Low-Energy Theorem for Compton Scattering*

S. SAITO
Department of Physics, University of Wisconsin, Madison, Wisconsin 53706
(Received 13 January 1969)

The low-energy theorem for Compton scattering on arbitrary spin targets is derived. Knowledge of the Born term of the amplitude, which is calculated explicitly, enables us to prove Singh's lemma, which allows us to calculate the threshold value of the amplitude from the gauge condition. Every multipole moment of the target is written down explicitly in terms of the low-energy limit of the amplitude. Up to linear order in photon energy ω, this theorem becomes a generalization to arbitrary spin of the theorem derived by Low and by Gell-Mann and Goldberger. To describe the spin-nonflip amplitude up to order ω^2, we need two structure-dependent parameters in addition to the charge and magnetic moment.

I. INTRODUCTION AND SUMMARY

THE low-energy theorem for Compton scattering was first derived by Low[1] and by Gell-Mann and Goldberger[2] who expressed the zero-energy limit of the Compton scattering amplitude of a spin-$\frac{1}{2}$ particle up to linear order in photon energy in terms of the charge and magnetic moment of the particle. The assumptions used in the derivation were the following: (i) The target is stable, and (ii) there is no virtual photon in intermediate states. The derivation does not depend on the internal structure of the target.

The theorem was extended to higher spin targets by Pais[3] and Bardakci and Pagels.[4] They showed the existence of a low-energy theorem which relates each multipole moment of the target to the low-energy limit of the amplitude. It has also been shown that to describe the amplitude of spin-$\frac{1}{2}$ particles up to ω^2 (ω= photon energy) it is necessary to introduce two structure-dependent parameters, electric and magnetic polarizability, in addition to the multipole moments.[5,6]

In this paper we reexamine the low-energy theorem for an arbitrary spin target. We use Low's method.[7] Each multipole moment of the target is given explicitly. If we restrict ourselves to linear order in ω, we obtain the total expression of the amplitude for all s (s= target spin), which is a generalization of the low-energy theorem of Low and Gell-Mann and Goldberger ($s=\frac{1}{2}$).[1,2] It is also shown that the electric and magnetic polarizabilities are sufficient to determine the spin-nonflip amplitude of an arbitrary spin target up to order ω^2. To describe the total amplitude to this order, we need two more parameters besides these.

The Compton scattering amplitude, written in the form

$$\sum_{\mu,\nu=1}^{4} \epsilon_\mu{}' \epsilon_\nu M_{\mu\nu}, \tag{1.1}$$

satisfies the gauge-invariance condition

$$\sum_{i=1}^{3} k_i{}' M_{i\nu} = -i\omega' M_{4\nu} \text{ and } \sum_{j=1}^{3} k_j M_{\mu j} = -i\omega M_{\mu 4}, \tag{1.2}$$

where ϵ_ν and $k_\nu = (\mathbf{k}, i\omega)$ [$\epsilon_\mu{}'$ and $k_\mu{}' = (\mathbf{k}', i\omega')$] are the polarization vector and the four-momentum of the incident (outgoing) photon. Each equation of (1.2) implies the other by time-reversal invariance. We choose the Coulomb gauge, so that only M_{ij} (i,j= 1, 2, 3) enter the cross-section formula. The proof of the theorem consists of showing that the excited intermediate states' contribution to the right-hand side of (1.2) is vanishingly small compared with the unexcited states' contribution at $\omega \approx 0$. This has first been proved by Singh[8] and then by Bell,[9] and it is called Singh's lemma.

As will be discussed in Sec. IV, we prove Singh's lemma as follows: First we calculate the Born term explicitly. Using the expression of the Born term in $M_{4\mu}$, we find that it dominates the excited-state contribution to $M_{4\mu}$ at $\omega = 0$ as the photon mass approaches zero.

Furthermore we see in Sec. IV that the Born-term contribution to

$$\sum_{i=1}^{3} k_i{}' M_{i\mu}$$

is higher in ω than its contribution to $\omega' M_{4\mu}$. Therefore, combined with Singh's lemma, we can write (1.2) in the form

$$\sum_{i=1}^{3} k_i{}' M_{i\mu}\big|_{\text{excited}} = -i\omega' M_{4\mu}\big|_{\text{Born}} \quad \text{at } \omega=0, \tag{1.3}$$

* Work supported in part by the University of Wisconsin Research Committee, with funds granted by the Wisconsin Alumni Research Foundation, and in part by the U. S. Atomic Energy Commission under Contract No. AT(11-1)-881, No. COO-881-214.

[1] F. E. Low, Phys. Rev. **96**, 1428 (1954).

[2] M. Gell-Mann and M. L. Goldberger, Phys. Rev. **96**, 1433, (1954). The problem was retreated in the following papers: A. Klein, *ibid.* **99**, 998 (1955); E. Kazers, Nuovo Cimento, **13**, 1226 (1959); L. I. Lapidus and Chou Kuang-Chao, Zh. Eksperim. i Teor. Fiz. **39**, 1286 (1960), [English transl.: Soviet Phys.—JETP **12**, 898 (1961)]; R. Jackiw, Phys. Rev. **168**, 1623 (1968).

[3] A. Pais, Phys. Rev. Letters **19**, 544 (1967); Nuovo Cimento **53A**, 433 (1968).

[4] K. Bardakci and H. Pagels, Phys. Rev. **166**, 1783 (1968).

[5] V. A. Petrun'kin, Zh. Eksperim. i Teor. Fiz. **40**, 1148 (1961) [English transl.: Soviet Phys.—JETP **13**, 808 (1961)].

[6] S. R. Choudhury and D. Z. Freedman, Phys. Rev. **168**, 1739 (1968). See also Ref. 13.

[7] The approach recently developed by H. Abarbanel and M. Goldberger employs the helicity amplitude. The low-energy theorem is obtained as the photon mass approaches its physical value. We see in Sec. IV how this leads to the low-energy theorem in Low's method. H. D. I. Abarbanel and M. L. Goldberger. Phys. Rev. **165**, 1594 (1968). See also Ref. 4.

[8] V. Singh, Phys. Rev. Letters **19**, 730 (1967).

[9] J. S. Bell, Nuovo Cimento **52A**, 635 (1967).

i.e., the linear combination of $M_{i\mu}$ on the left-hand side is given by the right-hand side, which is known. There are some amplitudes on the left-hand side of (1.3) which *a priori* are of lower order in ω than the right-hand side. Such terms must vanish.[10]

The calculation of the Born term of a general spin target is given in Sec. III. Various properties of the electromagnetic multipole moments are given in Appendix B. Another important problem is to know the kinematical structure of the amplitude. All the kinematical singularities and zeros should be removed before we take the low-energy limit. This is done for the amplitude which is irreducible under spatial rotations. We study the analyticity of this amplitude in ω in the Breit frame and the analyticity in t (the momentum transfer squared) in the c.m. frame. Combining these we find that the kinematical singularities in both variables can be removed from the amplitude in a simple way. We discuss this in Appendix A.

II. KINEMATICS

A. Irreducible Tensor Amplitudes

We define the T amplitude by

$$\langle \mathbf{p}'m';\mathbf{k}'\beta'|T|\mathbf{p}m;\mathbf{k}\beta\rangle =\frac{1}{(2\pi)^3(4\omega\omega')^{1/2}}\sum_{\mu,\nu=1}^{4}\epsilon_\mu(\mathbf{k}'\beta')\epsilon_\mu(\mathbf{k}\beta)M_{m'\mu;m\nu}(\mathbf{p},\mathbf{Q}) \quad (2.1)$$

and

$$M_{m'\mu;m\nu}(\mathbf{p},\mathbf{Q})=\int d^4x\, e^{iQx}\langle\mathbf{p}'m'|T(j_\mu(-\tfrac{1}{2}x),j_\nu(\tfrac{1}{2}x)) +i\delta(ix_4)\left[j_\mu(-\tfrac{1}{2}x),\frac{\partial}{\partial x_4}A_\nu(\tfrac{1}{2}x)\right]|\mathbf{p},m\rangle, \quad (2.2)$$

where $j_\mu(x)$ is the electromagnetic current density which satisfies $\Box A_\mu(x)=j_\mu(x)$, with $A_\mu(x)$, the electromagnetic vector potential, and

$$Q=\tfrac{1}{2}(k+k'). \quad (2.3)$$

$\mathbf{p}$ and m are the momentum and the z component of the spin of the target in the initial state. β denotes the helicity of the incident photon. Corresponding quantities of the particles in the final state are indicated by primes.

In order to discuss the low-energy phenomena we use amplitudes which are irreducible with respect to spatial rotations:

$$M_{m'a;mb}(\mathbf{p},\mathbf{Q})=\sum(-1)^{s-m'}\begin{pmatrix} s & u & s\\ -m' & \tau & m\end{pmatrix}\begin{pmatrix}1 & v & 1\\ a & \sigma & b\end{pmatrix} \times(-1)^{u-\tau}\begin{pmatrix}u & v & J\\ -\tau & \sigma & M\end{pmatrix}\mathfrak{F}_{JM}{}^{(uv)}(\mathbf{p},\mathbf{Q}). \quad (2.4)$$

Here we introduce the spherical notation $j_a(x)$, $j_b(x)$. The indices a and b take on the values 0 and ±1, and are related to the Cartesian ones $i=1$, 2, 3 by

$$j_{(\pm)}(x)=\mp(1/\sqrt{2})[j_1(x)\pm ij_2(x)],\quad j_{(0)}(x)=j_3(x). \quad (2.5)$$

We will also use the notation

$$j_{\mu=4}(x)=ij_0(x)\,, \quad (2.6)$$

with $a=0$, to represent the time component of the current density. This component transforms as a scalar under spatial rotations, and it should be distinguished from the third space component $j_{(0)}(x)$. In (2.5)

$$\begin{pmatrix} s & u & s\\ -m' & \tau & m\end{pmatrix},\quad \text{etc.}$$

are the Wigner 3-j symbols. In particular we denote the amplitude whose z axis is chosen along $\mathbf{p}$ by $F_{JM}{}^{(uv)}(k,t)$, where $k=|\mathbf{k}|$. In terms of this invariant amplitude, the differential cross section in the laboratory system is given by

$$\frac{d\sigma}{d\Omega_l}=\frac{\omega_l'^2}{(4\pi)^2M\omega_l^2}(M+\omega_l-\omega_l')\sum(2j+1) \times\begin{Bmatrix}s & j & s\\ u_f & s & u\end{Bmatrix}\begin{Bmatrix}s & j & s\\ u' & s & u_i\end{Bmatrix}(-1)^{J+J'+2s+\tau_f} \times\begin{pmatrix}u_f & u & j\\ \tau_f & \tau & \eta\end{pmatrix}\begin{pmatrix}u_i & u' & j\\ \tau_i & \tau' & -\eta\end{pmatrix}\begin{pmatrix}v & u & J\\ \sigma & -\tau & M\end{pmatrix}\begin{pmatrix}v' & u' & J'\\ \sigma' & \tau' & M'\end{pmatrix} \times\rho_{u_f}{}^{\tau_f}\rho_{u_i}{}^{\tau_i}\mathcal{E}_{v\sigma,v'\sigma'}(\theta_l)F_{JM}{}^{(uv)}(k_l,t)F_{J'M'}{}^{(u'v')*}(k_l,t)\,, \quad (2.7)$$

where the summation should be taken over all repeated indices.

$$\begin{Bmatrix}s & j & s\\ u_f & s & u\end{Bmatrix}$$

is the Wigner 6-j symbol,

$$\rho_{u_i}{}^{\tau_i}\ (\rho_{u_f}{}^{\tau_f})$$

is the statistical tensor describing the spin configuration of the initial- (final) state target and is defined by

$$\langle m'|\rho_i|m\rangle=(-1)^{2s} \times\sum_{u_i\tau_i}(-1)^{s-m'}\begin{pmatrix}s & u_i & s\\ -m' & \tau_i & m\end{pmatrix}\rho_{u_i}{}^{\tau_i}, \quad (2.8)$$

where ρ_i (ρ_f) is the density matrix of particle spin components in the initial state. $\mathcal{E}(\theta_l)$ contains all the photon polarization dependence and is a function of the scattering angle θ_l between the photons,

$$\mathcal{E}_{v\sigma,v'\sigma'}(\theta_l)=\sum_{\alpha\alpha'\beta\beta'aa'bb'}\langle\alpha'|\rho_f|\beta'\rangle\langle\beta|\rho_i|\alpha\rangle \times\epsilon_a(\mathbf{k}'\beta')\epsilon_b(\mathbf{k}\beta)\epsilon_{a'}(\mathbf{k}',-\alpha')\epsilon_{b'}(\mathbf{k},-\alpha) \times\begin{pmatrix}1 & 1 & v\\ a & b & \sigma\end{pmatrix}\begin{pmatrix}1 & 1 & v'\\ a' & b' & \sigma'\end{pmatrix}. \quad (2.9)$$

[10] This was also derived by A. Pais, Ref. 3, for a particular case, i.e., $A_2(0)=0$ in this paper.

In the following, we use the Breit frame rather than the laboratory frame. This is the frame which is specified by

$$\mathbf{p}'+\mathbf{p}=0. \tag{2.10}$$

Equations (2.3) and (2.10) show that

$$\mathbf{k}'=\mathbf{Q}+\mathbf{p},\quad \mathbf{k}=\mathbf{Q}-\mathbf{p},\quad \text{and}\quad \omega'=\omega, \tag{2.11}$$

which also implies

$$\mathbf{p}\cdot\mathbf{Q}=0 \tag{2.12}$$

B. Invariant Amplitudes and Their Kinematical Properties

We define the amplitude $M_{\mu\nu}^{(u\tau)}(\mathbf{p},\mathbf{Q})$ by

$$M_{\mu\nu}^{(u\tau)}(\mathbf{p},\mathbf{Q})=(2u+1)\times\sum_{m,m'}(-1)^{s-m'}\begin{pmatrix} s & u & s\\ -m' & \tau & m\end{pmatrix}M_{m'\mu;m\nu}(\mathbf{p},\mathbf{Q}). \tag{2.13}$$

This amplitude satisfies the following symmetry properties:

Under space reflection,

$$M_{\mu\nu}^{(u\tau)}(\mathbf{p},\mathbf{Q})=\xi_{\mu\nu}M_{\mu\nu}^{(u\tau)}(-\mathbf{p},-\mathbf{Q}),$$
$$\xi_{\mu\nu}=-1,\ \text{if one of } \mu \text{ or } \nu \text{ is the time component}$$
$$=1 \text{ otherwise.} \tag{2.14}$$

Under time reversal,

$$M_{\mu\nu}^{(u\tau)}(\mathbf{p},\mathbf{Q})=(-1)^{u}M_{\nu\mu}^{(u\tau)}(\mathbf{p},-\mathbf{Q}). \tag{2.15}$$

Under crossing,

$$M_{\mu\nu}^{(u\tau)}(\mathbf{p},\mathbf{Q};\omega)=M_{\nu\mu}^{(u\tau)}(\mathbf{p},-\mathbf{Q};-\omega), \tag{2.16}$$

where the change of the sign of the energy component is written explicitly. In terms of the invariant amplitude $F_{JM}^{(uv)}(k,t)$, these relations are:

space reflection,

$$F_{JM}^{(uv)}(k,t)=(-1)^{J-M}F_{J,-M}^{(uv)}(k,t), \tag{2.17}$$

time reversal,

$$F_{JM}^{(uv)}(k,t)=(-1)^{u+v+M}F_{JM}^{(uv)}(k,t), \tag{2.18}$$

and crossing,

$$F_{JM}^{(uv)}(k,t)=(-1)^{v+M}F_{JM}^{(uv)}(-k,t), \tag{2.19}$$

respectively. By expanding $M_{\mu\nu}^{(u\tau)}(\mathbf{p},\mathbf{Q})$ in powers of k, we find from (2.15) and (2.16)

$$M_{\mu\nu}^{(u\tau)}(\mathbf{p},\mathbf{Q})=\text{even (odd) power series of } k \text{ if } u=\text{even (odd)}. \tag{2.20}$$

In order to see more details, we expand the amplitude in terms of the harmonic polynomials of $\mathbf{k}'$ and $\mathbf{k}$:

$$\mathfrak{F}_{JM}^{(u\tau)}(\mathbf{p},\mathbf{Q})=\sum_{l,L=0}^{\infty}k^{l+L}\begin{pmatrix} l & L & J\\ l_3 & L_3 & M\end{pmatrix}\times Y_{ll_3}(\hat{k}')Y_{LL_3}(\hat{k})H_{J;lL}^{(uv)}(k,t). \tag{2.21}$$

$Y_{ll_3}(\hat{k}')$ is the spherical harmonic function whose arguments are the polar and azimuthal angles of $\hat{k}'$, the unit vector parallel to $\mathbf{k}'$. $H_{J;lL}^{(uv)}(k,t)$ is an invariant function of k and t. The space-reflection invariance (2.14) imposes

$$l+L=\text{even} \tag{2.22}$$

in (2.21). Combining this and (2.20) with the triangle condition $l+L\geq J$ due to the 3-j symbol in (2.21), we find

$$\mathfrak{F}_{JM}^{(uv)}(\mathbf{p},\mathbf{Q})\sim k^{J+e_J+e_u} \tag{2.23}$$

at small values of k, where

$$e_x=0 \text{ or } 1 \text{ corresponding to } x=\text{even or odd}. \tag{2.24}$$

Since this is the consequence of purely kinematical requirements, we expect that we can divide the amplitude by the power of k given in (2.23), without introducing any kinematical singularities at $k=0$.

The argument in Appendix A shows that the amplitude $F_{JM}^{(uv)}(k,t)$ is regular at $k=0$ (except for a possible dynamical singularity) after the factor

$$(\sin\theta)^{e_M} \tag{2.25}$$

is removed, where θ is the photon scattering angle.

In addition to $F_{JM}^{(uv)}(k,t)$, we introduce another invariant amplitude $G_{JM}^{(uv)}(k,t)$, which we will use to derive the low-energy theorem. This amplitude is specified by taking the z axis along $\mathbf{k}$, so that it is related to $F_{JM}^{(uv)}(k,t)$ by

$$G_{JM}^{(uv)}(k,t)=\sum_{M'}F_{JM'}^{(uv)}(k,t)d_{M'M}^{(J)}(-\Theta). \tag{2.26}$$

Here $d_{M'M}^{(J)}$ is the usual d function representing a rotation around the y axis; Θ is the angle between $\mathbf{p}$ and $\mathbf{k}$ and is related to θ by

$$\Theta=\tfrac{1}{2}(\theta+\pi). \tag{2.27}$$

To find the kinematical singularities of $G_{JM}(k,t)$ we first show that $H_{J;lL}$ in (2.21) is free of kinematical singularities. If we put $\mathbf{p}$ along the z axis in (2.21) and use $Y_{ll_3}(\hat{k}')=Y_{ll_3}(\pi-\Theta,0)$, $Y_{LL_3}(\hat{k})=Y_{LL_3}(\Theta,0)$, we obtain the expression for $F_{JM}(k,t)$:

$$F_{JM}^{(uv)}(k,t)=\sum_{lLJ'}k^{l+L}\begin{pmatrix} l & L & J\\ l_3 & L_3 & M\end{pmatrix}\begin{pmatrix} l & L & J'\\ l_3 & L_3 & M\end{pmatrix}\begin{pmatrix} l & L & J'\\ 0 & 0 & 0\end{pmatrix}\times(-1)^{l+L_3}\left[\frac{(2l+1)(2L+1)(J'-M)!}{(J'+M)!}\right]^{1/2}\frac{2J'+1}{4\pi}\times P_{J'}{}^{M}(\cos\Theta)H_{J;lL}^{(uv)}(k,t). \tag{2.28}$$

$P_{J'}{}^{M}(\cos\Theta)$ is the associated Legendre function. In (2.28),

$$\begin{pmatrix} l & L & J'\\ 0 & 0 & 0\end{pmatrix}=0$$

unless $l+L+J'$ is even, and so the condition (2.22) demands that J' is even. In this case $P_{J'}{}^{M}(\cos\Theta)$ is proportional to $\cos\Theta\sin\Theta=\frac{1}{2}\sin\theta$, if M is odd, and its remaining factor is a polynomial function of $\cos^2\Theta=\frac{1}{2}(1-\cos\theta)$. For M even, $P_{J'}{}^{M}(\cos\Theta)$ is just a polynomial function of $\cos^2\Theta$ (or $\cos\theta$). Thus $P_{J'M}(\cos\Theta)$ is proportional to the factor (2.27) which contains all kinematical singularities of $F_{JM}{}^{(uv)}(k,t)$. Since $\cos\theta$ is a linear function of t,

$$t=-2k^2(1-\cos\theta)\,, \tag{2.29}$$

a polynomial in it does not introduce any kinematical singularities. Therefore we conclude that $H_{J;lL}{}^{(uv)}(k,t)$ is a kinematical singularity-free amplitude.

The amplitude $G_{JM}{}^{(uv)}(k,t)$ is obtained from (2.21) by choosing the z axis along $\mathbf{k}$:

$$G_{JM}{}^{(uv)}(k,t)=\sum_{l,L=0}^{\infty}k^{l+L}\begin{pmatrix} l & L & J\\ -M & 0 & M\end{pmatrix}\times\frac{1}{4\pi}\left[\frac{(2l+1)(2L+1)(l-M)!}{(l+M)!}\right]^{1/2} P_l{}^M(\cos\theta)H_{J;lL}{}^{(uv)}(k,t)\,. \tag{2.30}$$

$P_{lM}(\cos\theta)$ is again proportional to (2.27) multiplied by a polynomial function of $\cos\theta$ (or t). Combining this with (2.23), we define the regularized amplitude $\tilde{G}_{JM}{}^{(uv)}(k,t)$ by

$$\tilde{G}_{JM}{}^{(uv)}(k,t)=k^{-J-e_J-e_u}(\sin\theta)^{-e_M}G_{JM}{}^{(uv)}(k,t)\,, \tag{2.31}$$

which does not have kinematical singularities or zeros at $k=0$.

III. BORN TERM

The Born term has to be known up to the required order in k in order to obtain the low-energy limit of the amplitude. We compute it explicitly in the Breit frame. The Born term, which is defined as the term due to the unexcited intermediate state in the Low equation, is given by

$$M_{m'\mu;m\nu}(\mathbf{p},\mathbf{Q})|_{\text{Born}}=i\left(\frac{R_{m'\mu;m\nu}(\mathbf{p},\mathbf{Q})}{\omega-\omega_p}-\frac{R_{m'\nu;m\mu}(\mathbf{p},-\mathbf{Q})}{\omega+\omega_p}\right), \tag{3.1}$$

where

$$R_{m'\mu;m\nu}(\mathbf{p},\mathbf{Q})=\sum_n\langle-\mathbf{p}m'|j_\mu(0)|\mathbf{Q}n\rangle\langle\mathbf{Q}n|j_\nu(0)|\mathbf{p}m\rangle \tag{3.2}$$

and $\sum_n$ denotes summation over the spin component of the intermediate state. ω_p is given by

$$\omega_p=E(\mathbf{Q})-E(\mathbf{p})=(M^2+\mathbf{Q}^2)^{1/2}-(M^2+\mathbf{p}^2)^{1/2}. \tag{3.3}$$

We denote the Lorentz transformation with velocity $\mathbf{p}/E(\mathbf{p})$ by $\Lambda(\mathbf{p})$; then each vertex function in (3.2) is written in the form

$$\langle-\mathbf{p}m'|j_\mu(0)|\mathbf{Q}n\rangle=\langle m'|\Lambda^{-1}(-\mathbf{p})j_\mu(0)\Lambda(\mathbf{Q})|n\rangle=\sum_{\mu'}A_{\mu\mu'}(\mathbf{p},-\phi)\langle m'|R(\phi)j_{\mu'}(0)\Lambda(\zeta)R(\psi)|n\rangle, \tag{3.4}$$

where $|m\rangle$ denotes the state of one particle at rest. By choosing the z and x axes along $\mathbf{p}$ and $\mathbf{Q}$, we can replace $\Lambda(\mathbf{p})\Lambda(\mathbf{Q})$ by the successive operations of the Lorentz transformation $\Lambda(\zeta)$ along the z axis and two rotations $R(\phi)$ and $R(\psi)$ around the y axis:

$$\Lambda(\mathbf{p})\Lambda(\mathbf{Q})=R(\phi)\Lambda(\zeta)R(\psi)\,. \tag{3.5}$$

$A_{\mu\mu'}(\mathbf{p},-\phi)$ is the matrix element describing the Lorentz transformation $\Lambda(\mathbf{p})$ after the spatial rotation around the y axis by the angle $-\phi$. The velocity of the Lorentz transformation $\tanh\zeta$ is given by

$$(\tanh\zeta)^2=1-M^4/E^2(\mathbf{p})E^2(\mathbf{Q})\,, \tag{3.6}$$

whereas the magnitudes of the rotations are given by

$$\cos\phi=|\mathbf{p}|E(\mathbf{Q})/M^2\sinh\zeta,\ \cos\psi=|\mathbf{p}|/M\sinh\zeta. \tag{3.7}$$

We define the electromagnetic multipole moment $\mathcal{T}_J{}^a$ by[11]

$$\langle m'|j_a(0)\Lambda(\zeta)|m\rangle=\sum_{J=|a|}^{\infty}(-1)^{s-m'}\begin{pmatrix} s & J & s\\ -m' & a & m\end{pmatrix}\mathcal{T}_J{}^a, \tag{3.8}$$

and

$$\langle m'|\Lambda(\zeta)j_a(0)|m\rangle=\sum_{J=|a|}^{\infty}(-1)^{s-m'}\begin{pmatrix} s & J & s\\ -m' & a & m\end{pmatrix}\bar{\mathcal{T}}_J{}^a.$$

The expression for the Born term (3.1) becomes

$$M_{m'\mu;m\nu}(\mathbf{p},\mathbf{Q})|_{\text{Born}}=i\sum_{u,\tau}(-1)^{s-m'}\begin{pmatrix} s & u & s\\ -m' & \tau & m\end{pmatrix}\left[\frac{1}{\omega-\omega_p}-\frac{(-1)^u}{\omega+\omega_p}\right]\times R_{\mu\nu}{}^{(u\tau)}(\mathbf{p},\mathbf{Q}), \tag{3.9}$$

$$R_{\mu\nu}{}^{(u\tau)}(\mathbf{p},\mathbf{Q})=\sum A_{\mu\mu'}(\mathbf{p},-\phi)A_{\nu\nu'}(-\mathbf{p},\phi)\eta_{\mu'a}\eta_{\nu'b}\times(2u+1)(-1)^{s-n}\begin{pmatrix} s & u & s\\ -m' & \tau & m\end{pmatrix}\begin{pmatrix} s & J_1 & s\\ -m' & a' & n'\end{pmatrix}\times\begin{pmatrix} s & J_2 & s\\ -n & b' & m\end{pmatrix}d_{a'a}{}^{(J_1)}(\phi)d_{b'b}{}^{(J_2)}(-\phi)\times d_{n'n}{}^{(s)}[2(\phi+\psi)]\mathcal{T}_{J_1}{}^a\bar{\mathcal{T}}_{J_2}{}^b, \tag{3.10}$$

where $\eta_{\mu a}$ is defined by

$$j_\mu(x)=\sum_a\eta_{\mu a}j_a(x)\,.$$

[11] L. Durand, III, P. C. DeCelles, and R. B. Marr, Phys. Rev. **126**, 1882 (1962).

We now consider the threshold behavior of the Born term. ω_p, defined by (3.3), behaves as

$$\omega_p \sim (k^2 \cos\theta)/2M \quad \text{at} \quad k \approx 0, \tag{3.11}$$

so that the singular factor of (3.9) becomes

$$\begin{aligned} \frac{1}{\omega-\omega_p}-\frac{(-1)^u}{\omega+\omega_p} &= \frac{2\omega_p}{\omega^2-\omega_p^2} \sim \frac{\cos\theta}{M}, \quad \text{if } u \text{ is even} \\ &= \frac{2\omega}{\omega^2-\omega_p^2} \sim \frac{2}{\omega}, \quad \text{if } u \text{ is odd}. \end{aligned} \tag{3.12}$$

The polynomial expansion of $R_{\mu\nu}^{(u\tau)}(\mathbf{p},\mathbf{Q})$ [similar to (2.21) and (2.14)] shows that $R_{4j}^{(u\tau)}(\mathbf{p},\mathbf{Q})$ and $R_{44}^{(u\tau)}(\mathbf{p},\mathbf{Q})$ behaves as $\sim k^{u+e_u-1}$ and $\sim k^{u+e_u}$, respectively, or as higher powers of k at $k \approx 0$. In order that $M_{\mu\nu}^{(u\tau)}(\mathbf{p},\mathbf{Q})|_{\text{Born}}$ satisfies (2.20), however, we have to have the following behavior:

$$R_{4\mu}^{(u\tau)}(\mathbf{p},\mathbf{Q}) \sim k^{u+e_u}, \quad \mu = 1 \sim 4 \tag{3.13}$$

and

$$M_{4\mu}^{(u\tau)}(\mathbf{p},\mathbf{Q})|_{\text{Born}} \sim k^u, \quad \mu = 1 \sim 4. \tag{3.14}$$

If the mass of the photon is finite, say m_γ, we obtain from (3.12)

$$M_{4\mu}^{(u\tau)}(\mathbf{p},\mathbf{Q})|_{\text{Born}} \sim k^u (k/m_\gamma)^{2-e_u}. \tag{3.15}$$

This result will be used for the proof of Singh's lemma.

IV. GAUGE INVARIANCE

The gauge-invariance condition (1.2) can be rewritten by using the irreducible tensor amplitude in the following form:

$$\sum \mathbf{k}_a' \begin{pmatrix} 1 & v & 1 \\ a & \sigma & b \end{pmatrix} (-1)^{u-\tau} \begin{pmatrix} u & v & J \\ -\tau & \sigma & M \end{pmatrix} \mathfrak{F}_{JM}^{(uv)}(\mathbf{p},\mathbf{Q}) = \omega M_{ob}^{(u\tau)}(\mathbf{p},\mathbf{Q}) \tag{4.1}$$

$$b = (0), (\pm), \quad u = 0, 1, 2 \cdots 2s.$$

It is easy to prove the following relation using the current conservation equation

$$\textstyle\sum_{\mu=1}^4 (p'-p)_\mu \langle \mathbf{p}' | j_\mu(0) | \mathbf{p} \rangle = 0:$$

$$R_{m'4;mj}(\mathbf{p},\mathbf{Q}) = i \sum_{i=1}^{3} \frac{k_i'}{\omega_p} R_{m'i;mj}(\mathbf{p},\mathbf{Q}). \tag{4.2}$$

Because of (3.11), we can see that the Born-term contribution to the right-hand side of (4.1) is smaller than its contribution to the right-hand side by the factor k/M.

Since the right-hand member of (4.1) behaves as $\sim k^{u+1}$ at $k \approx 0$, so must the left-hand member. When the photon mass is finite, the behavior of the Born term on the right-hand side changes to (3.15), whereas the behavior of the left-hand side is unchanged (i.e., k^{u+1}). This demands that the right-hand side of (4.1) also has to behave as

$$\omega M_{ob}^{(u\tau)}(\mathbf{p},\mathbf{Q}) \sim m_\gamma k^{u+1}, \quad \text{or higher order of } k \text{ at } k \approx 0,$$

$$b = (0), \quad (\pm). \tag{4.3}$$

As m_γ approaches zero, a singularity appears at $k=0$ which suppresses the extra k^{2-e_u} power of the Born term. We assume that the amplitude is a regular function of m_γ around $m_\gamma = 0$, so that the behavior (4.3), after the singular term is subtracted, does not change when we take the $m_\gamma \to 0$ limit. Taking (2.20) into account we finally conclude that

$$M_{ob}^{(u\tau)}(\mathbf{p},\mathbf{Q})|_{\text{excited}} \sim k^{u+2}, \quad b = (0), (\pm). \tag{4.4}$$

A similar discussion shows that (4.4) is true for $b=0$ too. This proves Singh's lemma, which claims that $M_{ob}^{(u\tau)}(\mathbf{p},\mathbf{Q})|_{\text{excited}}$ can be ignored compared with the Born term (3.14) as far as terms of order k^u.

V. LOW-ENERGY THEOREMS

According to the results discussed in the previous section, we can write (4.1) in the form

$$\sum \hat{k}_a' \begin{pmatrix} 1 & v & 1 \\ a & \sigma & b \end{pmatrix} (-1)^{u-\tau} \begin{pmatrix} u & v & J \\ -\tau & \sigma & M \end{pmatrix} \times \mathfrak{F}_{JM}^{(uv)}(\mathbf{p},\mathbf{Q})|_{\text{excited}} = M_{ob}^{(u\tau)}(\mathbf{p},\mathbf{Q})|_{\text{Born}}, \tag{5.1}$$

$$b = (0), (\pm), \quad u = 0, 1, 2 \cdots 2s,$$

up to order k^u. The space-reflection and time-reversal invariances show that there are $2u+2$ $(2u)$ independent equations in (5.1) corresponding to $b=(0)$, $(+)$ and $\tau = 0, 1 \cdots u$ $(\tau = 1, 2 \cdots u)$ for u even (odd). On the other hand, the number of independent $\mathfrak{F}_{JM}^{(uv)}(\mathbf{p},\mathbf{Q})$ in order k^u, after (2.14) and (2.15) are taken into account, is $3u+1$, $(3u)$ for u even (odd). Therefore (5.1) determines the values of $\mathfrak{F}_{JM}^{(uv)}(\mathbf{p},\mathbf{Q})|_{\text{excited}}(J=u, u-1, u-2)$ of order k^u leaving $u-1$ (u) functions of u even (odd) undetermined.

A. Low-Energy Theorem of Order k^{u-2}

There are terms on the left-hand side of (5.1) which behave as $\sim k^{u-2}$ at $k \approx 0$. Such terms should vanish since the right-hand side behaves as $\sim k^u$ (3.14). Thus, we obtain

$$\frac{1}{k^{u-2}} \sum \hat{k}_a' \begin{pmatrix} 1 & 2 & 1 \\ a & \sigma & b \end{pmatrix} (-1)^{u-\tau} \begin{pmatrix} u & 2 & u-2 \\ -\tau & \sigma & M \end{pmatrix} \times \mathfrak{F}_{u-2,M}^{(u-2)}(\mathbf{p},\mathbf{Q}) = 0 \tag{5.2}$$

at $k=0$. In particular

$$\mathfrak{F}_{00}^{(22)}(\mathbf{p},\mathbf{Q}) = 0 \tag{5.3}$$

and

$$(1/k^2)\mathfrak{F}_{2M}^{(4,2)}(\mathbf{p},\mathbf{Q}) = 0 \quad \text{at } k=0. \tag{5.4}$$

Equation (5.3) has been also derived by Pais,[10]

B. Low-Energy Theorem of Multipole Moments

Instead of (5.1), we use the gauge-invariance condition in the form

$$\sum \hat{k}_a{}'\hat{k}_b \begin{pmatrix} 1 & v & 1 \\ a & \sigma & b \end{pmatrix} (-1)^{u-\tau} \begin{pmatrix} u & v & J \\ -\tau & \sigma & M \end{pmatrix} \times \mathfrak{F}_{JM}{}^{(uv)}(\mathbf{p},\mathbf{Q})\,|_{\text{excited}} = M_{00}{}^{(u\tau)}(\mathbf{p},\mathbf{Q})\,|_{\text{Born}}, \quad u=0,\,1\cdots 2s \tag{5.5}$$

which is obtained by combining two equations of (1.2) and also by using the results obtained in Sec. IV. In order to separate the 2^u-pole moment from the others on the right-hand side of (5.5), we choose the z axis along $\mathbf{k}$ and put $\tau=u$. The appropriate invariant amplitude in this case is $\tilde{G}_{JM}{}^{(u\tau)}(k,t)$. Calculating the Born term explicitly from (3.9) and (3.10), we obtain

$$\begin{aligned} &\frac{\cos\theta}{[3(2u+1)]^{1/2}}\left\{-\tilde{G}_{uu}{}^{(0)}+\left[\frac{2u(2u-1)}{5(2u+3)(u+1)}\right]^{1/2}\tilde{G}_{uu}{}^{(2)}\right\} \\ &-\frac{\sin^2\theta}{2[(u+1)(2u+1)]^{1/2}}\left\{(\sqrt{\tfrac{1}{3}})\tilde{G}_{u,u-1}{}^{(1)} +\left[\frac{3(2u-1)}{5(2u+3)}\right]^{1/2}\tilde{G}_{u,u-1}{}^{(2)}\right\}+\frac{\sin^2\theta}{2(2u+1)^{1/2}} \\ &\times\left\{\frac{1}{\sqrt{3}}\tilde{G}_{u-1,u-1}{}^{(1)}+\left[\frac{u-1}{5(u+1)}\right]^{1/2}\tilde{G}_{u-1,u-1}{}^{(2)}\right\} \\ &= i\frac{[(2s+1)(2u)!]^{1/2}}{2^u u!}\cos\theta\,(\sin\theta)^u\frac{e}{M}\mathcal{Q}_u \end{aligned} \tag{5.6}$$

for u even and

$$\begin{aligned} &\frac{1}{2(2u+1)^{1/2}}\left\{\sqrt{\tfrac{1}{3}}\tilde{G}_{u-1,u-1}{}^{(1)}+\left[\frac{u-1}{5(u+1)}\right]^{1/2}\tilde{G}_{u-1,u-1}{}^{(2)}\right\} \\ &=2i(2s+1)^{1/2}\frac{[(2u-1)!]^{1/2}}{2^u(u-1)!}(\sin\theta)^{u-1} \\ &\times\left\{\left(\frac{s+1}{u+1}\right)^{1/2}\frac{e}{M}\mathfrak{M}_u+\tfrac{1}{2}u[\tfrac{1}{2}(2s+u+1)(2s-u+1)]^{1/2} \times\frac{1}{2u-1}\frac{1}{M}\left(\frac{\mu}{s}-\frac{e}{M}\right)\mathcal{Q}_{u-1}\right\} \end{aligned} \tag{5.7}$$

for u odd. The superscript (u) of $\tilde{G}_{JM}{}^{(uv)}$ is omitted because u is fixed. $\mathcal{Q}_u$ and $\mathfrak{M}_u$ are the electric and magnetic 2^u-pole moments defined by

$$\mathcal{T}_u{}^0=(2s+1)^{1/2}k^u\mathcal{Q}_u$$

and

$$\mathcal{T}_u{}^{(+)}=-[1/s(s+1)(2s+1)]^{1/2}k^u\mathfrak{M}_u, \tag{5.8}$$

and

$$e=\mathcal{Q}_0 \quad\text{and}\quad \mu=\mathfrak{M}_1 \tag{5.9}$$

are the total charge and the total magnetic dipole moment. If we use the invariant amplitude $H_{J;lL}{}^{(uv)}(k,t)$ related to $G_{JM}{}^{(uv)}(k,t)$ by (2.30), we can take the θ dependence into account explicitly and obtain

$$\begin{aligned} &(2u)^{1/2}H_{u;u,0}{}^{(0)}+\frac{1}{2(u+1)^{1/2}}H_{u;u,0}{}^{(1)} -\frac{4u-3}{2}\left[\frac{2u-1}{5(u+1)(2u+3)}\right]^{1/2}H_{u;u,0}{}^{(2)} \\ &=-4\pi i[(2u+1)6u(2s+1)]^{1/2}\frac{e}{M}\mathcal{Q}_u, \end{aligned} \tag{5.10}$$

and

$$\begin{aligned} &H_{u;u-1,1}{}^{(1)}+3\left[\frac{2u-1}{5(2u+3)}\right]^{1/2}H_{u;u-1,1}{}^{(2)} +\left[\frac{(u+1)(2u+1)}{(u-1)(2u-1)}\right]^{1/2}\left\{H_{u-1;u-1,1}{}^{(1)} +\left[\frac{3(u-1)}{5(u+1)}\right]^{1/2}H_{u-1;u-1,1}{}^{(2)}\right\}=0 \end{aligned} \tag{5.11}$$

for even u, and

$$\begin{aligned} &H_{u-1;u-1,1}{}^{(1)}+\left[\frac{3(u-1)}{5(u+1)}\right]^{1/2}H_{u-1;u-1,1}{}^{(2)} \\ &=8i\pi\left[\frac{u(4u^2-1)}{u-1}\right]^{1/2}\left[\frac{s+1}{s(u+1)}\right]^{1/2}\frac{e}{M}\mathfrak{M}_u +\tfrac{1}{2}u[\tfrac{1}{2}(2s+u+1)(2s-u+1)]^{1/2} \times\frac{1}{2u-1}\left(\frac{\mu}{s}-\frac{e}{M}\right)\mathcal{Q}_{u-1}\Big\} \end{aligned} \tag{5.12}$$

for odd u. The values of $H_{J;lL}{}^{(v)}$ should be evaluated at $k=0$. Equations (5.10) and (5.12) are the low-energy theorems for the electric and magnetic 2^u-pole moments, respectively. The $u=2$ case was derived by Pais[3] and by Bardakci and Pagels.[4] Equation (5.11) is a new type of low-energy theorem which does not depend on the static moments.

C. Low-Energy Theorem of the Total Amplitude

From (5.6) and (5.7), we immediately obtain

$$\tilde{G}_{00}{}^{(00)}(0,0)\,|_{\text{excited}}=-i[3(2s+1)]^{1/2}e^2/M \tag{5.13}$$

and

$$\tilde{G}_{00}{}^{(11)}(0,0)\,|_{\text{excited}} =i3\left[\frac{2(s+1)(2s+1)}{s}\right]^{1/2}\frac{1}{M}\left(2\mu-s\frac{e}{M}\right) \tag{5.14}$$

corresponding to $u=0$ and $u=1$. Equations (2.23) and (5.3) show that these are only the amplitudes which are left in the complete expression of the amplitude up to linear order in k besides the Born terms. Adding the Born terms calculated from (3.9) and (3.10), we arrive at the following:

$$\sum_{\mu,\nu=1}^{4} \epsilon_\mu(\mathbf{k}',\beta')\epsilon_\nu(\mathbf{k},\beta)M_{m'\mu;m\nu}(\mathbf{p},\mathbf{Q})$$
$$=i\frac{e^2}{M}(\boldsymbol{\varepsilon}'\cdot\boldsymbol{\varepsilon})\delta_{m'm}+\frac{e}{M}\left(2\mu-s\frac{e}{M}\right)k(\boldsymbol{\varepsilon}'\times\boldsymbol{\varepsilon})\cdot\mathbf{S}_{m'm}$$
$$-\frac{e\mu}{kM}[(\boldsymbol{\varepsilon}\cdot\mathbf{k}')(\boldsymbol{\varepsilon}'\times\mathbf{k}')-(\boldsymbol{\varepsilon}'\cdot\mathbf{k})(\boldsymbol{\varepsilon}\times\mathbf{k})]\cdot\mathbf{S}_{m'm}$$
$$-\frac{\mu^2}{ks}[(\boldsymbol{\varepsilon}'\times\mathbf{k}')\times(\epsilon+\mathbf{k})]\mathbf{S}_{m'm}+O(k^2), \quad (5.15)$$

where

$$(S_{m'm})_\tau\equiv\left[\frac{(s+1)(2s+1)}{s}\right]^{1/2}(-1)^{s-m'}\begin{pmatrix} s & 1 & s\\ -m' & \tau & m\end{pmatrix},$$
$$\boldsymbol{\varepsilon}\equiv\boldsymbol{\varepsilon}(\mathbf{k},\beta),\quad\text{and}\quad \boldsymbol{\varepsilon}'\equiv\boldsymbol{\varepsilon}(\mathbf{k}',\beta').$$

This is the generalization of the low-energy theorem of Low and Gell-Mann and Goldberger[1,2] to the arbitrary spin case.

VI. LOW-ENERGY THEOREM OF HIGHER ORDER

In order to determine the amplitude up to order k^2, we have to consider its dependence on the internal structure of the target, as was discussed in Sec. IV. We will show in the present section that two parameters, in addition to the static multipole moments, determine the spin-nonflip part of the amplitude completely up to this order, for arbitrary spin.

The spin-nonflip amplitude can be written in terms of the notation defined by (2.13)

$$M_{ij}^{(00)}(\mathbf{p},\mathbf{Q})=\delta_{ij}A(k)+(k_ik_j+k'_ik'_j)C+k'_ik_jD$$
$$-[2i/(2s+1)][k_ik_j'-(\mathbf{k}'\cdot\mathbf{k})\delta_{ij}]\tilde{\beta}, \quad (6.1)$$

which satisfies space reflection, time reversal, and crossing symmetry. The gauge-invariance condition demands

$$\sum_{i,j=1}^{3} k_i'k_jM_{ij}^{(00)}(\mathbf{p},\mathbf{Q})=(\mathbf{k}'\cdot\mathbf{k})A(k)+2k^2(\mathbf{k}'\cdot\mathbf{k})C+k^4D$$
$$=-k^JM_{44}^{(00)}(\mathbf{p},\mathbf{Q}). \quad (6.2)$$

The excited-state contribution to $M_{44}^{(00)}$ can be specified by the electric polarizability α [12] as

$$M_{44}^{(00)}(\mathbf{p},\mathbf{Q})\,|_{\text{excited}}=\frac{2i}{(2s+1)^{1/2}}(\mathbf{k}'\cdot\mathbf{k})\alpha. \quad (6.3)$$

The Born term is calculated from (3.9)

$$M_{44}^{(00)}(\mathbf{p},\mathbf{Q})\,|_{\text{Born}}=-i(2s+1)^{1/2}$$
$$\times\cos\theta\left(1-\frac{k^2}{4M^2}\sin^2\theta-\tfrac{1}{3}\langle r_e^2\rangle k^2\right)\frac{e^2}{M}+O(k^4), \quad (6.4)$$

where $\langle r_e^2\rangle$ is the mean-square radius of the charge. (See Appendix B.) Substituting (6.4) and (6.3) into (6.2) we find

$$D=0, \quad (6.5)$$

because all terms in (6.2) except D are proportional to $\cos\theta$. For the spin-$\frac{1}{2}$ case, (6.5) was proved by Singh.[13]

From the transversality condition of the photon we see that C does not appear in the amplitude $\sum_{i,j}\epsilon_i'\epsilon_jM_{ij}^{(00)}$. If we define the modified electric polarizability $\tilde{\alpha}$ by

$$\tilde{\alpha}=\alpha-i(2s+1)^{1/2}C+(e^2/M)\langle r_e^2\rangle\tfrac{1}{6}(2s+1), \quad (6.6)$$

we can write the spin-nonflip amplitude in the form

$$\sum_{i,j=1}^{3}\epsilon_i'\epsilon_jM_{ij}^{(00)}(\mathbf{p},\mathbf{Q})$$
$$=\frac{i}{(2s+1)^{1/2}}\left[\left(1-\frac{k^2}{4M^2}\sin^2\theta\right)\frac{(2s+1)e^2}{M}-2k^2\tilde{\alpha}\right](\boldsymbol{\varepsilon}'\cdot\boldsymbol{\varepsilon})$$
$$-\frac{i}{(2s+1)^{1/2}}(\boldsymbol{\varepsilon}'\times\mathbf{k}')\cdot(\boldsymbol{\varepsilon}\times\mathbf{k})\tilde{\beta}+O(k^4). \quad (6.7)$$

This formula contains another parameter $\tilde{\beta}$, which is called the magnetic polarizability of the target,[12] and cannot be determined from (6.2).

As was shown in Sec. V, Eq. (5.1) determines six amplitudes among seven independent ones $\mathfrak{F}_{JM}^{(uv)}(\mathbf{p},\mathbf{Q})$, to order k^2 for $u=2$. Other amplitudes, which can contribute to this order, were shown to vanish at $k=0$ in (5.4). Therefore, to determine the total amplitude up to order k^2, we have to introduce two more parameters which specify one of the amplitudes $\mathfrak{F}_{JM}^{(2v)}$ corresponding to the terms proportional to k^2 and $k^2\cos\theta$. This is the generalization of the low-energy theorem of k^2 order[5,6] to the arbitrary spin case.

ACKNOWLEDGMENTS

I would like to thank Professor B. Sakita for suggesting this problem and for much helpful advice and encouragement during the period of this work. I also wish to thank Professor C. J. Goebel for many illuminating discussions and his careful reading of this paper.

APPENDIX A: KINEMATICAL SINGULARITIES

We discuss the kinematical singularities of the invariant amplitudes $F_{JM}^{(uv)}(k,t)$ in the Breit frame. Since

$$t=-4\mathbf{p}^2 \quad (A1)$$

in this frame, the s dependence of the M function is contained only in e^{iQx} in (2.2). Equations (2.3), (2.10), and (2.11) show that

$$|\mathbf{Q}|=[((s-M^2)^2+st)/(4M^2-t)]^{1/2}, \quad (A2)$$

and therefore the kinematical singularities in s can

[12] A. M. Baldin, Nucl. Phys. **18**, 310 (1960). See also the paper by A. Klein in Ref. 2.

[13] V. Singh, Phys. Rev. **165**, 1532 (1968).

enter only through odd powers of $|\mathbf{Q}|$. This is equivalent to stating that the amplitude has the kinematical singularity given by (A2) if it is an odd function under the change of $\mathbf{Q}$ to $-\mathbf{Q}$. When $\mathbf{p}$ lies along $\hat{z}$ and $\mathbf{Q}$ along $\hat{x}$, (2.14) and (2.15) show that

$$\mathfrak{F}_{JM}^{(uv)}(\mathbf{p},-\mathbf{Q})=(-1)^M\mathfrak{F}_{JM}^{(uv)}(\mathbf{p},\mathbf{Q})=(-1)^M F_{JM}^{(uv)}(k,t).$$

From this we see that

$$F_{JM}^{(uv)}(k,t) \text{ is an even (odd) power function of } |Q| \text{ if } M \text{ is even (odd).} \quad \text{(A3)}$$

From (2.1) and (2.4), and by using

$$\epsilon_a(\mathbf{k}'\beta')\epsilon_b(\mathbf{k}\beta)=\sum_{a'b'}\epsilon_{a'}(\beta')\epsilon_{b'}(\beta)\times d_{a'a}^{(1)}\left(\frac{\pi-\theta}{2}\right)d_{b'b}^{(1)}\left(\frac{\pi+\theta}{2}\right), \quad \text{(A4)}$$

where $\epsilon_{(\pm)}(\beta)=\frac{1}{2}(\beta\pm1)$, $\epsilon_{(0)}(\beta)=0$, we can see that (A3) implies that

$$\langle -\mathbf{p}m';\mathbf{k}'\beta'|T|\mathbf{p}m;\mathbf{k}\beta\rangle \text{ is an even (odd) power function of } |\mathbf{Q}| \text{ if } m'-m \text{ is even (odd).} \quad \text{(A5)}$$

The kinematical singularities in t are known for the helicity amplitude in the c.m. frame. The T amplitude (2.1) is related to the helicity amplitude in the c.m. frame by

$$\langle -\mathbf{p}m';\mathbf{k}'\beta'|T|\mathbf{p}m;\mathbf{k}\beta\rangle=\sum_{\rho,\rho'}(-1)^{s+m'}d_{\rho',-m'}^{(s)}(-\chi)\times d_{\rho m}^{(s)}(\chi)\langle -\mathbf{p}_{\text{c.m.}}\rho';\mathbf{k}_{\text{c.m.}}'\beta'|T|\mathbf{p}_{\text{c.m.}}\rho;\mathbf{k}_{\text{c.m.}}\beta\rangle, \quad \text{(A6)}$$

where ρ (ρ') is the helicity of the initial- (final) state target in the c.m. frame, and $\mathbf{p}_{\text{c.m.}}$, $\mathbf{k}_{\text{c.m.}}$, etc., denote the three-momentum of the corresponding particles in this frame.[14] χ is the Wigner rotation angle which specifies the Lorentz transformation from the Breit frame to the c.m. frame along the x axis; its magnitude is given by

$$\cos\chi=\frac{s+M^2}{s-M^2}\frac{|\mathbf{p}|}{E(\mathbf{p})}, \quad \sin\chi=\frac{2M|\mathbf{Q}|}{s-M^2}. \quad \text{(A7)}$$

This angle χ is exactly the same angle which appears in the crossing matrix from the s-channel c.m. helicity amplitude to the t-channel one.[15] If we define the amplitude $\tilde{T}$ by

$$\langle -\mathbf{p}_{\text{c.m.}}\rho';\mathbf{k}_{\text{c.m.}}'\beta'|T|\mathbf{p}_{\text{c.m.}}\rho;\mathbf{k}\beta_{\text{c.m.}}\rangle\equiv(\sin\tfrac{1}{2}\theta_s)^{|\lambda'-\lambda|}\times(\cos\tfrac{1}{2}\theta_s)^{|\lambda'+\lambda|}\tilde{T}_{\rho'\beta';\rho\beta}(s,t), \quad \text{(A8)}$$

where θ_s is the scattering angle in the c.m. frame, and

$$\lambda'=\rho'-\beta', \quad \lambda=\rho-\beta, \quad \text{(A9)}$$

this amplitude is kinematical singularity free in t.[16] θ_s is given by

$$\cos\tfrac{1}{2}\theta_s=2|\mathbf{Q}|E(\mathbf{p})/(s-M^2)=[(s-M^2)^2-st]^{1/2}/(s-M^2)$$

$$\sin\tfrac{1}{2}\theta_s=2(\sqrt{s})|\mathbf{p}|/(s-M^2)=(-st)^{1/2}/(s-M^2). \quad \text{(A10)}$$

Substituting (A8) into (A6), and using (A7) and (A10), we see that the factor

$$(-1)^{s+m'}d_{\rho'-m'}^{(s)}(-\chi)d_{\rho m}^{(s)}(\chi)(s^{-1/2}\sin\tfrac{1}{2}\theta_s)^{|\lambda'-\lambda|}\times(\cos\tfrac{1}{2}\theta_s)^{|\lambda'+\lambda|} \quad \text{(A11)}$$

is proportional to $|\mathbf{Q}|^{|m'-m|}$. This is the only kinematical singularity in s which can be involved in the T amplitude, as can be seen in (A5). Thus the amplitude $(\sqrt{s})^{|\lambda'-\lambda|}\tilde{T}_{\rho'\beta';\rho\beta}(s,t)$ is kinematical singularity free in s as well as t. Equation (A11) contains all kinematical singularities of the T amplitude.

The kinematical singularities of (A11) in t come from odd powers of $|\mathbf{p}|$ and $E(\mathbf{p})=(M^2-\frac{1}{4}t)^{1/2}$, after those singularities coming from odd powers of $|\mathbf{Q}|$ are removed. Equations (A7), (A9), and (A10) show that (A11) is an even (odd) function of $E(\mathbf{p})$ for integer spin targets, and is an odd (even) function for half-integer-spin targets when (A11) is an even (odd) function of $|\mathbf{p}|$. The change of $|\mathbf{p}|$ to $-|\mathbf{p}|$ in (A11) is equivalent to changing the signs of m and m' in (A11). Since $\langle -\mathbf{p}m';\mathbf{k}'\beta'|T|\mathbf{p}m;\mathbf{k}\beta\rangle$ changes to $\langle -\mathbf{p},-m';\mathbf{k}'\beta'|T|\mathbf{p}-m;\mathbf{k}\beta\rangle$, and $\epsilon_a(\mathbf{k}'\beta')$ $\epsilon_b(\mathbf{k}\beta)$ in (A4) changes to $\epsilon_{-a}(\mathbf{k}'\beta')$ $\epsilon_{-b}(\mathbf{k}\beta)$ when $|\mathbf{p}|$ is replaced by $-|\mathbf{p}|$, we can see from the definition (2.1) that

$$F_{JM}^{(uv)}(k,t) \text{ is an even (odd) function of } |\mathbf{p}| \text{ if } M \text{ is even (odd).} \quad \text{(A12)}$$

If we ignore the kinematical singularity $(M^2-\frac{1}{4}t)^{1/2}$ which exists only in the amplitude of a half-integer-spin target, we can conclude from (A12) and (A3) that $F_{JM}^{(uv)}(k,t)$ is kinematical singularity-free in s and t around $k=0$ after the factor

$$(|\mathbf{p}|\ |\mathbf{Q}|)^{eM}\propto(\sin\theta)^{eM},$$

is removed.

APPENDIX B: PROPERTIES OF THE MULTIPOLE MOMENTS

From the definition of the multipole moment (3.8), we have

$$\mathcal{T}_J{}^a=(2J+1)\sum_{m,m'}(-1)^{s-m'}\begin{pmatrix} s & J & s \\ -m' & a & m\end{pmatrix}\times\langle m'|j_a(0)\Lambda(\zeta)|m\rangle. \quad \text{(B1)}$$

Writing $\Lambda(\zeta)$ in the form

$$\Lambda(\zeta)=e^{-i\zeta K_z}, \quad \text{(B2)}$$

[14] M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) **7**, 404 (1959).
[15] T. L. Trueman and G. C. Wick, Ann. Phys. (N. Y.) **26**, 322 (1964).
[16] M. Gell-Mann, M. L. Goldberger, F. E. Low, E. Marx, and F. Zachariasen, Phys. Rev. **133**, B145 (1964).

where ζ is given by (3.6) and K_z denotes the generator of the boost along the z axis, we obtain

$$\mathcal{T}_J{}^a=(2J+1)\sum_{n=J}^{\infty}\frac{(-i\zeta)^n}{n!}\sum_{m,m'}(-1)^{s-m'}\times\begin{pmatrix} s & J & s \\ -m' & a & m\end{pmatrix}\langle m'|j_a(0)K_z{}^n|m\rangle. \quad \text{(B3)}$$

Since $\zeta\propto k/M$ for small values of k,

$$\mathcal{T}_J{}^a\propto(k/M)^J \quad \text{at} \quad k\approx 0. \quad \text{(B4)}$$

Space reflection shows that

$$\langle m'|j_a(0)K_z{}^n|m\rangle=\mp(-1)^n\langle m'|j_a(0)K_z{}^n|m\rangle,$$
upper sign for $a=(0)$, $(\pm)$ (B5)
lower sign for $a=0$.

After the transformation $Y=e^{-i\pi J_y}P$, where P is the space reflection and J_y is the generator of a rotation around the y axis, we obtain

$$\langle m'|j_a(0)e^{-i\zeta K_z}|m\rangle=(-1)^{m'-m+a}\times\langle -m'|j_{-a}(0)e^{-i\zeta K_z}|-m\rangle. \quad \text{(B6)}$$

As a result of this, we get

$$\mathcal{T}_J{}^a=(-1)^J\mathcal{T}_J{}^{-a}. \quad \text{(B7)}$$

We use the time-reversal result

$$\langle m'|j_{(0)}(0)e^{-i\zeta K_z}|m\rangle=-(-1)^{m'-m}\times\langle -m'|j_{(0)}(0)e^{+i\zeta K_z}|-m\rangle^* \quad \text{(B8)}$$

in order to relate $\mathcal{T}_J{}^{(0)}$ to $\mathcal{T}_J{}^0$. Since both $j_{(0)}(0)$ and K_z are invariant under an arbitrary space rotation around the z axis, $m'=m$. By a comparison between (B6) and (B8), we get the following relation:

$$\langle m|j_{(0)}(0)e^{-i\zeta K_z}|m\rangle=-\langle m|e^{-i\zeta K_z}j_{(0)}(0)|m\rangle. \quad \text{(B9)}$$

In terms of the multipole moment, this is equivalent to

$$\mathcal{T}_J{}^{(0)}=\frac{\sinh\zeta}{1+\cosh\zeta}\mathcal{T}_J{}^0. \quad \text{(B10)}$$

Since K_z commutes with $j_{(\pm)}(0)$, $\bar{\mathcal{T}}_J{}^{(\pm)}=\mathcal{T}_J{}^{(\pm)}$, whereas (B9) shows that

$$\bar{\mathcal{T}}_J{}^{(0)}=-\mathcal{T}_J{}^{(0)}, \quad \text{(B11)}$$

which also implies that $\mathcal{T}_J{}^0=\bar{\mathcal{T}}_J{}^0$.

In order to relate the multipole moment defined by (B1) to the conventional one, we calculate (B1) for spin $\frac{1}{2}$ by using the Pauli and Dirac form factors F_1 and F_2 defined by

$$\langle\mathbf{p}'|j_\mu(0)|\mathbf{p}\rangle=i\bar{u}(\mathbf{p}')[e\gamma_\mu F_1((\mathbf{p}'-\mathbf{p}'^2)-\mu_A F_2((\mathbf{p}'-\mathbf{p})^2)\sigma_{\mu\nu}(p'-p)_\nu]u(\mathbf{p}), \quad \text{(B12)}$$

where $u(\mathbf{p})$ is the usual Dirac spinor, μ_A is the anomalous magnetic moment, and $\sigma_{\mu\nu}=(1/2i)(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu)$. Substitution of (B12) into (B1) gives us

$$\mathcal{T}_0{}^0=\sqrt{2}(e-\tfrac{1}{6}\langle r_e{}^2\rangle k^2e)+O(k^4), \quad \text{(B13)}$$

and

$$\mathcal{T}_1{}^{(+)}=-(\sqrt{6})\mu k+O(k^3), \quad \text{(B14)}$$

where $\mu=e/2M+\mu_A$ is the total magnetic dipole moment. The mean-square radius of the charge $\langle r_e{}^2\rangle$ is the one defined by[17]

$$\langle r_e{}^2\rangle=-6\frac{dF_1}{dk^2}+\frac{3}{4M^2}+\frac{3\mu_A}{eM}. \quad \text{(B15)}$$

Equations (B13) and (B14) hold for the arbitrary spin case if $\sqrt{2}$ and $\sqrt{6}$ in these formulas are replaced by $(2s+1)^{1/2}$ and $[(s+1)(2s+1)/s]^{1/2}$, respectively.

[17] F. J. Ernst, R. G. Sachs, and K. C. Wali, Phys. Rev. 119, 1105 (1960).