract No. 1866 (55).
† Parker predoctoral fellow.
[1] G. Veneziano, Nuovo Cimento **57A**, 190 (1968).
[2] C. Lovelace, Phys. Letters **28B**, 264 (1968).
[3] M. Ademollo, G. Veneziano, and S. Weinberg, Phys. Rev. Letters **22**, 83 (1969).
[4] G. Veneziano, in Proceedings of the Sixth Coral Gables Conference on Symmetry Principles at High Energy, 1969 (unpublished); S. Weinberg, Comments Nucl. Particle Phys. (to be published).

[5] Our coupling constants are summarized by

$$\mathcal{L} = g\epsilon^{abc}\rho_\mu{}^a\pi^b\partial^\mu\pi^c + g_\omega\epsilon_{\mu\nu\lambda\sigma}\partial^\nu\rho_a{}^\mu\omega^\nu\partial^\lambda\pi_e\delta^{ae} + \epsilon^{abc}(aA_\mu{}^a\rho^{\mu b}\pi^c - bA_\mu{}^a\rho_\nu{}^b\partial^\mu\partial^\nu\pi^c) + \lambda_{\sigma\pi\pi}\sigma\pi^a\pi^a + g_{A\sigma\pi}A_\mu{}^a\partial^\mu\pi^a\sigma + \lambda_{\sigma\tau\tau}\sigma\tau^a\tau^a + \lambda_{\sigma\tau\pi}\sigma\pi^a\tau^a + (\lambda_{\sigma\sigma\sigma}/3!)\sigma\sigma\sigma + g_{\rho\tau\tau}\epsilon^{abc}\rho_\mu{}^a\tau^b\partial^\mu\tau^c + g_{\rho\tau\pi}\epsilon^{abc}\rho_\mu{}^a\tau^b\partial^\mu\pi^c - \tfrac{1}{2}\epsilon^{abc}g_{\rho AA}A_\mu{}^a\partial^\nu A^{\mu b}\rho_\nu{}^c + (\text{two other } \rho AA \text{ couplings}).$$

[6] The relation of our coupling constants to those of Gilman and Harari is $a = \frac{1}{8}g_T m^2$ and $b = \frac{3}{4}g_T + g_L$. For $m_\pi = 0$, $\Gamma(A \to \rho\pi) = \frac{1}{16}\pi \times (a^2/\sqrt{2}m)\{1 + (1/24)[1 - 3bm^2/a + \frac{1}{4}b^2m^4/a^2]\}$. Note that $a$ controls the width unless $bm^2/a$ is very large; also note that the finite $m_\pi$ correction factor to the width is as large as 1.6.