Unequal-Mass Bootstrap Using Finite-Energy Sum Rules*

MATTHEW A. JACOBS AND MARK H. VAUGHN
University of California, San Diego, La Jolla, California 92037
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Mandelstam's $N\bar{N}$ bootstrap of the ρ, A_1, and π trajectories, employing finite-energy sum rules, is generalized to unequal-mass kinematics. Various problems associated with the unequal-mass kinematics are discussed: the explicit form of the crossing matrix, the unequal-mass invariant amplitudes, the mass dependence of the Regge residues, and the kinematic constraints. A bootstrap is described for slightly unequal masses to obtain the stability of the bootstrap against broken-mass asymmetry. It is found to be comparatively stable against mass splittings.

I. INTRODUCTION

THERE have been a number of bootstrap models introduced in elementary-particle physics which have predicted the masses and coupling constants of sets of particles. In general, these were reasonably successful in their mass predictions in single-channel calculations or for small groups of particles. Except for the technique of considering large numbers of particles as small numbers of multiplets of degenerate-mass particles with their coupling constants given group-theoretically, the application of these bootstraps to large sets of particles has not been too successful.

Following the work of de Alfaro *et al.*[1] on superconvergence relations, several people [2–4] derived the finite-energy sum rules (FESR) for scattering amplitudes. Mandelstam,[5] Gross,[6] Freund,[7] Schmid,[8] Igi,[9] and Desai *et al.*[10] have used the FESR's as a method for obtaining bootstrap equations for Regge parameters in various reactions. All of these models have considered a few particles or a few degenerate multiplets of particles and have reasonable results (note, however, some of the considerations of Desai *et al.*). Since the method is easily generalized, in principle, to the bootstrap of many independent particles, it is interesting to consider whether in this case its predictions are still reasonable. We have chosen to generalize Mandelstam's bootstrap. He considered the "elastic" scattering of degenerate nucleon-antinucleon octets via three degenerate meson nonets (ρ, A_1, and π). We have generalized this to unequal-mass nucleon and meson multiplets (although we require the coupling constants to retain their degenerate values) as a test of the further applicability of the FESR bootstraps and because some of the details of this implementation are particularly interesting. One of these is the close relationship among the equal-mass and unequal-mass constraints and conspiracies and has been treated elsewhere by Stack[11] and by us.[12]

First we present, in Sec. II, a review of Mandelstam's method of applying the FESR's to obtain the equal-mass $N\bar{N}$ bootstrap. In Sec. III, we find the explicit form of the unequal-mass $N\bar{N}\to N\bar{N}$ crossing matrix between regularized helicity amplitudes. We use as a starting point the general prescriptions of many authors[13,14] for the helicity crossing matrix. Section IV contains a discussion of the contributions in the t channel to the FESR's; in the small-t region, over which the integral of the FESR's is evaluated, the contributions are taken to be resonances (or bound states). The evaluation of these contributions is facilitated by the introduction of the unequal-mass invariant amplitudes.[15] s-channel Regge contributions are discussed in Sec. V. Kinematic constraints must be imposed on the s-channel amplitudes because the FESR's are evaluated at $s=0$. It is not necessary to impose the constraints on the t-channel amplitudes, because the constraints are automatically satisfied when t-channel amplitudes are crossed into the s-channel by the crossing matrix. Section VI contains the group theory used. The results of the unequal-mass bootstrap are presented in Sec. VII.

II. EQUAL-MASS FESR BOOTSTRAP

Superconvergence relations[1] for amplitudes with Regge asymptotic contributions subtracted have been used by various authors[2–4] to derive FESR's:

$$\frac{1}{N^{n+1}}\int^{N} dt\, t^n\, \mathrm{Im}\bar{f}_{\lambda\mu}(s,t)=\sum_r \frac{\gamma_r(s)N^{\alpha_r(s)-\lambda_m}}{\alpha_r(s)-\lambda_m+n+1}, \quad (2.1)$$

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[9] K. Igi, Phys. Rev. Letters 21, 184 (1968).
[10] B. R. Desai, P. E. Kaus, and Y. Shan, Phys. Rev. 179, 1595 (1969).
[11] J. D. Stack, Phys. Rev. 173, 1644 (1968).
[12] M. A. Jacobs and M. H. Vaughn, University of California at San Diego Report No. UCSD-10P10-51 (unpublished).
[13] T. L. Trueman and G. C. Wick, Ann. Phys. (N. Y.) 26, 322 (1964); I. J. Muzinich, J. Math. Phys. 5, 1486 (1964).
[14] G. Cohen-Tannoudji, A. Morel, and H. Navelet, Ann. Phys. (N. Y.) 46, 239 (1968).
[15] The invariant amplitudes for unequal-mass $N\bar{N}$ scattering have been considered recently by B. H. Kellett [Nuovo Cimento 56A, 1003 (1968)], whose results agree with ours.

where $\bar{f}_{\lambda\mu}$ is the reduced amplitude, $\lambda_m = \max(|\lambda|, |\mu|)$, and $\gamma_r(s)$ is related to the Regge residue.

Equation (2.1) is strictly true only if the sum on the right-hand side is interpreted as a generalized sum over the background integral and all cuts and poles to the right of the background contour. If only the highest Regge poles are kept in the sum, Eq. (2.1) is still approximately true only for large N and only if cuts can be neglected. It has been hypothesized by Dolen, Horn, and Schmid[4,16] that the dominant Regge poles at high energy represent an average of the amplitude at all energies or, equivalently, that Eq. (2.1), with the sum over high-lying trajectories only, is approximately true to low values of the cutoff for a smoothed-out function $\mathrm{Im}\bar{f}_{\lambda\mu}$. Since any smoothing out will affect the higher t-moments more, it is expected that the lower-moment equations will retain this approximate validity to smaller N.

If we assume, for low energies, that the amplitude is given by the sum of the contributions of direct- (t-) channel resonances and that the resonances lie on Regge trajectories, and if we choose a reaction for which the trajectories summed over in Eq. (2.1) (s and u channels) are the same as the resonance trajectories, then Eq. (2.1) can be used to obtain consistency conditions on the Regge parameters. Mandelstam[5] has used the FESR's at $s=0$ with low cutoff in an equal-mass $N\bar{N} \to N\bar{N}$ bootstrap calculation. The $N\bar{N} \to N\bar{N}$ reaction is not strictly crossing-identical, since the u channel ($NN \to NN$) is not the same as the s and t channels. However, since there is no strong binding in the u channel, its contributions to the FESR's may be ignored.

There are five independent amplitudes for the equal-mass $N\bar{N}$ scattering process. At $s=0$ and for equal masses, the half-angle factors reduce the high-t behavior; hence high spin-flip amplitudes will more readily satisfy superconvergence relations and we might expect the FESR's derived from them to retain their validity to lower cutoffs. Since the lower-moment sum rules are expected to be better approximations, Mandelstam uses the zeroth moment of the higher spin-flip amplitudes to obtain his bootstrap. For unequal-mass $N\bar{N}$ scattering (where there are eight independent amplitudes), the cosine of the scattering angle remains finite as $t \to \infty$ ($z_s = 1$). Nevertheless, it has been shown by Jacobs and Vaughn[17] that the effect of daughter and conspirator trajectories is to restore the expected t^α dependence at high t. Since the FESR's are to be evaluated at $s=0$, s-channel conspiracies are important. There is an equal-mass conspiracy, which Mandelstam uses, in which the pion is usually assumed to be involved. It has been shown by Stack[11] and by Jacobs and Vaughn[12] that the equal-mass limit of unequal-mass conspiracies and kinematic constraints exists and is smooth, and that the regularized helicity amplitudes are smooth functions of the mass differences. We expect, therefore, that the unequal-mass bootstrap should provide correlations between the meson and baryon mass asymmetries.

In Mandelstam's bootstrap, one trajectory is included for each type of intermediate particle (ρ, A_1, and π). The scale of energy is taken to correspond to $M_{\mathrm{baryon}} = 1$, and the slopes of all trajectories are taken phenomenologically to be $1/M$. Since there are only three equal-mass equations, he chose a single mass parameter $m_\pi^2 = m_\rho^2 = m_{A_1}^2 - 0.5M^2$ (compared with $m_\rho^2 - m_\pi^2 \sim 0.38M^2$ and $m_{A_1}^2 - m_\rho^2 \sim 0.58M^2$ and two coupling-constant ratios. The equality of the ρ, π multiplet masses is suggested by certain $SU(6)_{JJ}$ bootstrap schemes. With these three trajectories included in the model, the first omitted trajectory is the ρ daughter, so that the resonance contributions to $\mathrm{Im}\bar{f}_{\lambda\mu}$ are known up to $t \sim m_\pi^2 + 1.0M^2$. A convenient place for the cutoff N is at the mass of the A_1. Consistent with the smoothing requirement, only one-half the contribution to $\mathrm{Im}\bar{f}_{\lambda\mu}$ from the A_1 is included.

One uses the kinematic and group-theoretic crossing matrices to relate the s-channel Regge contributions to the t-channel resonance contributions. The kinematic crossing matrix for the equal-mass case has been given explicitly by Goldberger, Grisaru, MacDowell, and Wong[18] (GGMW) and, implicitly, for general masses by various other people.[13,14] The $SU(3)$ crossing matrix for $\{8\}\otimes\{8\} \to \{8\}\otimes\{8\}$ has been given by DeSwart.[19] An interesting aspect of the group crossing matrix is that it has an eigenvector with eigenvalue $+1$ which involves only octets and singlets of intermediate (meson) states. Although this eigenvector does not satisfy factorization (it should for any single Regge-trajectory contribution), nevertheless the advantage of considering only nonet mesons warrants its use.

Mandelstam solved the resulting equations and obtained reasonable values for m_π^2, g_π^2/g_{VM}^2, and g_A^2/g_{VM}^2. We present in the following sections a generalization of this model in which FESR's are used to obtain conditions on the $SU(3)$ mass breaking of the meson multiplets as a function of small baryon splittings.

III. UNEQUAL-MASS $N\bar{N} \to N\bar{N}$ CROSSING MATRIX

In the unequal $N\bar{N} \to N\bar{N}$ reaction,[20] there are eight linearly independent amplitudes (assuming parity). We use the Jacob-Wick[21] phase conventions for the helicity amplitudes $f_{\lambda_c\lambda_d,\lambda_a\lambda_b}$ for the reaction $a+b \to c+d$. The

[16] R. Dolen, D. Horn, and C. Schmid, Phys. Rev. **166**, 1768 (1968).

[17] M. A. Jacobs and M. H. Vaughn, Phys. Rev. **172**, 1677 (1968).

[18] M. L. Goldberger, M. T. Grisaru, S. W. MacDowell, and D. Y. Wong, Phys. Rev. **120**, 2250 (1960).

[19] J. J. DeSwart, Nuovo Cimento **31**, 420 (1964).

[20] The authors would like to express their appreciation to Dr. M. Levine for the use of his symbol manipulation program ASHMEDAI, which was used in the preliminary algebraic analyses of this section.

[21] M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) **7**, 404 (1959).

crossing matrix for helicity amplitudes[13,14] is

$$f_{\lambda_c\lambda_d,\lambda_a\lambda_b}(s,t)=-(-1)^{\lambda_c-\lambda_d}\times\sum_{\lambda_a',\lambda_b',\lambda_c',\lambda_d'}(-1)^{\lambda_a'-\lambda_b'}d_{\lambda_a'\lambda_a}{}^{1/2}(\chi_a)d_{\lambda_b'\lambda_b}{}^{1/2}(\chi_b)\otimes d_{\lambda_c'\lambda_c}{}^{1/2}(\chi_c)d_{\lambda_d'\lambda_d}{}^{1/2}(\chi_d)f_{\lambda_c'\lambda_a',\lambda_d'\lambda_b'}{}^{t}(s,t)\,,\quad(3.1)$$

where

$$\cos\chi_a=-\frac{(s+a^2-b^2)(t+a^2-c^2)+2a^2\Delta}{S_{ab}T_{ac}},$$

$$\cos\chi_b=\frac{(s+b^2-a^2)(t+b^2-d^2)-2b^2\Delta}{S_{ab}T_{bd}},$$

$$\cos\chi_c=\frac{(s+c^2-d^2)(t+c^2-a^2)-2c^2\Delta}{S_{cd}T_{ac}},$$

$$\cos\chi_d=-\frac{(s+d^2-a^2)(t+d^2-b^2)+2d^2\Delta}{S_{cd}T_{bd}},$$

$$\sin\chi_a=\frac{2a\sqrt{\Phi}}{S_{ab}T_{ac}},\qquad\sin\chi_b=\frac{2b\sqrt{\Phi}}{S_{ab}T_{bd}},$$

$$\sin\chi_c=-\frac{2c\sqrt{\Phi}}{S_{cd}T_{ac}},\qquad\sin\chi_d=-\frac{2d\sqrt{\Phi}}{S_{cd}T_{bd}},$$

$$\Delta=-a^2+b^2+c^2-d^2,$$

$$S_{ij}=(s-\Delta_{ij}{}^2)^{1/2}(s-\sigma_{ij}{}^2)^{1/2},$$

$$T_{ij}=(t-\Delta_{ij}{}^2)^{1/2}(t-\sigma_{ij}{}^2)^{1/2},$$

$$d^{1/2}(\chi)=\frac{1}{\sqrt{2}}\begin{pmatrix}(1+\cos\chi)^{1/2} & -(1-\cos\chi)^{1/2}\\ (1-\cos\chi)^{1/2} & (1+\cos\chi)^{1/2}\end{pmatrix},$$

$$\Delta_{ij}=i-j\,,\quad\sigma_{ij}=i+j\,,$$

$\Phi(s,t)$ is the Kibble function,[22] and the cut in $\sqrt{\Phi}$ is taken along the positive Φ axis. The cuts in S_{ab}, etc., are taken from the pseudothresholds to the thresholds, with s evaluated at $s+i|\epsilon|$, t at $t-i|\epsilon|$, and u real.

The eight regularized helicity amplitudes (all kinematic poles, zeros, and branch cuts removed) for the $N\bar{N}$ reaction may be defined as

$$\begin{aligned}
\hat{f}_1{}^t&=(f_{++++}{}^t-f_{++--}{}^t)\mathcal{P}_t\,,\\
\hat{f}_2{}^t&=(f_{++++}{}^t+f_{++--}{}^t)\mathcal{T}_t\,,\\
\hat{f}_3{}^t&=[(1+z_t)^{-1}f_{+-+-}{}^t-(1-z_t)^{-1}f_{+--+}{}^t]t\mathcal{T}_t{}^{-1}\,,\\
\hat{f}_4{}^t&=[(1+z_t)^{-1}f_{+-+-}{}^t+(1-z_t)^{-1}f_{+--+}{}^t]t\mathcal{P}_t{}^{-1}\,,\\
\hat{f}_5{}^t&=(f_{+++-}{}^t+f_{++-+}{}^t)(1-z_t{}^2)^{-1/2}t^{1/2}\mathcal{T}_t{}^{-1}\,,\\
\hat{f}_6{}^t&=(f_{+++-}{}^t-f_{++-+}{}^t)(1-z_t{}^2)^{-1/2}t^{1/2}\mathcal{P}_t{}^{-1}\,,\\
\hat{f}_7{}^t&=(f_{+-++}{}^t-f_{+---}{}^t)(1-z_t{}^2)^{-1/2}t^{1/2}\mathcal{T}_t{}^{-1}\,,\\
\hat{f}_8{}^t&=(f_{+-++}{}^t+f_{+---}{}^t)(1-z_t{}^2)^{-1/2}t^{1/2}\mathcal{P}_t{}^{-1}\,,
\end{aligned}\quad(3.2)$$

where

$$\mathcal{T}_t=(t-\sigma_{ac}{}^2)^{1/2}(t-\sigma_{bd}{}^2)^{1/2}\,,$$

$$\mathcal{P}_t=(t-\Delta_{ac}{}^2)^{1/2}(t-\Delta_{bd}{}^2)^{1/2}.$$

Using parity, $f_{\lambda_c\lambda_d,\lambda_a\lambda_b}=(-1)^{\lambda-\mu}f_{-\lambda_c-\lambda_d,\lambda_a-\lambda_b}$, and the definitions of f, it is easy to obtain the regularized helicity crossing matrix in terms of the quantities $\mathcal{T}_t$, $\mathcal{P}_t$, $\mathcal{T}_s$, $\mathcal{P}_s$, z_s, z_t, and

$$C_{\beta\gamma}=\cos\tfrac{1}{2}(\chi_a+\beta\chi_b+\gamma\chi_c+\beta\gamma\chi_d)\,,$$

$$S_{\beta\gamma}=\sin\tfrac{1}{2}(\chi_a+\beta\chi_b+\gamma\chi_c+\beta\gamma\chi_d)\,,\quad\beta,\gamma=\pm1\,.$$

We shall obtain explicit forms for the eight quantities $C_{\beta\gamma}$, $S_{\beta\gamma}$. Consider, for example, the crossing matrix element

$$X_{1,6}=\frac{\mathcal{P}_s\mathcal{P}_t(1-z_t{}^2)^{1/2}}{2t^{1/2}}\sin\tfrac{1}{2}(\chi_a+\chi_b+\chi_c+\chi_d)=(\sqrt{\Phi})\frac{\mathcal{P}_s}{\mathcal{T}_t}\sin\tfrac{1}{2}(\chi_a+\chi_b+\chi_c+\chi_d)\,.$$

The following properties of S_{++} may be obtained from Eqs. (3.1) and (3.2): (1) S_{++} contains a factor $\sqrt{\Phi}$, since each $\sin\chi_{a,b,c,\mathrm{or}\,d}$ contains $\sqrt{\Phi}$, and (2) its denominator is $\mathcal{P}_s\mathcal{P}_t\mathcal{T}_s\mathcal{T}_t$. Combining these, we have that $X_{1,6}$ is proportional to $\Phi/\mathcal{T}_t{}^2\mathcal{P}_t\mathcal{T}_s$. Because of the analyticity of $X_{1,6}$, the proportionality must be a polynomial in s, t, a, b, c, and d multiplied by $\mathcal{T}_s\mathcal{P}_t$. It is a polynomial of dimension $(\mathrm{mass})^{+1}$ and can be written $p_1a+p_2b+p_3c+p_4d$. It is easy to obtain the ratios $p_1:p_2:p_3:p_4$ by considering the terms of highest power in s and t in $\sin(\chi_a+\chi_b+\chi_c+\chi_d)\propto(a-b+c-d)[-(s+t)/st]^{1/2}$. The factor $a-b+c-d$ cannot occur in C_{++}, since it would imply a proportionality between $\sqrt{\Phi}$ and $\mathcal{P}_s\mathcal{T}_t$ (for $a-b+c-d=0$) in S_{++}, which does not occur. Therefore, the polynomial in $X_{1,6}$ is $\eta_{++}{}^{(s)}(a-b+c-d)$. In an analogous fashion, we obtain

$$\sin\tfrac{1}{2}(\chi_a+\beta\chi_b+\gamma\chi_c+\beta\gamma\chi_d)=\eta_{\beta\gamma}{}^{(s)}(a-\beta b+\gamma c-\beta\gamma d)(\sqrt{\Phi})/[s-(a-\beta b)^2]^{1/2}[s-(c-\beta d)^2]^{1/2}\times[t-(a+\gamma c)^2]^{1/2}[t-(b+\gamma d)^2]^{1/2},\quad(3.3)$$

where the dimensionless constants $\eta_{\beta\gamma}{}^{(s)}$ will be determined later. $\cos\frac{1}{2}(\chi_a+\chi_b+\chi_c+\chi_d)$ must be of the form $N/\mathcal{P}_s\mathcal{T}_t$, where N is a polynomial of dimension $(\mathrm{mass})^4$ in order not to violate Eq. (3.3) and the kinematic structure of the crossing matrix element $X_{1,2}=-C_{++}\mathcal{P}_s/2\mathcal{T}_t$. The most general polynomial of that degree is $h_0[(s+s_0)(t-t_0)+r_0]$, where h_0, s_0, t_0, r_0 are polynomials in a, b, c, and d of degree 0, 2, 2, and 4, respectively. $h_0=\pm1$ because

$$\lim_{s=t\to\infty}C_{++}=h_0,\quad\lim_{s=t\to\infty}S_{++}=0.$$

We can find s_0, t_0 and by considering the limits $(t\to\infty$, $s=\mathrm{const})$ and $(s\to\infty$, $t=\mathrm{const})$, respectively, for S_{++}

[22] T. Kibble, Phys. Rev. **117**, 1159 (1959).

Table I. Unequal-mass crossing matrix for the regularized helicity amplitudes.[a]

$\frac{N_B}{2\mathcal{P}_t^2}$	$\frac{-N_A}{2\mathcal{T}_t^2}$	$\frac{\mathcal{P}_t^2N_A-Z_tN_B}{2t\mathcal{P}_t^2}$	$\frac{-\mathcal{T}_t^2N_B+Z_tN_A}{2t\mathcal{T}_t^2}$	$\frac{(a-b-c+d)\Phi}{\mathcal{P}_t^2}$	$\frac{-(a-b+c-d)\Phi}{\mathcal{T}_t^2}$	$\frac{(a-b-c+d)\Phi}{\mathcal{P}_t^2}$	$\frac{-(a-b+c-d)\Phi}{\mathcal{T}_t^2}$
$\frac{-N_D}{2\mathcal{P}_t^2}$	$\frac{N_C}{2\mathcal{T}_t^2}$	$\frac{\mathcal{P}_t^2N_C-Z_tN_D}{2t\mathcal{P}_t^2}$	$\frac{-\mathcal{T}_t^2N_D+Z_tN_C}{2t\mathcal{T}_t^2}$	$\frac{(a+b-c-d)\Phi}{\mathcal{P}_t^2}$	$\frac{-(a+b+c+d)\Phi}{\mathcal{T}_t^2}$	$\frac{-(a+b-c-d)\Phi}{\mathcal{P}_t^2}$	$\frac{(a+b+c+d)\Phi}{\mathcal{T}_t^2}$
$\frac{-[s-(a-b)(c-d)]}{4\mathcal{P}_t^2}$	$\frac{-[s+(a-b)(c-d)]}{4\mathcal{T}_t^2}$	$\frac{-\mathcal{P}_s^2[t+(a-c)(b-d)]+Z_s[t-(a-c)(b-d)]}{4t\mathcal{P}_t^2}$	$\frac{-\mathcal{P}_s^2[t+(a+c)(b+d)]+Z_s[t-(a+c)(b+d)]}{4t\mathcal{T}_t^2}$	$\frac{-\mathcal{P}_s^2(a+b-c-d)+Z_s(a-b-c+d)}{4\mathcal{P}_t^2}$	$\frac{-\mathcal{P}_s^2(a+b+c+d)+Z_s(a-b+c-d)}{4\mathcal{T}_t^2}$	$\frac{\mathcal{P}_s^2(a+b-c-d)+Z_s(a-b-c+d)}{4\mathcal{P}_t^2}$	$\frac{\mathcal{P}_s^2(a+b+c+d)+Z(a-b+c-d)}{4\mathcal{T}_t^2}$
$\frac{[s-(a+b)(c+d)]}{4\mathcal{P}_t^2}$	$\frac{[s+(a+b)(c+d)]}{4\mathcal{T}_t^2}$	$\frac{-\mathcal{T}_s^2[t-(a-c)(b-d)]+Z_s[t+(a-c)(b-d)]}{4t\mathcal{P}_t^2}$	$\frac{-\mathcal{T}_s^2[t-(a+c)(b+d)]+Z_s[t+(a+c)(b+d)]}{4t\mathcal{T}_t^2}$	$\frac{-\mathcal{T}_s^2(a-b-c+d)+Z_s(a+b-c-d)]}{4\mathcal{P}_t^2}$	$\frac{-\mathcal{T}_s^2(a-b+c-d)+Z_s(a+b+c+d)]}{4\mathcal{T}_t^2}$	$\frac{-\mathcal{T}_s^2(a-b-c+d)-Z_s(a+b-c-d)}{4\mathcal{P}_t^2}$	$\frac{\mathcal{T}_s^2(a-b+c-d)-Z_s(a+b+c+d)}{4\mathcal{T}_t^2}$
$\frac{a-b-c+d}{4\mathcal{P}_t^2}$	$\frac{-(a-b+c-d)}{4\mathcal{T}_t^2}$	$\frac{\mathcal{P}_t^2(a-b+c-d)-Z_t(a-b-c+d)}{4t\mathcal{P}_t^2}$	$\frac{-\mathcal{T}_t^2(a-b-c+d)+Z_t(a-b+c-d)}{4t\mathcal{T}_t^2}$	$\frac{-N_B}{2\mathcal{P}_t^2}$	$\frac{N_A}{2\mathcal{T}_t^2}$	$\frac{-N_B}{2\mathcal{P}_t^2}$	$\frac{N_A}{2\mathcal{T}_t^2}$
$\frac{a+b-c-d}{4\mathcal{P}_t^2}$	$\frac{-(a+b+c+d)}{4\mathcal{T}_t^2}$	$\frac{-\mathcal{P}_t^2(a+b+c+d)+Z_t(a+b-c-d)}{4t\mathcal{P}_t^2}$	$\frac{\mathcal{T}_t^2(a+b-c-d)-Z_t(a+b+c+d)}{4t\mathcal{T}_t^2}$	$\frac{N_D}{2\mathcal{P}_t^2}$	$\frac{-N_C}{2\mathcal{T}_t^2}$	$\frac{-N_D}{2\mathcal{P}_t^2}$	$\frac{N_C}{2\mathcal{T}_t^2}$
$\frac{a-b-c+d}{4\mathcal{P}_t^2}$	$\frac{a-c+b-d}{4\mathcal{T}_t^2}$	$\frac{-\mathcal{P}_t^2(a-b+c-d)-Z_t(a-b-c+d)}{4t\mathcal{P}_t^2}$	$\frac{-\mathcal{T}_t^2(a-b-c+d)-Z_t(a-b+c-d)}{4t\mathcal{T}_t^2}$	$\frac{-N_B}{2\mathcal{P}_t^2}$	$\frac{-N_A}{2\mathcal{T}_t^2}$	$\frac{-N_B}{2\mathcal{P}_t^2}$	$\frac{-N_A}{2\mathcal{T}_t^2}$
$\frac{a+b-c-d}{4\mathcal{P}_t^2}$	$\frac{a+b+c+d}{4\mathcal{T}_t^2}$	$\frac{\mathcal{P}_t^2(a+b+c+d)+Z_t(a+b-c-d)}{4\mathcal{T}\mathcal{P}_t^2}$	$\frac{\mathcal{T}_t^2(a+b-c-d)+Z_t(a+b+c+d)}{4t\mathcal{T}_t^2}$	$\frac{N_D}{2\mathcal{P}_t^2}$	$\frac{N_C}{2\mathcal{T}_t^2}$	$\frac{-N_D}{2\mathcal{P}_t^2}$	$\frac{-N_C}{2\mathcal{T}_t^2}$

[a] $f_h{}^s = X_{hh'} f_{h'}{}^t$,

$N_A = +(s+\Delta_{ab}\Delta_{cd})(t-\sigma_{ac}\sigma_{bd}) + 2\Delta_{ab}\Delta_{cd}\sigma_{ac}\sigma_{bd} + \delta$,
$N_B = -(s-\Delta_{ab}\Delta_{cd})(t-\Delta_{ac}\Delta_{bd}) - 2\Delta_{ab}\Delta_{cd}\Delta_{ac}\Delta_{bd} - \delta$,
$N_C = -(s+\sigma_{ab}\sigma_{cd})(t+\sigma_{ac}\sigma_{bd}) - 2\sigma_{ab}\sigma_{cd}\sigma_{ac}\sigma_{bd} - \delta$,
$N_D = +(s-\sigma_{ab}\sigma_{cd})(t+\Delta_{ac}\Delta_{bd}) + 2\sigma_{ab}\sigma_{cd}\Delta_{ac}\Delta_{bd} + \delta$,
$\delta = (a^2-b^2-c^2+d^2)(ad-bc)$.

and C_{++}, and obtain $s_0=(a-b)(c-d)$ and $t_0=(a+c)(b+d)$. A bit of elementary algebra then yields $r_0=2s_0t_0+(a^2-b^2-c^2+d^2)(ad-bc)$. The corresponding formulas for the other cosines are

$$\cos\tfrac{1}{2}(\chi_a+\beta\chi_b+\gamma\chi_c+\beta\gamma\chi_d)$$
$$=\eta_{\beta\gamma}{}^{(c)}\frac{(s+\gamma s_0)(t-\beta t_0)+\beta\gamma r_0}{[s-(a-\beta b)^2]^{1/2}[s-(c-\beta d)^2]^{1/2}}$$
$$\otimes\frac{1}{[t-(a+\gamma c)^2]^{1/2}[t-(b+\gamma d)^2]^{1/2}},\qquad(3.4)$$

where

$$s_0=(a-\beta b)(c-\beta d),$$
$$t_0=(a+\gamma c)(b+\gamma d),$$
$$r_0=2s_0t_0+(a^2-b^2-c^2+d^2)(ad-bc),$$
$$\eta_{\beta\gamma}{}^{(c)}=\pm 1.$$

The product $\eta_{\beta\gamma}{}^{(s)}\eta_{\beta\gamma}{}^{(c)}$ can be determined from $\sin(\chi_a+\beta\chi_b+\gamma\chi_c+\beta\gamma\chi_d)$ by comparing the highest powers of s and t, and is -1. $\eta_{\beta\gamma}{}^{(c)}$ can then be determined within the phase conventions for crossing-matrix angles of Cohen-Tannoudji *et al.*[14] (Secs. II B and IV). Evaluating χ_i in the limit $s=t\to\infty$, we obtain $\eta_{\beta\gamma}{}^{(c)}=\beta\gamma$. The $N\bar{N}$ unequal-mass crossing matrix is given explicitly in Table I.

IV. RESONANCE CONTRIBUTIONS

The contributions to the resonance side of the FESR's are from the lowest multiplet on the ρ, A_1, and π trajectories. As the simplest approximation, we assume that the invariant amplitudes, which contain the dynamics, are constants. The mass dependence of the ρ, A_1, and π contributions is then given by the matrix elements of the operators (between initial and final states) associated with the invariant amplitudes.

For equal-mass scattering, GGMW[18] and others have shown that the five invariant amplitudes can be taken to be S, P, V, A, and T. For the unequal-mass case, we consider the following eight amplitudes (t channel)[15]:

$$A_h{}^1=\tfrac{1}{2}\;\bar{u}_{\lambda_c}\;1\;v_{\lambda_a}\bar{v}_{\lambda_b}\;1\;u_{\lambda_d}I^{(1)}(s,t),$$
$$A_h{}^2=\tfrac{1}{2}\;\bar{u}_{\lambda_c}\;\gamma_5\;v_{\lambda_a}\bar{v}_{\lambda_b}\;\gamma_5\;u_{\lambda_d}I^{(2)}(s,t),$$
$$A_h{}^3=\tfrac{1}{2}\;\bar{u}_{\lambda_c}\;\gamma^\mu\;v_{\lambda_a}\bar{v}_{\lambda_b}\;\gamma_\mu\;u_{\lambda_d}I^{(3)}(s,t),$$
$$A_h{}^4=\tfrac{1}{2}\;\bar{u}_{\lambda_c}\;\gamma_5\gamma^\mu\;v_{\lambda_a}\bar{v}_{\lambda_b}\;\gamma_5\gamma_\mu\;u_{\lambda_d}I^{(4)}(s,t),$$
$$A_h{}^5=\tfrac{1}{8}\;\bar{u}_{\lambda_c}\;[\gamma^\mu,\gamma^\nu]\;v_{\lambda_a}\bar{v}_{\lambda_b}\;[\gamma_\nu,\gamma_\mu]\;u_{\lambda_d}I^{(5)}(s,t),$$
$$A_h{}^6=\tfrac{1}{2}\;\bar{u}_{\lambda_c}\;\gamma^\mu\;v_{\lambda_a}\bar{v}_{\lambda_b}\;(P_{b_\mu}-P_{d_\mu})u_{\lambda_d}I^{(6)}(s,t),$$
$$A_h{}^7=\tfrac{1}{2}\;\bar{u}_{\lambda_c}(P_{a\mu}-P_{c\mu})\;v_{\lambda_a}\bar{v}_{\lambda_b}\;\gamma^\mu\;u_{\lambda_d}I^{(7)}(s,t),$$
$$A_h{}^8=\tfrac{1}{2}\{\bar{u}_{\lambda_c}\;\gamma_5\gamma^\mu\;v_{\lambda_a}\bar{v}_{\lambda_b}\;\gamma_5(P_{b_\mu}-P_{d_\mu})u_{\lambda_d}$$
$$-\bar{u}_{\lambda_c}\gamma_5(P_{a_\mu}-P_{c_\mu})v_{\lambda_a}\bar{v}_{\lambda_b}\gamma_5\gamma^\mu u_{\lambda_d}\}\;I^{(8)}(s,t),$$

where

$$h=\{\lambda_c\lambda_a,\lambda_d\lambda_b\},$$
$$\gamma_5=\begin{pmatrix}0&1\\-1&0\end{pmatrix},\quad\gamma_0=\begin{pmatrix}1&0\\0&-1\end{pmatrix},$$
$$\gamma^i=\begin{pmatrix}0&\sigma^i\\-\sigma^i&0\end{pmatrix},$$
$$v_{\lambda_a}=\frac{2\lambda_a}{\sqrt{2}t^{1/4}}\begin{pmatrix}-2\lambda_a(t^{1/2}-\sigma_{ac})^{1/2}(t^{1/2}-\Delta_{ac})^{1/2}\\(t^{1/2}+\sigma_{ac})^{1/2}(t^{1/2}+\Delta_{ac})^{1/2}\end{pmatrix}R\chi_{\lambda_a},$$
$$v_{\lambda_b}=\frac{2\lambda_b}{\sqrt{2}t^{1/4}}\begin{pmatrix}-2\lambda_b(t^{1/2}-\sigma_{bd})^{1/2}(t^{1/2}-\Delta_{bd})^{1/2}\\(t^{1/2}+\sigma_{bd})^{1/2}(t^{1/2}+\Delta_{bd})^{1/2}\end{pmatrix}\chi_{\lambda_b},$$
$$u_{\lambda_c}=\frac{1}{\sqrt{2}t^{1/4}}\begin{pmatrix}(t^{1/2}+\sigma_{ac})^{1/2}(t^{1/2}-\Delta_{ac})^{1/2}\\2\lambda_c(t^{1/2}-\sigma_{ac})^{1/2}(t^{1/2}+\Delta_{ac})^{1/2}\end{pmatrix}R\chi_{\lambda_c},$$
$$u_{\lambda_d}=\frac{1}{\sqrt{2}t^{1/4}}\begin{pmatrix}(t^{1/2}+\sigma_{bd})^{1/2}(t^{1/2}-\Delta_{bd})^{1/2}\\2\lambda_d(t^{1/2}-\sigma_{bd})^{1/2}(t^{1/2}+\Delta_{bd})^{1/2}\end{pmatrix}\chi_{\lambda_d},$$
$$\sigma_z\chi_\lambda=2\lambda\chi_\lambda,$$
$$R=e^{-1/2i\theta_t\sigma_y}.$$

The z direction has been taken in the direction of the three-momentum of particle d, and the scattering has occurred in the xz plane. With the convention $\chi_\lambda{}^\dagger\chi_\lambda=1$, the wave functions are normalized to $\pm(2m_i)^{1/2}$. The contributions to the eight regularized amplitudes are given in Table II. For arbitrary mass assignments, it is possible to show that the contributions of $I^{(1)},\cdots I^{(8)}$ are linearly independent. The contributions also display the proper t-kinematic structure and satisfy all constraints and conspiracies. They cannot be multiplied by any function with t-singularities without violating these conditions.

The $I^{(4)}$ amplitude is a sum of contributions from two charge-conjugation quanrum numbers. We may take $A_h{}^2$, $A_h{}^3$, and that portion of $A_h{}^4$ with the proper charge-conjugation quantum number to be unequal-

TABLE II. Contributions of invariant amplitudes to the regularized helicity amplitudes.

$$f_1{}^t=-\mathcal{P}_t{}^2I^{(2)}+t^{-1}(\mathcal{P}_t{}^2\sigma_{ac}\sigma_{bd}-\Delta_{ac}\Delta_{bd}\mathcal{P}_t\mathcal{T}_zz_t)I^{(4)}+\mathcal{P}_t\mathcal{T}_tz_tI^{(5)}$$
$$+[(a+b-c-d)/t][\mathrm{P}_t{}^2\sigma_{ac}\sigma_{bd}+\mathcal{P}_t\mathcal{T}_tz_t(t-\Delta_{ac}\Delta_{bd})]I^{(8)}$$
$$f_2{}^t=\mathcal{T}_t{}^2I^{(1)}+t^{-1}(\mathcal{T}_t{}^2\Delta_{ac}\Delta_{bd}-\mathcal{P}_t\mathcal{T}_tz_t\sigma_{ac}\sigma_{bd})I^{(3)}-\mathcal{P}_t\mathcal{T}_tz_tI^{(5)}$$
$$+t^{-1}[\mathcal{T}_t{}^2\Delta_{ac}\Delta_{bd}\sigma_{bd}+\mathcal{P}_t\mathcal{T}_tz_t\sigma_{ac}(t-\sigma_{bd})]I^{(6)}$$
$$+t^{-1}[\mathcal{T}_t{}^2\Delta_{ac}\Delta_{bd}\sigma_{ac}+\mathcal{P}_t\mathcal{T}_tz_t\sigma_{bd}(t-\sigma_{ac}{}^2)]I^{(7)}$$
$$f_3{}^t=-tI^{(4)}+\Delta_{ac}\Delta_{bd}I^{(5)}$$
$$f_4{}^t=-tI^{(3)}-\sigma_{ac}\sigma_{bd}I^{(5)}$$
$$f_5{}^t=\Delta_{ac}I^{(4)}-\Delta_{bd}I^{(5)}-(t-\Delta_{ac}{}^2)I^{(8)}$$
$$f_6{}^t=\sigma_{ac}I^{(3)}+\sigma_{bd}I^{(5)}-(t-\sigma_{ac}{}^2)I^{(7)}$$
$$f_7{}^t=-\Delta_{bd}I^{(4)}+\Delta_{ac}I^{(5)}+(t-\Delta_{bd}{}^2)I^{(8)}$$
$$f_8{}^t=-\sigma_{bd}I^{(3)}-\sigma_{ac}I^{(5)}+(t-\sigma_{bd}{}^2)I^{(6)}$$

mass generalizations of the contributions of the π, ρ, and A_1 when their trajectories pass through $\alpha=0$, 1, and 1, respectively. Explicitly, in the narrow-resonance approximation, the contributions to $\mathrm{Im}f^t$ are obtained from Table II by evaluation at $s=0$ with the substitution of $-\frac{1}{2}(g_{\mathrm{meson}}^2/4\pi)\,\pi\delta(t-m_{\mathrm{meson}}^2)$ for I^{meson}. The $(g_{\mathrm{meson}}^2/4\pi$ are constants which will be determined in the bootstrap. We retain the distinction between the electric and magnetic couplings of the vector meson by using g_{VE}^2, g_{VEM}^2, and g_{VM}^2 in the amplitudes f_2^t, $f_{6,8}$, and f_4^t, which contain the electric-electric, electric-magnetic, and magnetic-magnetic vertices, respectively.

V. REGGE CONTRIBUTIONS

The Regge contribution of a single (t-channel) trajectory to an amplitude $f_h^t(s,t)$ is

$$f_h^t(s,t)=\frac{[2\alpha(t)+1]\beta_h(t)}{\sin\pi(\alpha(t)-\lambda)}d_{\lambda,\mu}^{\alpha(t)}(-z_t)\,,$$

where $\beta_h(t)$ is the residue of the pole in J at $\alpha(t)$ of the partial-wave amplitude. In the narrow-resonance approximation, in which $\alpha(t)$ has an imaginary part only at a resonance, the usual partial-wave expansion can be converted to the following expression for $\mathrm{Im}f_h^t$:

$$\mathrm{Im}f_h^t(s,t)=\sum_J(2J+1)\pi\delta(J-\alpha(t))\beta_h(t)d_{\lambda,\mu}^J(z_t)\,.$$

Since we have evaluated the contribution of the resonances to the amplitudes, we can now relate $\beta_h(t=m_{\mathrm{meson}}^2)$ to the coupling constants and obtain

$$\beta_{++,++}{}^{\pi}(\mu_\pi^2)=-\frac{g_\pi^2}{16\pi}(\mu_\pi^2-\Delta_{ac}^2)^{1/2}(\mu_\pi^2-\Delta_{bd}^2)^{1/2}\,,$$

$$\beta_{+-,+-}{}^{A}(\mu_A^2)=-\frac{g_{A_1}^2}{24\pi}(\mu_A^2-\sigma_{ac}^2)^{1/2}(\mu_A^2-\sigma_{bd}^2)^{1/2}\,,$$

$$\beta_{+-,+-}{}^{\rho}(\mu_\rho^2)=-\frac{g_{\rho M}^2}{24\pi}(\mu_\rho^2-\Delta_{ac}^2)^{1/2}(\mu_\rho^2-\Delta_{bd}^2)^{1/2}\,,\quad(5.1)$$

$$\beta_{++,+-}{}^{\rho}(\mu_\rho^2)=\frac{g_{\rho EM}^2}{24\sqrt{2}\pi}(\mu_\rho^2-\Delta_{ac}^2)^{1/2}(\mu_\rho^2-\Delta_{bd}^2)^{1/2}\frac{\sigma_{ac}}{\mu_\rho}\,,$$

$$\beta_{+-,++}{}^{\rho}(\mu_\rho^2)=\frac{g_{\rho EM}^2}{24\sqrt{2}\pi}(\mu_\rho^2-\Delta_{ac}^2)^{1/2}(\mu_\rho^2-\Delta_{bd}^2)^{1/2}\frac{\sigma_{bd}}{\mu_\rho}\,.$$

Since the trajectories in the s channel are the same as those in the t channel, these may also be interpreted as the residues for the s-channel trajectories, with the appropriate renaming of the particles ($a\leftrightarrow d$).

It is known that the analytic structure of β_h^s is complicated. It is more convenient to consider the function $E(s)$ defined by

$$\beta_h^s(s)=\left(\frac{4p_{ab}p_{cd}}{e}\right)^{\alpha(s)-\lambda_m}\frac{\mathfrak{N}(\alpha(s))\mathcal{K}(s)E(s)}{\Gamma(\alpha(s)+\frac{3}{2})}\,.$$

The factor $(4p_{ab}p_{cd})^{\alpha-\lambda_m}$ reproduces the correct threshold behavior. $1/\Gamma(\alpha+\frac{3}{2})$ ensures that zeros of β_h will occur at the negative half-odd-integers. $(1/e)^{\alpha-\lambda_m}$ has been included to ensure that $\beta(s)$ is not exponential in s. $\mathfrak{N}(\alpha)$ is the required sense-nonsense factor and is determined by the dynamics. $\mathcal{K}(s)$ is the usual kinematic factor of Wang, and other authors.[23] The function $E(s)$ is an entire function. It is also known that $\beta_h^s(s)$ is factorizable. The choice of three-particle vertices $\gamma_{i,j}{}^{\mathrm{meson}}$, given below, reproduces the above analytic structure for $\beta_h^s(s)$:

$$\gamma_{i,j}{}^{\pi}=(\lambda_i+\lambda_j)s^{1/2}[E_{1,ij}{}^{\pi}(s)\alpha_\pi(s)+E_{2,ij}{}^{\pi}(s)\Psi_{ij}{}^2]\times\left(\frac{\Psi_{ij}\Phi_{ij}}{(se)^{1/2}}\right)^{\alpha_\pi(s)}\frac{\Psi_{ij}}{[\Gamma(\alpha_\pi(s)+\frac{3}{2})]^{1/2}}\,,$$

$$\gamma_{i,j}{}^{A}=(\lambda_i+\lambda_j)E_{ij}{}^{A}(s)\left(\frac{\Psi_{ij}\Phi_{ij}}{(se)^{1/2}}\right)^{\alpha_A(s)}\times\frac{(\alpha_A+1)^{1/2}}{\Psi_{ij}[\Gamma(\alpha_A(s)+\frac{3}{2})]^{1/2}}+(\lambda_i-\lambda_j)E_{ij}{}^{A}(s)\times\left(\frac{\Psi_{ij}\Phi_{ij}}{(se)^{1/2}}\right)^{\alpha_A(s)-1}\frac{\Phi_{ij}(\alpha_A)^{1/2}}{[\Gamma(\alpha_A(s)+\frac{3}{2})]^{1/2}}\,,\quad(5.2)$$

$$\gamma_{i,j}{}^{\rho}=|\lambda_i+\lambda_j|E_{ij}{}^{\rho,E}(s)\left(\frac{\Psi_{ij}\Phi_{ij}}{(se)^{1/2}}\right)^{\alpha_\rho(s)}\times\frac{(\alpha_\rho+1)^{1/2}}{\Phi_{ij}[\Gamma(\alpha_\rho(s)+\frac{3}{2})]^{1/2}}+|\lambda_i-\lambda_j|E_{ij}{}^{\rho,M}(s)\times\left(\frac{\Psi_{ij}\Phi_{ij}}{(se)^{1/2}}\right)^{\alpha_\rho(s)}\frac{\Psi_{ij}(\alpha_\rho)^{1/2}}{[\Gamma(\alpha_\rho(s)+\frac{3}{2})]^{1/2}}\,,$$

where $\Psi_{ij}=(s-\Delta_{ij}^2)^{1/2}$, $\Phi_{ij}=(s-\sigma_{ij}^2)^{1/2}$, and the E_{ij}'s are entire functions. We have allowed two types of vertices for the pion, representing $M=1$ and $M=0$ couplings, respectively, as suggested by the model of Frazer, Lipinski, and Snider[24] which involves the crossing of an $M=0$ and $M=1$ trajectory between $s=0$ and μ_π^2. Phenomenologically, Ball, Frazer, and Jacob[25] have shown that the $N\bar{N}$ vertex is proportional to $1-\lambda(s-\mu_\pi)/\mu_\pi^2$, with a value of 0.4 for λ obtained from photoproduction. The choice of $\gamma_{i,j}{}^{\pi}$ in (5.2) will reproduce this dependence of the pion's residue where λ is given in

[23] Y. Hara, Phys. Rev. **136**, B507 (1964); L. L. C. Wang, *ibid.* **142**, 1187 (1966); see also Ref. 14.

[24] W. R. Frazer, H. M. Lipinski, and D. R. Snider, Phys. Rev. **174**, 1932 (1968).

[25] J. S. Ball, W. R. Frazer, and M. Jacob, Phys. Rev. Letters **20**, 518 (1968).

terms of the E's by

$$\lambda_{ij}=\frac{E_{1,ij}^{\pi}+E_{2,ij}^{\pi}}{E_{2,ij}^{\pi}(1-\Delta_{ij}^2/\mu_\pi^2)}.$$

As a first approximation, we take all of the E's of (5.2) to be constants in s. The dependence of these constants on the masses can be obtained from Eqs. (5.1), (5.2), and $\beta_h^{\text{meson}}(s)=\gamma_{\lambda_a\lambda_b}^{\text{meson}}(s)\gamma_{\lambda_c\lambda_d}^{\text{meson}}(s)$.

At $s=0$, we obtain for $\int^N \operatorname{Im}\hat{f}_i^s(0,t)dt$:

$j=3$ (the axial-vector contribution):

$$\tfrac{1}{4}(g_{A_1}^2)_{ab,cd}\frac{e}{8\Gamma(2-\mu_A^2)}\left(\frac{4N}{e}\right)^{1-\mu_A^2}; \quad (5.3a)$$

$j=4$ (vector contribution):

$$0;$$

$j=6$ (vector contribution);

$$-\tfrac{1}{4}(g_{\rho EM}^2)_{ab,cd}\frac{\sigma_{cd}e}{8\Gamma(2-\mu_\rho^2)}\left(\frac{4N}{e}\right)^{1-\mu_\rho^2}; \quad (5.3b)$$

$j=8$ (vector contribution);

$$\tfrac{1}{4}(g_{\rho EM}^2)_{ab,cd}\frac{\sigma_{ab}e}{8\Gamma(2-\mu_\rho^2)}\left(\frac{4N}{e}\right)^{1-\mu_\rho^2}.$$

It is not consistent to neglect other contributions to the amplitude $j=4$. At $s=0$ and equal masses, there is a well-known conspiracy[18,26] involving $\hat{f}_1^s$, $\hat{f}_3^s$, and $\hat{f}_4^s$. For the pion to be involved in this conspiracy ($M=1$) it must contribute to $I^{(5)}$. We can obtain the highest-t behavior of its contribution to $I^{(5)}$ by comparing the predictions if Eq. (5.2) and Table II at the pseudothresholds:

$$I^{(5)}\underset{t\to\infty}{\longrightarrow}E_{1,ab}^{\pi}E_{2,cd}^{\pi}\frac{2}{e\Gamma(\alpha_\pi(s)+\frac{3}{2})}\left(\frac{2t}{e}\right)^{\alpha_\pi(s)-1}.$$

We can then obtain the contribution of the pion conspirator to $\int^N\operatorname{Im}\hat{f}_4^s(0,t)dt$;

$j=4$ (pion conspirator):

$$\tfrac{1}{4}(g_\pi^2)_{ab,cd}\frac{\sigma_{ab}\sigma_{cd}(1-\lambda_{ab})(1-\lambda_{cd})}{4\Gamma(1-\mu_\pi^2)}\left(\frac{4N}{e}\right)^{-\mu_\pi^2}. \quad (5.3c)$$

Equations (5.3) contain the highest t dependence of the contributions to the amplitudes which we use in the FESR's for $j=3$, 4, 6, 8.

VI. GROUP THEORY

The eigenvector of the $\{8\}\otimes\{8\}\to\{8\}\otimes\{8\}$ crossing matrix which we use is

$$T_{ab,cd}=5d_{ab}{}^i d_{cd}{}^i+9f_{ab}{}^i f_{cd}{}^i+16C_{ab}C_{cd},$$

where $d_{ab}{}^i$ are the Clebsch-Gordan coefficients of $\{8\}_a\otimes\{8\}_b\to\{8\}_i$ symmetric under $a\leftrightarrow b$; $f_{ab}{}^i$ are antisymmetric under $a\leftrightarrow b$; and C_{ab} are Clebsch-Gordan coefficients of $\{8\}_a\otimes\{8\}_b\to\{1\}$. The sum over i is over the meson octet states. We do not allow an admixture of the other eigenvectors of the crossing matrix in either the equal- or unequal-mass problems in order to retain the nonet assignments of the mesons (the only other eigenvector involving only 8's and 1's has -1 eigenvalue and so does not give a positive-definite contribution) and to keep the calculation as simple as possible. We assume that the mass operator is diagonal in the usual $SU(3)$ I, I_z, and Y labels and do not consider φ-ω mixing.

The kinematic dependences of the contributions of degenerate-mass mesons to equal-mass scattering are, of course, equal and the group-theoretic crossing can be accomplished separately from the kinematic crossing (the eigenvalue is $+1$, so $T_{ab,cd}=T_{\bar{d}b,c\bar{a}}$). We parametrize the mass splitting in the baryon octet as α_N times the symmetric octet splitting plus β_N times the hypercharge. For mesons, charge conjugation eliminates the second type of splitting and we allow one asymmetry parameter each: ϵ_ρ, ϵ_{A_1}, and ϵ_π. The group and kinematic crossings are not separable in the unequal-mass case.

VII. CALCULATIONS AND RESULTS

With the above mass splittings in the baryon octet, there are 120 kinematically distinct $N\bar{N}\to N\bar{N}$ reactions. For each of these reactions there are four amplitudes which we use in the FESR's. For equal masses, the 120 reactions and two of the four FESR's are identical. If we follow Mandelstam, there are three parameters which can be used to solve the equal-mass equations: $m_\pi^2=m_\rho^2=m_{A_1}^2-0.5M^2$, g_π^2/g_{VM}^2, and g_A^2/g_{VM}^2. We allow for the possibility of trajectory crossing of the pion $M=0$, 1 trajectories and have the additional parameter λ (although the results of the calculation show that only the value of g_π^2/g_{VM}^2 is significantly affected by λ) defined in Sec. V. It is also possible to take m_ρ^2 and $m_{A_1}^2$ to be independent parameters, as well as $g_{VE}{}^2$and g_{VEM}^2. Since we are concerned primarily with the interdependences of the mass perturbations within the model, we expect that the inclusion of these as additional equal-mass parameters would not significantly affect the perturbation calculation. Following Mandelstam, we take $g_{VE}^2/g_{VM}^2=3/31$ and $g_{VEM}^2/g_{VM}^2=6/31$. The solution sof the three equal-mass equa-

TABLE III. Equal-mass solutions for various values of λ.

λ	m_π^2 (M^2)	g_π^2/g_{VM}^2	g_A^2/g_{VM}^2
0.0	0.559	0.793	0.260
0.1	0.559	0.979	0.260
0.2	0.559	1.239	0.260
0.3	0.559	1.619	0.260
0.4	0.559	2.204	0.260

[26] D. V. Volkov and V. M. Gribov, Zh. Eksperim. i Teor. Fiz. **44**, 1068 (1963) [English transl.: Soviet Phys.—JETP **17**, 720 (1963)].

TABLE IV. Components of the minimum-eigenvalue eigendirection, the minimum eigenvalue, and next larger eigenvalue of various values of λ.

	Direction									
λ	m_π^2/M^2	g_π^2/g_{VM}^2	g_A^2/g_{VM}^2	ϵ_π	ϵ_ρ	ϵ_{A_1}	α_N	β_N	Eigen-value	Next larger eigen-value
0.0	0.33226	0.93722	0.10592	0.00025	0.00165	0.00000	0.00065	−0.00061	0.061	6.0
0.1	0.27587	0.95700	0.08989	−0.00038	0.00235	−0.00056	0.00096	−0.00037	0.049	5.8
0.2	0.22244	0.97215	0.07377	0.00011	0.00187	−0.00062	0.00009	−0.00055	0.031	5.7
0.3	0.17333	0.98311	0.05872	0.00016	0.00146	−0.00001	0.00028	0.00002	0.021	5.6
0.4	0.12942	0.99073	0.04128	0.00014	0.00183	−0.00017	−0.00014	0.00063	0.016	5.6

tions are given in Table III. For $\lambda=0$ (pure $M=1$ pion), we obtain

$$m_\pi^2=0.56M^2,\quad g_\pi^2/g_{VM}^2=0.79,\quad g_A^2/g_{VM}^2=0.26,$$

which compare with Mandelstam's

$$0.29M^2,\quad 0.21,\quad 0.18.$$

We differ from Mandelstam's values because we have included the analytic structure of the regularized helicity amplitudes near thresholds as well as pseudothresholds. We note that the solutions of the equal-mass problem are critically dependent upon the exact form of the equations. Thus, inclusion of corrections to the model due to daughter trajectories, lower-lying leading trajectories, Regge cuts, etc., may significantly affect the results. Presumably, also, the effects of other members of the {56} as well as non-Fermi-Yang contributions to the meson masses are important for the same reason. Nevertheless, since both forms for the analytic structure of the amplitudes when used in the FESR bootstrap give reasonable values of the equal-mass parameters, we can ask what type of mass asymmetries are consistent within the model.

Since there are 480 equations, it is possible, in principle, to have 480 parameters which describe the mass splittings and coupling-constant perturbations to solve the 480 equations. As a method of solution of these equations with fewer parameters, we consider the single function

$$F(m_\pi^2,g_\pi^2/g_{VM}^2,g_A^2/g_{VM}^2,\epsilon_\pi,\epsilon_\rho,\epsilon_{A1},\alpha_N,\beta_N)$$
$$=\sum_{i=1}^{120}\sum_{j=3,4,6,8}\left(\int^N \mathrm{Im} f_j{}^s\,dt-\int^N X_{jk}\,\mathrm{Im} f_k{}^t\,dt\right).\quad(7.1)$$

About the value $F(m_0^2,g_{p0}^2,g_{A0}^2,0,0,0,0,0)=0$, the function is well approximated by $F(\{X\})=(X_i-\bar{X}_i)(X_j-\bar{X}_j)a_{ij}$, where $\{X_i\}$ are the eight arguments of the functions, $\{\bar{X}_i\}$ are the equal-mass solutions, and $a_{ij}=a_{ji}$ were determined by a computer. It is obvious that this function $F(\{X)\}$ is zero only at $\{X\}=\{\bar{X}\}$, since otherwise it would correspond to solving all 480 equations with only eight parameters. However, if our choice of the parameters to describe the perturbations is good, then there may be a direction in $\{X\}$ space along which $F(\{X\})$ remains small, and hence along which the bootstrap is least stable against perturbations. Within this model it is possible to determine only the direction of instability and not the actual magnitude of the asymmetry along this direction or even its sign. The direction, which corresponds to the minimum-eigenvalue eigenvector of the matrix a_{ij}, is given in Table IV as a function of λ, along with its eigenvalue. Moving in this direction a distance Δ from $\{\bar{X}_i\}$ introduces an average error of $\sim 8\times 10^{-3}\Delta$ in each of the 480 equations. In any other direction, the corresponding error is 10–500 times as large. Since this eigenvector consistently involves only the equal-mass parameters, the model is stable against unequal-mass perturbations. That is, within this model the nucleon octet and each of the meson nonets are required to be degenerate in mass.

Experimentally the mass asymmetries are

$$\epsilon_\pi\sim-0.045,\quad \epsilon_\rho\sim-0.025,\quad \epsilon_{A_1}\sim-0.067$$
$$\alpha_N\sim 0.009,\quad \beta_N\sim 0.165,$$

measured in units in which the nucleon octet average mass is 1. Since it is impossible to determine the "shift" of the equal-mass parameters from their "degenerate" values, and since, in this model, the instability unfortunately lies almost wholly in the direction of the equal-mass parameters, we cannot determine the agreement of the model with experimental data.

The fact that the solutions to the equal-mass bootstrap equations obtained by Mandelstam and us seem to be reasonable may be fortuitous. One would hope that the solution of the bootstrap would not be particularly sensitive to the ρ, π and A_1, π mass differences. Nevertheless, for input values of these differences taken from experiment, we find that no solution to the equations exists. Further evidence to the sensitivity of the results is, of course, the previously mentioned large change in solution for a slightly different analytic structure.