Double Spectral Approximation for Pion Scattering

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The double spectral functions in the Mandelstam representation are approximated by a zero-width approximation that reproduces elastic crossed-channel resonances. The partial-wave potential generated by this double-spectral function has a right-hand cut which, for pion scattering, must correspond to inelastic scattering in the direct channel. The Frye-Warnock modification of the ND^{-1} method for inelasticity can then be used. The potential is convergent enough that no cutoff is required. Self-consistency in a bootstrap calculation is achieved by attaining proper threshold behavior rather than using the zero-width approximation in the direct channel.

I. INTRODUCTION

THERE have been a great number of dynamical S-matrix calculations based on the Mandelstam representation.[1] At the heart of this representation are double integrals over "double spectral functions" that contain all the dynamical information. Yet in their approach, many authors forget about the double spectral functions at an early stage and often forget about the branch points which they imply for the scattering amplitudes. There have been attempts[2] to deal with the double spectral functions by the iteration scheme first suggested by Mandelstam, but this technique has proved to be unwieldy and particularly difficult to apply to a bootstrap calculation.

In this paper, we deal directly with the double spectral functions and choose a zero-width approximation for them that reproduces crossed-channel resonances. Then, by partial-wave projection of the Mandelstam integral over the double spectral function, a generalized partial-wave potential is generated that has a right-hand cut due to the double spectral function. We then use the well-known property that double spectral functions can contribute to elastic scattering of pions in only one channel. That is, if a double spectral function contributes to an elastic resonance in a crossed channel, it can only contribute to inelastic scattering in the direct channel. This is because of the pseudoscalar nature of the pion and the fact that conservation of parity in strong interactions forbids a strong three-pion vertex which would be required for a doubly elastic process. This means that the right-hand cut of the potential must correspond to inelastic scattering and we can use the Frye-Warnock[3] inelastic modification of the N/D method[4,5] to solve for the scattering amplitude. We note also that it is inconsistent to do a calculation based on the Mandelstam representation using purely elastic unitarity. The right-hand cut of the potential defined by the double spectral approximation corresponds to the minimum inelasticity required for consistency within the Mandelstam representation.

The double spectral approximation could be applied to any scattering process involving pions (or any pseudoscalar particle). We outline the method for p-wave pion-pion scattering (the ρ bootstrap) but only minor changes would be required to do pion-nucleon or pion-omega scattering.

II. DOUBLE SPECTRAL APPROXIMATION

The Mandelstam representation for the $I=1$ pion-pion scattering amplitude can be written[1]

$$A^1(s,t)=\sum_{I'} c_{1I'}\left(\frac{1}{\pi}\int_4^\infty \frac{dt' A_t{}^{I'}(t',s)}{t'-t} - \frac{(-1)^{I'}}{\pi}\int_4^\infty \frac{du' A_u{}^{I'}(u',s)}{u'-u}\right), \quad (1)$$

with the absorptive part $A_t{}^I(t,s)$ given by

$$A_t{}^I(t,s)=\frac{1}{\pi}\int \frac{ds' \rho_{ts}{}^I(t,s')}{s'-s}+\frac{1}{\pi}\int \frac{du' \rho_{tu}{}^I(t,u')}{u'-u} \quad (2)$$

and, similarly, for $A_u{}^I(u,s)$. Here s, t, and u are the usual Mandelstam variables with $u=4-s-t$, and $\rho_{ts}{}^I(t,s')$ and $\rho_{tu}{}^I(t,u')$ are double spectral functions for isospin I in the t channel. The coefficients $c_{II'}$ (with $c_{11}=\frac{1}{2}$) form the pion-pion (s-t) crossing matrix.[4] The limits of integration in Eq. (2) are defined by the boundaries of the double spectral functions. As it stands, Eq. (1) defines the full amplitude. If we restrict ourselves to the elastic absorptive parts $A_t{}^{I(\mathrm{el})}(t',s)$ and $A_u{}^{I(\mathrm{el})}(u',s)$ in the crossed channel [given by integrals over $\rho_{ts}{}^{I(\mathrm{el})}(t',s')$, etc.], then the right-hand side of Eq. (1) defines a generalized potential since it cannot include the direct-channel elastic unitarity cut. This cut can then by put in by the N/D method, after partial-wave projection. The amplitude defined by this procedure will not include any double spectral contributions that are inelastic in both direct and crossed channels simultaneously. We assume that these are negligible since they must be quite far away from the physical region in both channels. A large part of their effect would be included in the sub-

[1] S. Mandelstam, Phys. Rev. **112**, 1344 (1958).
[2] N. F. Bali, Phys. Rev. **150**, 1358 (1966).
[3] G. Frye and R. L. Warnock, Phys. Rev. **130**, 478 (1963); see also P. W. Coulter, A. Scotti, and G. L. Shaw, *ibid.* **136**, B1399 (1964).
[4] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).
[5] J. L. Uretsky, Phys. Rev. **123**, 1459 (1961).

traction constant required for s waves. The subtraction does not appear for $I=1$ and we have left it out of Eq. (1). This breakup of the double spectral functions defines a generalization of the strip concept[6] although no strip width or cutoff need be introduced. The two double spectral functions $\rho_{ts}{}^{I(\mathrm{el})}(t,s)$ and $\rho_{st}{}^{I(\mathrm{el})}(s,t)$, in fact, overlap but they are easily distinguishable.[7]

We approximate the elastic double spectral function by

$$\rho_{ts}{}^{I(\mathrm{el})}(t,s)=\pi\gamma\delta(t-m^2)(s-4m^2)^{1/2}\theta(s-4m^2)\delta_{I1}, \quad (3)$$

with m being the input ρ mass and γ being related to the input ρ width. This form leads to a zero-width approximation to the ρ in the t channel. In the s channel, it leads to a cut starting at $s=4m^2$ corresponding to inelastic scattering. We take the effective inelastic threshold to be $4m^2$ corresponding to the ρ-ρ threshold. This is the lowest threshold associated with the first elastic unitarity correction of a ρ-exchange potential in the crossed channel. The particular diagram that corresponds to this double spectral function is shown in Fig. 1. There will also be higher inelastic thresholds corresponding to more iterations of the potential in the crossed channel, but these branch points will not make much of a difference since they will be quite high in energy. The contribution of these higher inelastic states is included, since γ is chosen to give the correct ρ width. It is just the exact locations of the higher inelastic thresholds that are changed to all coincide at $4m^2$, which already is quite high in energy. The factor $(s-4m^2)^{1/2}$ provides the proper threshold behavior for ρ-ρ production. This factor also happens, fortuitously, to give the potential a good asymptotic behavior that corresponds to the observed $s^{\sim 1/2}$ behavior for ρ exchange in the forward direction.

In an exact calculation, this $(s-4m^2)^{1/2}$ dependence would arise from the shape of the double spectral boundary curve rather than being put in as a factor. Also, the sharp ρ peak would arise, most likely, as a result of the slow convergence of the integral over the double spectral function in the ρ region rather than from the δ function that we have put in explicitly. This means that the double spectral function that we have chosen does not look very much like the exact double spectral function. But this need not be an important consideration since the double spectral function is only nonvanishing far from the physical region, and we are concerned only with its giving a reasonable approximation when integrated over. We emphasize that after approximating the crossed-channel double spectral function by Eq. (3) we make no further approximations and the unitarization of the amplitude (including appropriate inelastic unitarity) is exact.

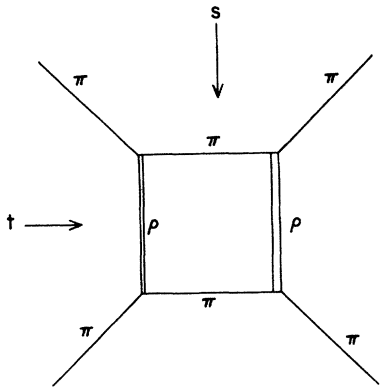

FIG. 1. Diagram for double spectral contribution to elastic unitarity in the t channel for a ρ-exchange, t-channel potential.

We now look at the absorptive part of the amplitude in the t channel. Using Eq. (3) for the double spectral function in Eq. (2) and using the crossing property of the double spectral functions, we find

$$A_t{}^{1(\mathrm{el})}(t,s)=\gamma\delta(t-m^2)(t-4)z_t\int_{4m^2}^{\infty}\frac{ds'(s'-4m^2)^{1/2}}{(s'-s)(s'+s+t-4)}, \quad (4)$$

where $z_t=(s-u)/(t-4)$, is the cosine of the t-channel scattering angle. Since the physical range of s (for $t=m^2$) is $4-m^2\leq s\leq 0$, the magnitude of physical s in Eq. (4) is much smaller than $4m^2$ and we could neglect the s dependence of the integral in Eq. (4) as long as we stayed in the physical t-channel region. This would leave a pure p-wave dependence for $A_t{}^{1(\mathrm{el})}(t,s)$ corresponding to the ρ-resonance angular state. We see, then, in the derivation of Eq. (4) that any double spectral function (without correlated oscillations) will produce p-wave behavior in z_t provided its s-channel threshold is considerably higher than the value of t being considered. We note that the asymptotic behavior in s does not enter at all, provided only that the integral in Eq. (4) converge.

We can see this more exactly by doing the integral in Eq. (4), resulting in

$$A_t{}^{1(\mathrm{el})}(t,s)=\pi\gamma\delta(t-m^2)[(s+5m^2-4)^{1/2}-(4m^2-s)^{1/2}]. \quad (5)$$

The partial-wave expansion of Eq. (5) is

$$A_t{}^{1(\mathrm{el})}(t,s)=2\pi\gamma\delta(t-m^2)\left(\frac{h}{m^2-4}\right)^{1/2}\sum_{\mathrm{odd}\,l}P_l(z_t)(-h)^l \times\left[9m^2-4-(m^2-4)\left(\frac{l}{h(2l-1)}+\frac{h(l+1)}{(2l+3)}\right)\right], \quad (6)$$

with the constant h given by

$$h=(m^2-4)/[9m^2-4+4m(5m^2-4)^{1/2}]. \quad (7)$$

The convergence of Eq. (6) is extremely rapid because $|h|\lesssim 1/18$ for physical m values ($m^2>4$). For any

[6] G. F. Chew and S. C. Frautschi, Phys. Rev. Letters **123**, 1478 (1961).

[7] This approach is similar to that of N. Masuda, Phys. Rev. **175**, 2087 (1968). Masuda assumed Regge-pole behavior instead of approximating the double spectral functions.

$m^2 \gg 4$ (we note that $m_\rho^2 \simeq 30$), it is a very good approximation to set $h=1/18$. This leads to the approximate form

$$A_t^{1(\mathrm{el})}(t,s) \approx \frac{\pi\gamma(m^2-4)\delta(t-m^2)}{3\sqrt{2}m} \times \sum_{\text{odd}\, l} \frac{P_l(z_t)(-1/18)^{l-1}}{2l-1}, \quad (6')$$

which shows how closely the double spectral approximation to $A_t^{1(\mathrm{el})}(t,s)$ approximates a p-wave resonance.

To define the generalized potential, we substitute the t-channel absorptive part given by Eq. (4) (keeping the full s dependence) into Eq. (1) and project out the direct-channel p-wave part. In doing so, we note that the u-channel contribution [the second integral in Eq. (1)] has the same direct-channel p-wave projection as the t-channel contribution and can be included by simply doubling the t-channel contribution. The result is

$$V(s)=\gamma v(s)[(s+5m^2-4)^{1/2}-(4m^2-s)^{1/2}], \quad (8)$$

with

$$v(s)=\frac{1}{s-4}\left[\left(1+\frac{2m^2}{s-4}\right)\ln\left(1+\frac{s-4}{m^2}\right)-2\right]. \quad (9)$$

To use the Frye-Warnock method, we require the right-hand discontinuity of $V(s)$, considered as the inelastic unitarity contribution. This discontinuity follows from inspection of Eq. (8):

$$[V(s)]_R = 2\gamma v(s)(s-4m^2)^{1/2}\theta(s-4m^2) \quad (10a)$$

$$= [1-\eta^2(s)]/2[(s-4)/s]^{1/2}, \quad (10b)$$

where $\eta(s)$ is the inelasticity function to be used in the Frye-Warnock method. Then $\eta(s)$ is given by

$$\eta(s)=\{1-4\gamma v(s)[(s-4m^2)(s-4)/s]^{1/2}\theta(s-4m^2)\}^{1/2}. \quad (11)$$

The p-wave amplitude is given by

$$A(s)=U(s)+V(s), \quad (12)$$

where $U(s)$ is the contribution of those double spectral functions that represent elastic unitarity in the s channel. We can write $U(s)$ in terms of integrals over its right- and left-hand cuts:

$$U(s)=U_R(s)+U_L(s), \quad (13)$$

with

$$U_R(s)=\frac{1}{\pi}\int_4^\infty \frac{ds'[(s'-4)/s']^{1/2}|A(s')|^2}{s'-s} \quad (14R)$$

and

$$U_L(s)=\frac{1}{2\pi}\int_{-\infty}^{4-m^2} \frac{ds'[U(s')]_L}{s'-s}. \quad (14L)$$

We have used elastic unitarity for the integral over the right-hand cut. The integral over the left-hand cut must also be important because it is required to cancel the elastic unitarity integral exactly at threshold. This is because $A(s)$ and $V(s)$ have been defined by p-wave projection and therefore vanish at threshold.[8] Therefore, from Eq. (12), $U(s)$ must also vanish at threshold. This is frequently done, in the absence of a better procedure, by assuming $U_L(s)$ to be a constant. Then a threshold subtraction applied to Eq. (13) would result in an explicit factor of $(s-4)$ for $U(s)$. The factor $(s-4)$ could then be divided out of $A(s)$, $V(s)$, and $U(s)$. In our case, however, we can use the double spectral approximation to identify the left-hand discontinuity of $U(s)$ and calculate $U_L(s)$.

We use the form given by Eq. (3) (with the interchange $s \leftrightarrow t$) for $\rho_{st}^{1(\mathrm{el})}(s,t)$ in Eqs. (1) and (2), and then partial-wave project. This gives the representation

$$U(s)=\bar{U}(s)/(m^2-s), \quad (15)$$

where

$$\bar{U}(s)=\frac{\gamma}{s-4}\int_{4-s}^{0} dt\left(1+\frac{2t}{s-4}\right) \times[(t+5m^2-4)^{1/2}-(4m^2-t)^{1/2}]$$

$$=\frac{32\gamma m^3}{15(s-4)^2}\left[16m^2+5(s-4) -\left(1+\frac{s-4}{4m^2}\right)^{3/2}(16m^2-s+4)\right]. \quad (16)$$

From Eqs. (13)–(16) we see that the double spectral approximation for $U_L(s)$ leads to

$$U_L(s)=U(s)-U_R(s)=[\bar{U}(s)-\bar{U}(m^2)]/(m^2-s). \quad (17)$$

We note that the pole due to the zero-width approximation in $U(s)$ has been removed and that $U_L(s)$ has smooth behavior near $s=m^2$, which it clearly [from Eq. (14L)] must have.

The combination $V(s)+U_L(s)$ now includes the p-wave projection of the full Mandelstam representation [Eq. (1)] except for the right-hand elastic unitarity cut [Eq. (14R)]. This cut will be put in using the Frye-Warnock N/D method. For completeness, we list these equations here[3]:

$$A(s)=N(s)/D(s)=[\eta(s)e^{2i\delta(s)}-1]/2i[(s-4)/s]^{1/2}, \quad (18)$$

$$D(s)=1-\frac{s}{\pi}\int_4^\infty \frac{[(s'-4)/s']^{1/2}2\,\mathrm{Re}N(s')ds'}{s'(s'-s)[1+\eta(s')]}, \quad (19)$$

$$\frac{2\eta(s)\,\mathrm{Re}N(s)}{1+\eta(s)}=B(s)+\frac{1}{\pi}\int_4^\infty \frac{[s'B(s')-sB(s)][(s'-4)/s']^{1/2}2\,\mathrm{Re}N(s')ds'}{s'(s'-s)[1+\eta(s')]}. \quad (20)$$

[8] J. Franklin, Phys. Rev. **139**, B912 (1965).

The Frye-Warnock effective potential is given by[3]

$$B(s)=\mathrm{Re}U_L(s)+\mathrm{Re}V_L(s)+\frac{P}{2\pi}\int_{4m^2}^{\infty}\frac{[1-\eta(s')]ds'}{(s'-s)[(s'-4)/s']^{1/2}},\quad(21)$$

where $V_L(s)$ is the left-hand cut contribution to $V(s)$ and is given by

$$V_L(s)=V(s)-\frac{1}{2\pi}\int_{4m^2}^{\infty}\frac{[V(s')]_R ds'}{s'-s}.\quad(22)$$

Then, using Eq. (10b) for $[V(s)]_R$, our final form for the effective potential is

$$B(s)=\mathrm{Re}U_L(s)+\mathrm{Re}V(s)+\frac{P}{4\pi}\int_{4m^2}^{\infty}\frac{[1-\eta(s')]^2 ds'}{(s'-s)[(s'-4)/s']^{1/2}},\quad(23)$$

with $U_L(s)$ given by Eq. (17), $V(s)$ by Eq. (8), and $\eta(s)$ by Eq. (11). After the integral equation (20) is solved for $\mathrm{Re}N(s)$, this can be used in Eq. (19) to find $D(s)$ and then the phase shift is given by

$$\tan\delta(s)=-\mathrm{Im}D(s)/\mathrm{Re}D(s)=[(s-4)/s]^{1/2}2\,\mathrm{Re}N(s)/\{[1+\eta(s)]\,\mathrm{Re}D(s)\}.\quad(24)$$

If the zero-width approximation to the N/D solution were used to evaluate $U_R(4)$ and the calculation were self-consistent (input ρ mass and width equal to output ρ mass and width), we would necessarily get the result $U_R(4)+U_L(4)=0$ [this follows directly from Eqs. (13), (15), and (16) for $U(s)$], and the p-wave amplitude would have the proper threshold behavior. Instead of using the zero-width approximation in the direct channel, however, we can use the fact that $A(s)$ must vanish at threshold, to determine self-consistency of input and output. The procedure is the following.

(a) A trial m_{in}^2 and γ_{in} are picked to determine $V_L(s)$, $U_L(s)$, and $\eta(s)$.

(b) The Frye-Warnock N/D method is used to determine $N(4)$.

(c) γ_{in} is varied until $N(4)=0$.

(d) m_{out}^2 is found from $D(m_{\mathrm{out}}^2)=0$.

(e) m_{in}^2 is varied and steps (a)–(d) repeated until $m_{\mathrm{out}}^2=m_{\mathrm{in}}^2$, which determines the self-consistent solution.

This procedure does not require determining the output from[9] $\Gamma_{\mathrm{out}}=-N(m^2)/D'(m^2)$ which is a weak link of many bootstrap calculations. The γ_{out} determined by this equation should not be vastly different from γ_{in} if the zero-width approximation is to be considered a good one, but the procedure of steps (a)–(e) should be much better than having to use the above estimate of Γ.

III. DISCUSSION

The method of double spectral approximation described here has many advantages over previous bootstrap calculations. The potential that we define does not mutilate the Mandelstam representation and includes the basic analytic properties of the Mandelstam amplitude. It includes the inelasticity required for pions by the Mandelstam representation and has a good asymptotic behavior that does not require a cutoff or any arbitrary parameter. Not all of the inelasticity is accounted for. The details of inelastic resonances and some final states (for instance, π-ω in π-π scattering[10]) are not included, but these can be added by doing a two-channel ND^{-1} matrix calculation.[11] The important left-hand cut due to the direct-channel double spectral functions is included correctly and this provides proper threshold behavior and a good method for achieving self-consistency.

As for calculation time, the double spectral approximation should not take appreciably longer on a computer than purely elastic N/D bootstrap calculations. The calculation of the effective potential $[B(s)]$ for the Frye-Warnock method would take somewhat longer than a simple no-spectral-function potential, but the major time consumer on a computer is inverting the matrix of the kernel in the integral equation for N, which will be the same for any method. There is a minor simplification in our only requiring $N(4)=0$ to determine a self-consistent γ, rather than integrating over $N(s)$ to look for $D(m_{\mathrm{out}}^2)=0$. Of course, we will want to find m_{out}^2 once we have a self-consistent γ.

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[9] F. Zachariasen, Phys. Rev. Letters **7**, 112 (1961); **7**, 268(E) (1961). The width Γ is related to our constant γ by $\Gamma\simeq[(m^2-4)/9\sqrt{2}m]\gamma$.

[10] Conservation of G parity prevents the right-hand cut of the equivalent potential from corresponding to π-ω intermediate states in the same way that conservation of parity excluded π-π intermediate states.

[11] J. D. Bjorken, Phys. Rev. Letters **4**, 473 (1960); R. L. Warnock, Phys. Rev. **146**, 1109 (1966).