PHYSICAL REVIEW VOLUME 184, NUMBER 5 25 AUGUST 1969

Off-Energy-Shell Partial-Wave Amplitudes

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(Received 25 April 1969)

We present dispersion relations which give the full off-energy-shell T-matrix elements $T_{ll'}(p,p';s)$ for *all* values of the parametric energy s in terms of bound-state form factors, a subtraction constant, and the half-off-shell T-matrix elements $T_{ll'}(p,s^{1/2};s)$ in the scattering ($s>0$) region. Study of the half-off-shell T matrix for $s>0$ shows that it can be written as the product of a *real* matrix $H_{ll'}(p,s)$ and the on-shell T matrix. We combine these results to obtain a representation of the full off-energy-shell T-matrix elements in terms of experimental on-shell T-matrix elements, the real half-off-shell factors $H_{ll'}(p,s)$, a subtraction constant and bound-state form factors. Our results are based *only* on assumptions of time-reversal invariance, off-energy-shell unitarity, analyticity, and asymptotic behavior. The results are independent of any specific dynamical assumptions. We conclude with a discussion of the special case of uncoupled partial waves and the advantages of a separable representation of the half-off-shell factors.

I. INTRODUCTION

THE full off-energy-shell partial-wave two-body T (transition) matrix elements $T_{ll'}(p,p';s)$ are necessary inputs to many multiparticle-scattering calculations. Omnès and Basdevant[1] have commented upon the shortcomings of the usual approximations to these T-matrix elements. We are therefore led to further consideration of the T-matrix elements $T_{ll'}(p,p';s)$ for elastic two-body scattering in a state of total angular momentum J, for the case where N partial waves are coupled. If the reduced mass of the two particles is μ, we use units in which $\hbar=2\mu=1$, so that $k^2=s$, the c.m. energy.

We shall make the following assumptions about the amplitudes $T_{ll'}(p,p';s)$:

(i) *Time-reversal invariance and unitarity.* We assume that the T matrix is time-reversal invariant and satisfies off-energy-shell two-body elastic unitarity. This enables us to write the expression

$$\mathrm{Im}T_{ll'}(p,p';s) = -\pi\rho(s)\sum_{j=1}^{N} T_{lj}(p,k;s)T_{jl'}^*(k,p';s) \quad (1)$$

for the discontinuity of the T-matrix element across the unitarity cut.

(ii) *Analyticity.* We assume that the only singularities of the T-matrix elements in the complex energy variable s are the bound state and resonance poles and the unitarity cut prescribed by Eq. (1).

(iii) *Asymptotic behavior.* We assume that the off-shell T-matrix element $T_{ll'}(p,p';s)$ approaches a constant as $s\to\pm\infty$. This is not a critical assumption. If $T_{ll'}(p,p';s)\to s^n$ as $s\to\pm\infty$, we need only make $(n+1)$ subtractions, instead of one subtraction, in the dispersion relations set forth below.

All these assumptions are true, for example, for T-matrix elements obtained from the Lippmann-Schwinger (LS) equation

$$T_{ll'}(p,p';s)=V_{ll'}(p,p') + \sum_{j=1}^{N}\int_0^\infty \frac{q^2dq\, V_{lj}(p,q)T_{lj'}(q,p';s)}{s-q^2+i\epsilon},$$

which satisfies the nonrelativistic form of two-particle elastic unitarity with $\rho(s)=s^{1/2}$. The assumptions are also true for T-matrix elements derived from the Blankenbecler-Sugar (BS) equation

$$T_{ll'}(p,p';s)=V_{ll'}(p,p') + \sum_{j=1}^{N}\int_0^\infty \frac{q^2dq}{(q^2+1)^{1/2}}\frac{V_{lj}(p,q)T_{jl'}(q,p';s)}{s-q^2+i\epsilon},$$

which satisfies the relativistic form of two-particle elastic unitarity with $\rho(s)=[s/(s+1)]^{1/2}$. To be specific, we shall proceed under the assumption that $\rho(s)$ has either the LS or the BS form.

II. DISPERSION RELATIONS

We have assumed that the T-matrix element $T_{ll'}(p,p';s)$ approaches a constant as $s\to\pm\infty$. The Sugawara-Kanazawa theorem,[2] coupled with our analyticity assumption, then tells us that $T_{ll'}(p,p';s)$ approaches a constant as $|s|\to\infty$ in *any* direction in the complex s plane. Let us consider a state of total angular momentum J in which there are M bound states at energies $s=-s_m$. Then, using the unitarity relation (1), we may write the following once-subtracted dispersion relation for the full off-energy-shell T-matrix element:

$$T_{ll'}(p,p';s) = T_{ll'}(p,p';s_s)+\sum_{m=1}^{M}\frac{a_{ml}(p)a_{ml'}(p')}{s+s_m}\left(\frac{s-s_s}{-s_m-s_s}\right) - (s-s_s)\int_0^\infty d\xi\frac{\rho(\xi)}{(\xi-s)(\xi-s_s)}\sum_{j=1}^{N}T_{lj}(p,\xi^{1/2};\xi) \times T_{jl'}^*(\xi^{1/2},p';\xi), \quad (2)$$

[1] R. Omnès and J. L. Basdevant, Phys. Rev. Letters 14, 775 (1966).

[2] M. Sugawara and A. Kanazawa, Phys. Rev. 133, 1895 (1961); G. Barton, *Introduction to Dispersion Techniques in Field Theory* (W. A. Benjamin, Inc., New York, 1965).

where the subtraction has been made at $s=s_s$ on the physical sheet of the complex-energy Riemann surface. We have used the fact that the residue of the mth bound-state pole can be factored as $a_{ml}(p)a_{ml'}(p')$, where the bound-state form factor satisfies $a_{ml}(p) = -(s_m+p^2)\psi_{ml}(p)$ and $\psi_{ml}(p)$ is the l component of the bound-state wave function in momentum space.

If we make the subtraction at $s_s=0$, we find

$$T_{ll'}(p,p';s) = T_{ll'}(p,p';0)+\sum_{m=1}^{M}\frac{a_{ml}(p)a_{ml'}(p')}{s+s_m}\left(\frac{-s}{s_m}\right) - s\int_0^\infty d\xi\frac{\rho(\xi)}{\xi(\xi-s)}\sum_{j=1}^{N}T_{lj}(p,\xi^{1/2};\xi) \times T_{jl'}^*(\xi^{1/2},p';\xi). \quad (3)$$

With the forms we are considering for $\rho(\xi)$, the off-shell unitarity relation (1) tells us that the subtraction constant in Eq. (3) is real provided the half-off-shell amplitude $T_{ll'}(p,k;s)=T_{ll'}(p,s^{1/2};s)$ is finite for all p when $s=0$.

If, however, we assume that $\mathrm{Im}T_{ll'}(p,p';s)\to 0$ as $s\to\infty$ along the real axis, we can make the subtraction in Eq. (2) at $s_s=\infty$ and obtain

$$T_{ll'}(p,p';s) = T_{ll'}(p,p';\infty)+\sum_{m=1}^{M}\frac{a_{ml}(p)a_{ml'}(p')}{s+s_m}-\int_0^\infty d\xi\frac{\rho(\xi)}{\xi-s} \times\sum_{j=1}^{N}T_{lj}(p,\xi^{1/2};\xi)T_{jl'}^*(\xi^{1/2},p';\xi), \quad (4)$$

where $T_{ll'}(p,p';\infty)$ is real by assumption.

Now, let us consider the possibility of making the subtraction at the position of a bound state or resonance pole at $s=s_R$. Using the fact that the residue at this pole will factor in the form $g_{Rl}(p)\, g_{Rl'}(p')$, where $g_{Rl}(p)$ denotes the bound-state or resonance form factor, the resulting once-subtracted dispersion relation is

$$T_{ll'}(p,p';s) = \pm g_{Rl}(p)g_{Rl'}(p')/(s-s_R)+\sum_{m=1}^{M}\frac{a_{ml}(p)a_{ml'}(p')}{s+s_m} \times\left(\frac{s-s_R}{-s_m-s_R}\right)-(s-s_R)\int_0^\infty d\xi\frac{\rho(\xi)}{(\xi-s)(\xi-s_R)} \times\sum_{j=1}^{N}T_{lj}(p,\xi^{1/2};\xi)T_{jl'}^*(\xi^{1/2},p';\xi). \quad (5)$$

The plus sign in the first term holds if s_R is chosen as a bound-state pole, in which case the sum in the second term is over the remaining bound-state poles. The minus sign in the first term is appropriate if s_R is chosen as a resonance pole. The sign difference between the bound-state and resonance cases arises because the bound-state and resonance poles are, respectively, *inside* and *outside* the large circular contour (indented to exclude the right-hand unitarity cut) on the physical sheet of the complex-energy Riemann surface which was used to derive the dispersion relations (2) and (5). This is because the resonance poles lie on the second sheet of the complex-energy Riemann surface, just below the physical region which is reached by approaching the real s axis from above in the first, or physical, sheet.

We wish to stress that these dispersion relations provide a representation of $T_{ll'}(p,p';s)$ that is *exact* for *all* values of the energy s on the physical sheet, including the bound-state region (s real and negative) and the physical scattering region ($s=k^2+i\epsilon$, where k^2 and ϵ are real and positive). Thus, if these representations are used in three-body scattering calculations, the fact that values of the parametric energy s from some real, positive value to $-\infty$ are required will cause no difficulty.[1]

Furthermore, these dispersion representations show that the full off-energy-shell two-body elastic T-matrix element $T_{ll'}(p,p;s)$ is determined, to within a subtraction constant, at *all* energies by the half-off-shell T-matrix elements in the scattering region (s real and positive) and the bound-state form factors. This conclusion is based *only* on the assumptions we made about unitarity, analyticity, time reversal, and asymptotic behavior, and is independent of any specific dynamical assumptions.

If we assume that we are operating within the dynamical framework of the LS or BS equations, we see that

$$T_{ll'}(p,p';s)\to V_{ll'}(p,p'),$$

which is constant as a function of s, when $s\to\infty$. In other words, the Born approximation becomes exact as we go to very high energies. Furthermore, when we deal with a real potential, we have $\mathrm{Im}T_{ll'}(p,p';s)\to 0$ as $s\to\infty$, $T_{ll'}(p,p';\infty)=V_{ll'}(p,p')$, and we may use Eq. (4). We have therefore recovered the result that Noyes[3] derived from the Low equation, which may be written symbolically in the form $T=V+VGV$.

Finally, we believe it is possible to obtain similar dispersion relations for inelastic T-matrix elements, using inelastic unitarity extended off the energy shell.

III. HALF-OFF-SHELL AMPLITUDES

Since we have shown that the full-off-shell T-matrix elements are largely determined by the half-off-shell T-matrix elements in the scattering region, we shall digress to discuss the half-off-shell amplitudes.

We begin with the uncoupled wave case ($N=1$) and reproduce a well-known result. We assume that the

[3] H. P. Noyes, paper presented at the International Colloquium on Polarized Targets and Beams, C.E.N. Saclay, France, December, 1966 (unpublished).

on-shell transition amplitude

$$T_l(k,k;s)\equiv T_l(s^{1/2},s^{1/2};s)\equiv T_l(s) =(-2/\pi s^{1/2})e^{i\delta_l(s)}\sin\delta_l(s),$$

has no zeros for s real and positive. That is, we assume that the phase shift $\delta_l(s)$ does not equal $n\pi$ for s real and positive. Then we define the half-off-shell function $H_l(p,s)$ by

$$H_l(p,s)\equiv T_l(p,k;s)/T_l(s), \quad (6)$$

or

$$T_l(p,k;s)\equiv H_l(p,s)T_l(s). \quad (7)$$

Now, the off-shell unitarity relation (1) for the half-off-shell amplitude $T_l(p,k;s)$ reads

$$\mathrm{Im}T_l(p,k;s)=-\pi\rho(s)T_l(p,k;s)T_l^*(s),$$

or, using relation (7),

$$\mathrm{Im}H_l(p,s)T_l(s)=-\pi\rho(s)H_l(p,s)|T_l(s)|^2. \quad (8)$$

The on-shell unitarity relation satisfied by the on-shell amplitude is a special case of relation (1) and can be written

$$\mathrm{Im}T_l(s)=-\pi\rho(s)|T_l(s)|^2. \quad (9)$$

If we insert Eq. (9) into Eq. (8), we find

$$\mathrm{Im}H_l(p,s)T_l(s)=H_l(p,s)\ \mathrm{Im}T_l(s),$$

and this relation can only be satisfied if $H_l(p,s)$ is a *real* function. We have thus reproduced the familiar result, attributed to Sobel,[4] that, based on time-reversal invariance and unitarity alone, the half-off-shell transition amplitude can be written as a real function times the on-shell amplitude. That is,

$$T_l(p,k;s)=-H_l(p,s)(2/\pi s^{1/2})e^{i\delta_l(s)}\sin\delta_l(s),$$

and $T_l(p,k;s)$ is completely determined by the real function $H_l(p,s)$ and the experimentally measurable scattering phase shifts $\delta_l(s)$.

We now consider the uncoupled-wave case where the phase shift $\delta_l(s)$, and thus the transition amplitude $T_l(s)$, has a zero at $s=s_0$. Then, for $s\approx s_0$, $T_l(s)$ is given by

$$T_l(s)\approx T_l'(s_0)(s-s_0)\approx(-2\pi/s_0^{1/2})\delta_l'(s_0)(s-s_0), \quad (10)$$

where $T_l'(s_0)$ is the first derivative of the transition amplitude and $\delta_l'(s_0)$ is the first derivative of the phase shift at $s=s_0$. If we now insert Eq. (10) into Eq. (6), we see that the half-off-shell functoin $H_l(p,s)$ has poles at the values of s corresponding to the zeros of $\delta_l(s)$. To handle this conveniently in a case where we have N_z zeros of the phase shift $\delta_l(s)$ [and therefore in the transition amplitude $T_l(s)$] at the energies s_i, we simply write

$$H_l(p,s)=h_l(p,s)\prod_{i=1}^{N_z}\frac{p^2-s_i}{s-s_i},$$

where, according to Eq. (6), the real function $h_l(p,s)$ must be such that $h_l(s^{1/2},s)=1$. Then, the half-off-shell amplitude can be written in the form (7), which, in the vicinity of the mth zero of the phase shift ($s\approx s_m$), becomes

$$T_l(p,k;s)\approx h_l(p,s)\prod_{i=1,i\neq m}^{N_z}\frac{p^2-s_i}{s-s_i}[(-2\pi/s_m^{1/2})\delta_l'(s_m)].$$

We see that the half-off-shell amplitude can again be written as a real function times a quantity depending only on the on-shell scattering data.

Now, consider the coupled-wave case, where $N>1$. We assume that the on-shell T matrix $T_{ll'}(s)$ has an inverse for all values of s real and positive. This is true unless there exist real, positive values of s such that $\det T_{ll'}(s)=0$. For example, in the case of triplet nucleon-nucleon scattering, if the T matrix is parametrized in terms of the Stapp nuclear-bar or Blatt-Biedenharn phase parameters, the condition $\det T_{ll'}(s)=0$ requires all three of the phase parameters (two phase shifts and a mixing parameter) to be zero at the same energy. Since we regard this as an unlikely occurence, we consider in this paper only the case where $T_{ll'}(s)$ has an inverse. We define the half-off-shell matrix $H_{ll'}(p,s)$ by

$$H_{ll'}(p,s)\equiv\sum_{j=1}^{N}T_{lj}(p,k;s)T^{-1}{}_{jl'}(s),$$

where, by definition, $H_{ll'}(k,s)\equiv H_{ll'}(s^{1/2},s)=I$, where I is the unit matrix, and we thus have

$$T_{ll'}(p,k;s)=\sum_{j=1}^{N}H_{lj}(p,s)T_{jl'}(s). \quad (11)$$

If we write the off-shell unitarity relation (1) for the half-off-shell T-matrix element $T_{ll'}(p,k;s)$, we find

$$\mathrm{Im}T_{ll'}(p,k;s)=-\pi\rho(s)\sum_{j=1}^{N}T_{lj}(p,k;s)T_{jl'}^*(s),$$

or, using Eq. (11),

$$\mathrm{Im}\sum_{\alpha=1}^{N}H_{l\alpha}(p,s)T_{\alpha l'}(s)=-\pi\rho(s)\sum_{\alpha=1}^{N}\sum_{\beta=1}^{N}H_{l\alpha}(p,s)\times T_{\alpha\beta}(s)T_{\beta l'}^*(s). \quad (12)$$

Now, the on-shell T-matrix element $T_{ll'}(s)$ satisfies a special case of the unitarity relation (1) which can be written

$$\mathrm{Im}T_{ll'}(s)=-\pi\rho(s)\sum_{j=1}^{N}T_{lj}(s)T_{jl'}^*(s). \quad (13)$$

If we put Eq. (13) into Eq. (12), we find

$$\mathrm{Im}\sum_{\alpha=1}^{N}H_{l\alpha}(p,s)T_{\alpha l'}(s)=\sum_{\alpha=1}^{N}H_{l\alpha}(p,s)\ \mathrm{Im}T_{\alpha l'}(s),$$

which can only be satisfied if the $H_{l\alpha}(p,s)$ are real. This shows that $H_{ll'}(p,s)$ is a real matrix. Thus, based

[4] M. I. Sobel, Phys. Rev. **156**, 1553 (1967).

only on the assumptions of time-reversal invariance and off-energy-shell unitarity, the half-off-energy-shell T matrix can be written as the product of a *real* matrix $H_{ll'}(p,s)$ and the experimentally determined on-shell T matrix. Therefore, the off-energy-shell behavior of the half-off-shell T matrix is completely contained in the real matrix elements $H_{ll'}(p,s)$.

Based on the same assumptions of time-reversal invariance and unitarity, Sobel[4] has presented a different parametrization of the half-off-shell T matrix in nucleon-nucleon scattering. However, our approach is more useful for our present purposes and it seems that it is more easily extended to the case of $N>2$ coupled waves than Sobel's approach.

Our relation (11) for the half-off-shell T-matrix element $T_{ll'}(p,k;s)$ lends itself readily to an approximation when p is not far off the energy shell ($p^2\approx k^2=s$). In this case, since $H_{ll'}(k,s)\equiv I$, we may write, for $p^2\approx s$,

$$T_{ll'}(p,k;s)\approx H_{ll}(p,s)T_{ll'}(s).$$

Kowalski[5] and Amadzadeh and Chung[6] have shown that, by performing a Fredholm reduction on the LS or the BS equation in uncoupled waves, the half-off-shell function $H_l(p,s)$ can be obtained as the solution of the nonsingular integral equation

$$H_l(p,s)=V_l(p,k)/V_l(k,k)+\int_0^\infty dq\,\frac{w(q)}{s-q^2}\times\left[V_l(p,q)-\frac{V_l(p,k)V_l(k,q)}{V_l(k,k)}\right]H_l(q,k).\quad(14)$$

Here, $w(q)=q^2$ if the equation is derived from the LS equation, $w(q)=q^2/[(q^2+1)^{1/2}]$ if the equation is derived from the BS equation, and $k^2=s$. To derive Eq. (14), we must assume that both $T_l(s)$ and $V_l(k,k)$ are nonzero. Thus, within the dynamical framework provided by the LS equation or the BS equation, Eq. (14) provides a means of calculating the real half-off-shell function $H_l(p,s)$. If we assume the existence of $T_{ll'}^{-1}(s)$ *and* $V^{-1}{}_{ll'}(k,k)$, we can derive the coupled-wave analog of Eq. (14) by performing a Fredholm reduction on the coupled-wave LS or BS equations. The result is a set of coupled nonsingular integral equations for the matrix elements of the real half-off-shell matrix $H_{ll'}(p,s)$, which can be written

$$H_{ll'}(p,s)=\sum_{\alpha=1}^{N}V_{l\alpha}(p,k)V^{-1}{}_{\alpha l'}(k,k)+\sum_{j=1}^{N}\int_0^\infty dq\frac{w(q)}{s-q^2}[V_{lj}(p,q)-\sum_{\alpha=1}^{N}\sum_{\beta=1}^{N}V_{l\alpha}(p,k)V^{-1}{}_{\alpha\beta}(k,k)V_{\beta j}(k,q)]H_{jl'}(q,s).$$

[5] K. L. Kowalski, Phys. Rev. Letters **15**, 798 (1965).
[6] A. Amadzadeh and V. Chung, Phys. Rev. **161**, 1602 (1967).

IV. FULL-OFF-SHELL AMPLITUDES FROM ON-SHELL DATA AND HALF-OFF-SHELL FACTORS

By the methods of Sec. III, we can show that the half-off-shell amplitude $T_{ll'}(k,p;s)$ can be written in the form

$$T_{ll'}(k,p;s)=\sum_{i=1}^{N}T_{li}(s)G_{il'}(p,s),$$

where the matrix elements $G_{il'}(p,s)$ are real, provided that $T^{-1}{}_{ll'}(s)$ exists. Then, time-reversal invariance demands that

$$T_{ll'}(k,p;s)=T_{l'l}(p,k;s)$$

and

$$T_{ll'}(s)=T_{l'l}(s),$$

whence we can show that

$$G_{ll'}(p,s)=H_{l'l}(p,s)$$

or

$$G_{ll'}(p,s)=H^T{}_{ll'}(p,s),$$

where $H_{ll'}(p,s)$ is the real matrix defined by Eq. (6), and H^T is the transpose of H. Thus we have

$$T_{ll'}{}^*(k,p;s)=\sum_{i=1}^{N}T_{li}{}^*(s)H^T{}_{il'}(p,s).\quad(15)$$

We may now insert Eqs. (11) and (15) into the dispersion relations (2)–(5) set forth in Sec. II. To avoid repetition, we shall display only the result for the case of subtraction at a bound-state or resonance pole. Beginning with Eq. (5), we find

$$T_{ll'}(p,p';s)=\pm\frac{g_{Rl}(p)g_{Rl'}(p')}{s-s_R}+\sum_{m=1}^{M}\frac{a_{ml}(p)a^m{}_{l'}(p')}{s+s_m}\left(\frac{s-s_R}{-s_m-s_R}\right)-(s-s_R)\int_0^\infty\frac{d\xi\rho(\xi)}{(\xi-s)(\xi-s_R)}\sum_{\alpha=1}^{N}\sum_{\beta=1}^{N}\sum_{\gamma=1}^{N}H_{l\alpha}(p,\xi)\times T_{\alpha\beta}(\xi)T_{\beta\gamma}{}^*(\xi)H^T{}_{\gamma l'}(p',\xi),\quad(16)$$

where the minus sign in the first term occurs if we subtract at a resonance pole. If we subtract at a bound-state pole, the first term in Eq. (16) has a plus sign and the sum in the second term is over the *remaining* bound-state poles.

We may now insert a representation of $T_l(s)$ in terms of real phase parameters into Eq. (16). For example, if we are considering the coupled waves in nucleon-nucleon scattering, where $N=2$, we can write $T_l(s)$ in terms of the Stapp nuclear-bar or the Blatt-Biedenharn phase shifts and mixing parameters and put the resulting expression into Eq. (16).

We have shown in Eq. (16) that, if we write a dispersion relation subtracted at a bound-state or resonance pole, the full off-energy T-matrix element for *all* energies can be expressed in terms of the bound-state and resonance form factors, the experimentally measured phase parameters and the real half-off-shell factors $H_{ll'}(p,s)$. We repeat that this result is based *only* on the assumptions of time-reversal invariance, unitarity, analyticity, and asymptotic behavior (which were announced in Sec. I), and is independent of any specific dynamical assumptions.

Furthermore, if we have a separable representation or approximation for $H_{ll'}(p,s)$ of the form

$$H_{ll'}(p,s)=\alpha_l(p)\beta_{l'}(s),$$

where $\alpha_l(p)$ and $\beta_{l'}(s)$ are real functions, Eq. (16) shows that we then obtain a separable representation or approximation to the full off-energy-shell T-matrix element.

V. UNCOUPLED WAVES

We now specialize to the case $N=1$, where we deal with uncoupled waves and the relations we have presented take on an appealing simplicity. This case serves as an example and a "laboratory" of considerable practical importance, because off-shell amplitudes of the form $T_l(p,p';s)$ are a basic ingredient in the optical model of pion-nucleus scattering as well as being important in the three-nucleon problem.

For uncoupled waves, Eq. (16) becomes

$$T_l(p,p';s)=\pm g_{Rl}(p)g_{Rl}(p')/(s-s_R) + \sum_{m=1}^{M}\frac{a_{ml}(p)a_{ml}(p')}{s+s_m}\left(\frac{s-s_R}{-s_m-s_R}\right)-(s-s_R)\frac{4}{\pi^2}\int_0^\infty d\xi\frac{\rho(\xi)}{\xi} \times\frac{H_l(p,\xi)\sin^2\delta_l(\xi)H_l(p,'\xi)}{(\xi-s)(\xi-s_R)}, \quad (17)$$

where $\delta_l(s)$ is the phase shift. We see again that the bound-state and resonance form factors and the real function $H_l(p,s)$ contains all the information about the off-energy-shell behavior of $T_l(p,p';s)$.

At this stage, we wish to mention an approximation, which we call the single-wave-dominance model. This model may be useful when we can assume that a single partial wave dominates the scattering process, for example, in π-N scattering, where the p wave dominates scattering. We know that the total elastic cross section $\sigma_E(s)$ is the sum of the partial cross sections $\sigma_l(s)$,

$$\sigma_E(s)=\sum_{l=1}^{\infty}\sigma_l(s),$$

where

$$\sigma_l(s)=(2l+1)(4\pi/s)\sin^2\delta_l(s).$$

If we assume that the L wave dominates scattering, we can make the approximation

$$\sigma_E(s)\approx\sigma_L(s)$$

or

$$\sin^2\delta_L(s)=[s/4\pi(2L+1)]\sigma_E(s).$$

If we insert this equation into Eq. (17), we have the single-wave-dominance model of the off-energy-shell scattering amplitude. This is more general than the usual single-pole dominance model, wherein we assume that a single bound-state or resonance pole in a single partial wave dominates the scattering process. The single-wave-dominance model has two other attractive features. First, it may be used for processes for which the experimental situation is insufficiently advanced to permit a phase-shift analysis. Second, it overestimates the contribution of the dominant partial wave in such a way as to approximate the contributions of the other partial waves to the reaction.

In an earlier paper,[7] we denoted the half-off-shell function $H_l(p,s)$ by $f_l(p,k)$, and compared the half-off-shell functions resulting from several potential models of the nucleon-nucleon interaction. In particular, we compared the half-off-shell functions resulting from some separable-potential models of the nucleon-ncuelon interaction[8] with the half-off-shell functions generated by local potential models. In the remainder of this section, we shall discuss the half-off-shell functions $H_l(p,s)$. We pay particular attention to a separable representation or approximations for the half-off-shell functions, because Eq. (17) shows that a separable representation for $H_l(p,s)$ of the form $H_l(p,s)=\alpha_l(p)\beta_l(s)$ yields a separable representation of the full-off-energy-shell amplitude *which is valid at all energies.*

It is clear from the definition of $H_l(p,s)$ that $H_l(k,s)\equiv 1$, where $k=s^{1/2}$. Furthermore, it is well known that $H_l(p,s)$ behaves as p^l as $p\to 0$ and as k^{-l} as $k\to 0$, and that $H_l(p,s)$ has cuts for $p^2<0$. Also, $H_l(p,s)$ has poles at values of s where the transition amplitude is zero, i.e., where the phase shift $\delta_l(s)=\pm n\pi$.

If there are N_z zeros of the transition amplitude in the partial wave l, parametrizations of the form

$$H_l(p,s)=\prod_{i=1}^{N_z}\frac{p^2-s_i}{s-s_i}\sum_{j=1}^{N_T}\left\{\frac{G_j}{p^{l+2+2N_z}}Q_l\left(1+\frac{\mu_j^2}{2p^2}\right)\Big/\left[\frac{G_j}{s^{\frac{1}{2}l+1+N_z}}Q_l\left(1+\frac{\mu_j^2}{2s}\right)\right]\right\}$$

meet the above requirements and afford the additional advantage of generating a separable off-energy-shell transition amplitude from Eq. (17). Here, Q_l is the Legendre function of the second kind, s_i is the position of the ith real, positive zero of the transition amplitude, and N_T is the number of cuts in the parametrization form.

Separable potentials[8] provide another means of obtaining a separable off-shell factor $H_l(p,s)$. For partial waves in which the experimental data can be adequately represented by a single-term separable potential of the form

$$V_l(p,p')=\lambda g_l(p)g_l(p'),$$

[7] T. R. Mongan, Phys. Rev. **180**, 1514 (1969).
[8] T. R. Mongan, Phys. Rev. **175**, 1260 (1968); **178**, 1597 (1969).

the off-shell factor $H_l(p,s)$ takes the form

$$H_l(p,s)=g_l(p)/g_l(s^{1/2}). \quad (18)$$

In the case where a two-term separable potential is needed to represent the experimental situation in a given partial wave, the half-off-shell transition amplitude has the form[8]

$$T_l(p,k;s)=N_l(p,k;s)/D_l(s).$$

Then, the half-off-shell factor $H_l(p,s)$ has the form

$$H_l(p,s)=N_l(p,k;s)/N_l(k,k;s),$$

which can be written

$$H_l(p,s)=\alpha_1(p)\beta_1(s)-\alpha_2(p)\beta_2(s).$$

A separable form for $H_l(p,s)$ of this type leads to a separable representation for $T_l(p,p';s)$ from Eq. (17), but the number of separable terms increases over the number of terms produced by a representation of the form (18).

If we assume that the off-shell behavior of a given transition amplitude is identical to the off-shell behavior arising from scattering mediated by a hard-core potential with core radius a,[9] the off-shell factor will take the form

$$H_l(p,s)=j_l(pa)/j_l(s^{1/2}a),$$

where j_l is the spherical Bessel function. This representation of $H_l(p,s)$ is separable in p and s and may serve as a useful starting point for approximations to the off-shell amplitudes based on Eq. (17).

If we deal with amplitudes arising from a Yukawa potential, the method of Choudhury[10] allows us to make a separable approximation to the Yukawa potential which results in a separable approximation to the off-shell factor of the form (18).

Finally, in the $^1S_0$ partial wave of nucleon-nucleon scattering, Tabakin[11] has presented a single-term separable potential which fits the $^1S_0$ phase-shift data. As we have pointed out in an earlier paper,[7] the value of using this potential to generate transition amplitudes for scattering calculations is uncertain. However, in the same paper, we point out that Tabakin's potential leads to the following representation for the half-off-shell factor:

$$H_l(p,s)=\left\{\left(\frac{k_c^2-p^2}{p^4+a^4}\right)\left(\frac{p^2+d^2}{p^2+b^2}\right)\right\}\Big/\left\{\left(\frac{k_c^2-s}{s^2+a^4}\right)\left(\frac{s+d^2}{s+b^2}\right)\right\},$$

where $k_c=1.7$ F^{-1}, $a=4.05$ F^{-1}, $b=1.08548$ F^{-1}, and $d=1.683$ F^{-1}. Then, by comparing with the half-off-shell factors generated by other potentials fitted to the $^1S_0$ partial wave, we have shown that Tabakin's form provides a separable representation of the half-off-shell factor which is in qualitative agreement with the half-off-shell factors produced by other potential models. This indicates that Tabakin's work does at least provide a parametrization of the $^1S_0$ half-off-shell factor which is well suited for insertion into the dispersion relation (17).

Again, the advantage of a separable representation of the half-off-shell factor $H_l(p,s)$ is that it provides a separable representation of $T_l(p,p';s)$, by way of Eq. (17), that is equally valid for *all* values of the parametric energy s. This is especially important for three-body scattering calculations, where one must integrate over values of the two-body parametric energy s extending down to $-\infty$.

VI. CONCLUSION

In this paper, we have *not* provided a theory which enables us to calculate off-energy-shell transition-matrix elements. We have simply set forth a representation which allows the maximum incorporation of experimental two-body scattering data into calculations of multiparticle processes. Because we have *not* based our development on dynamical equations, our results are valid independent of any particular dynamical assumptions.

[9] O. Brander, Arkiv Fysik **24**, 439 (1963).
[10] M. H. Choudhury, Nuovo Cimento **57**, 601 (1968).
[11] F. Tabakin, Phys. Rev. **164**, 1208 (1968).