satisfied for all s; i.e. the reduction of $\bar{U}(s,J)$ to $\bar{U}(s)$ when $J \to 0$, 2 would not automatically guarantee the corresponding reduction of $U(s,J)$ to $U(s)$. This type of failure of unitarity is partially corrected in I by adding some terms to the input amplitudes so that unitarity is exactly satisfied when $J \to 0$ only at the resonant energy. Further discussion of this technical point is unwarranted here and rightly belongs to the subject matter of latter work, since it clearly involves highly specialized approximations.

IV. CONCLUSION

We have established the basic formalism for the three-channel generalization of a ρ-bootstrap model in $\pi\pi$ scattering proposed for investigating external spin continuation. The three-channel model was motivated as a means of obtaining a statement of unitarity that was more compatible with the infinite set of conventional channels introduced by external spin continuation. The formalism hinges around the set of amplitudes constructed so as to be free of kinematic zeros throughout the spin continuation, and concerns the formulation of constraints, crossing relations, and unitarity for these amplitudes.

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Interference Effects and Corrections in $A_1 \to 3\pi$ and $A_2 \to 3\pi$ Decays

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Corrections due to overlapping ρ and σ bands in the decay $A_1 \to 3\pi$, and ρ bands in $A_2 \to 3\pi$, are evaluated to first order in the ρ and σ widths. The corrections in the case of the A_1 indicate the use of a smaller anomalous magnetic moment λ_A (or δ) than was previously needed to fit the width of the A_1.

I. $A_1 \to 3\pi$

RECENT current-algebraic "hard-pion" calculations[1] have established a correlation between the decay rates for $\rho \to \pi\pi$ and $A_1 \to 3\pi$, namely,

$$\Gamma_\rho = 141(1-\tfrac{1}{4}\lambda_A)^2 \text{ MeV}, \quad (1)$$

$$\Gamma_{A_1} = \Gamma_{A_1\to\rho\pi} + \Gamma_{A_1\to\sigma\pi} + \Gamma_C, \quad (2)$$

where $\Gamma_{A_1\to\rho\pi}$ is the decay rate into $\rho\pi$,

$$\Gamma_{A_1\to\rho\pi} = 7.0(8+12\lambda_A+5\lambda_A^2) \text{ MeV}, \quad (3)$$

$\Gamma_{A_1\to\sigma\pi}$ is the decay rate into $\sigma\pi$ (undetermined by current algebra, Γ_C is the nonresonant "seagull" contribution, and λ_A is the anomalous magnetic moment of the charged A_1 particle. The experimental value $\Gamma_\rho = 111 \pm 17$ MeV is obtained by choosing

$$\lambda_A = 0.4 \pm 0.3. \quad (4)$$

Equation (4), combined with Eqs. (2) and (3), provides for minimum values of Γ_{A_1}. For example, if $\Gamma_\rho = 120$ MeV, then $\lambda_A = 0.3$ and $\Gamma_{A_1} \geq 78$ MeV. This is to be compared with the most recent experimental compilation,[2] in which $\Gamma_{A_1} = 80 \pm 35$ MeV. The implication is either that the other modes are small or that significant interference effects occur, so that the overall width stays within experimental limits. It is to these interference effects that we address ourselves in this section.

The effect of finite widths may influence current-algebra results in essentially two different ways: (1) The inclusion of a spread in the two-point spectral functions will alter the longitudinal constraints on the vertex functions due to the generalized Ward identities,[3,4] and (2) the use of the altered two-point functions (propagators) will affect the calculation of the four-point tree diagrams, such as $A_1 \to 3\pi$ via ρ mesons. The first type of correction is constrained by minimal-coupling principles to a replacement of the mass m^2 in the inverse σ and ρ propagators by $m^2 - i\Gamma m$.[3] Since quantities such as $\Delta^{-1}(p^2) - \Delta^{-1}(q^2)$ enter the Ward identities, this type of replacement will have no effect within the minimal-coupling framework.

The second type of correction will have quite discernible effects. We will calculate these in the approxi-

[1] H. Schnitzer and S. Weinberg, Phys. Rev. **164**, 1828 (1967); S. G. Brown and G. B. West, Phys. Rev. Letters **19**, 812 (1967); R. Arnowitt, M. H. Friedman, and P. Nath, *ibid.* **19**, 1085 (1967); Phys. Rev. **174**, 1999 (1968); **174**, 2008 (1968); J. Schwinger, Phys. Letters **24B**, 473 (1967); J. Wess and B. Zuméno, Phys. Rev. **163**, 1727 (1967); B. W. Lee and N. T. Nieh, *ibid.* **166**, 1507 (1968); I. S. Gerstein and H. J. Schnitzer, *ibid.* **170**, 1638 (1968); **175**, 1876 (1968).

[2] A. H. Rosenfeld *et al.*, Rev. Mod. Phys. **40**, 77 (1968).

[3] Schnitzer and Weinberg (Ref. 1).

[4] Gerstein and Schnitzer (Ref. 1).

mation of (i) replacing the mass m^2 in the inverse propagator by $m^2 - i\Gamma m$ (Breit-Wigner), and (ii) then going to δ-function limits for all quantities of the form $(\Gamma m/\pi)[(m^2-q^2)^2+(\Gamma m)^2]$ in the integrations over the Dalitz plot. To this order, we find the following: (1) There is no interference among the different decay modes given in Eq. (2) and (2) there is a sizable correction to $A_1 \to 3\pi$ via both the ρ and σ intermediate states. In the case of $A_1 \to 3\pi$ via the ρ, $\lambda_A = 0.3$ will give a correction of about 38%.

We consider the decay of the A_1 into 3π as proceeding via the three diagrams shown in Fig. 1 for the case of a ρ intermediate state, plus three more similar diagrams with the ρ replaced by σ. There is also a direct nonresonant contribution,[4] which we discuss later on. The general Feynman amplitude for the decay $A_1{}^A(P) \to \pi^a(k_1)+\pi^b(k_2)+\pi^c(k_3)$ may be written

$$\epsilon_\mu{}^{(\lambda)}(P,A)M^\mu(k_1a,k_2b,k_3c),$$

where λ is the polarization of the A_1, A, a, b, c are isotopic indices, and $P=k_1+k_2+k_3$. Isotopic-spin invariance permits the reduction

$$M_\mu = A_\mu \delta_{Aa}\delta_{bc} + B_\mu \delta_{Ab}\delta_{ac} + C_\mu \delta_{Ac}\delta_{ab}. \quad (5)$$

Defining $A_1\rho\pi$, $\rho\pi\pi$, $A_1\sigma\pi$, and $\sigma\pi\pi$ couplings by

$$\begin{aligned}
&\mathcal{L}(A_{1\mu}{}^a \to \rho_\nu{}^b(q)+\pi^c(k)) \\
&\quad = \epsilon_{abc}G_{A\rho\pi}[\tfrac{1}{2}m_\rho{}^2(1+\lambda_A)g_{\mu\nu}+(1-\lambda_A)q_\mu k_\nu], \\
&\mathcal{L}(\rho_\lambda{}^a \to \pi^b(k)+\pi^c(k')) = i\epsilon_{abc}G_{\rho\pi\pi}(k-k')_\lambda, \quad (6)\\
&\mathcal{L}(A_{1\lambda}{}^a \to \sigma(q)+\pi^b(k)) = i\delta_{ab}G_{A\sigma\pi}(q-k)_\lambda, \\
&\quad \mathcal{L}(\sigma \to \pi^a(k)+\pi^b(k')) = \delta_{ab}G_{\sigma\pi\pi},
\end{aligned}$$

we find that

$$\begin{aligned}
A_\mu &= (R_\mu{}^{(2)}-R_\mu{}^{(3)})+S_\mu{}^{(1)}, \\
B_\mu &= (R_\mu{}^{(3)}-R_\mu{}^{(1)})+S_\mu{}^{(2)}, \quad (7)\\
C_\mu &= (R_\mu{}^{(1)}-R_\mu{}^{(2)})+S_\mu{}^{(3)},
\end{aligned}$$

where

$$\begin{aligned}
R_\mu{}^{(1)} = \tfrac{1}{2}iG_{\rho\pi\pi}[&m_\rho{}^2(1+\lambda_A)(k_2-k_3) \\
&+(1-\lambda_A)(\sigma_2-\sigma_3)(k_2+k_3)_\mu] \\
&\times 1/(m_\rho{}^2-\sigma_1-i\Gamma_\rho m_\rho), \quad (8)
\end{aligned}$$

$$S_\mu{}^{(1)} = iG_{\sigma\pi\pi}G_{A\sigma\pi}\frac{(2k_1-P)_\mu}{m_\sigma{}^2-\sigma_1-i\Gamma_\sigma m_\sigma}, \quad (9)$$

and the other amplitudes are defined by cyclic permutation of k_1, k_2, and k_3. We have defined $\sigma_i = (P-k_i)^2$ $(i=1,2,3)$. The contact term is omitted for the moment.

From the Appendix, the A_1 decay width is given by

$$\Gamma_{A_1\to 3\pi} = \int d\rho_{3\pi}\,[\tfrac{3}{2}(A,A)+\mathrm{Re}(A,B)], \quad (10)$$

Where (X,Y) denotes the spin-averaged quantity

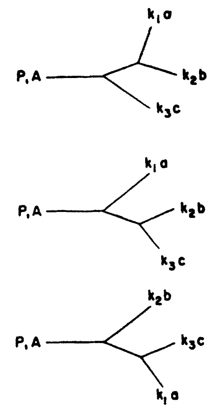

FIG. 1. Diagrams contributing to $A_1 \to 3\pi$ and $A_2 \to 3\pi$ via ρ as the intermediate state.

$\frac{1}{3}X_\alpha{}^*\mathcal{P}^{\alpha\beta}Y_\beta$, where

$$\begin{aligned}
\mathcal{P}^{\alpha\beta} &= \text{spin-1 projection operator} \\
&= -g^{\alpha\beta}+P^\alpha P^\beta/m_{A_1}{}^2.
\end{aligned} \quad (11)$$

The notation $\int d\rho_{3\pi}$ is shorthand for the integration over the three-particle phase space

$$d\rho_{3\pi} \equiv (2\pi)^{-5}(16s^{1/2}\omega_1\omega_2\omega_3)^{-1}d^3k_1d^3k_2d^3k_3. \quad (12)$$

The symmetries existing in the phase-space integrations

$$\begin{aligned}
&\int d\rho_{3\pi}(R^{(i)},R^{(i)}) = I_1, \quad \text{independent of } i \\
&\int d\rho_{3\pi}(R^{(i)},R^{(j)}) = I_2, \quad \text{independent of } i,\ j\ (i\neq j) \\
&\int d\rho_{3\pi}(R^{(i)},S^{(i)}) = I_3, \quad \text{independent of } i \\
&\int d\rho_{3\pi}(R^{(i)},S^{(j)}) = I_4, \quad \text{independent of } i,\ j\ (i\neq j) \quad (13)\\
&\int d\rho_{3\pi}(S^{(i)},S^{(i)}) = I_5, \quad \text{independent of } i \\
&\int d\rho_{3\pi}(S^{(i)},S^{(j)}) = I_6, \quad \text{independent of } i,\ j\ (i\neq j)
\end{aligned}$$

allow us to write (again, omitting the contact contribution for now)

$$\begin{aligned}
\Gamma_{A_1\to 3\pi} &= 2\int d\rho_{3\pi}(R^{(1)},R^{(1)}) - 2\,\mathrm{Re}\int d\rho_{3\pi}(R^{(1)},R^{(2)}) \\
&\quad + \frac{3}{2}\int d\rho_{3\pi}\left(S^{(1)},S^{(1)}+\mathrm{Re}\int d\rho_{3\pi}(S^{(1)},S^{(2)}\right) \quad (14)\\
&\equiv \Gamma_{A_1\rho\pi}{}^{(0)}+\Gamma_{A_1\rho\pi}{}^{(1)}+\Gamma_{A_1\sigma\pi}{}^{(0)}+\Gamma_{A_1\sigma\pi}{}^{(1)}. \quad (15)
\end{aligned}$$

Note that there is no ρ-σ interference.

We now implement the limits

$$\left|\frac{1}{m^2-\sigma-i\Gamma m}\right|^2 \to \frac{\pi}{\Gamma m}\delta(m^2-\sigma)\,, \tag{16}$$

$$\mathrm{Re}\left(\frac{1}{m_1^2-\sigma_1-i\Gamma_1 m_1}\right)^* \frac{1}{m_2^2-\sigma_2-i\Gamma_2 m_2} \to \pi^2\delta(m_1^2-\sigma_1)\delta(m_2^2-\sigma_2)\,, \tag{17}$$

while making use of

$$\int d\rho_{3\pi}(\ )=(2\pi)^{-5}(2m_A)^{-3}\pi^2\int d\sigma_1 d\sigma_2(\ ) \tag{18}$$

for the phase-space integration of the interference term.[5] It should be noted that to sustain this approximation, we must make sure that the intersection of the ρ (or σ) bands lies within the Dalitz plot. Amusingly enough, this is fulfilled if $m_{A_1}^2 \geq 2m_{\rho,\sigma}^2+m_\pi^2$; this means that for canonical values $m_{A_1}=\sqrt{2}m_{\rho,\sigma}$, the intersection lies just at the edge of the Dalitz plot. The *prescription we follow is to calculate the interference terms as if the intersection point lay within the Dalitz plot, and then take* $\frac{1}{2}$ *of the result obtained.*

We can now state our results. Neglecting all terms of $O(m_\pi^2/m_\rho^2)$, and setting $m_{A_1}=\sqrt{2}m_\rho$, and, for simplicity, $m_{A_1}=\sqrt{2}m_\sigma$, we obtain

$$\Gamma_{A_1\rho\pi}{}^{(0)}=\frac{\sqrt{2}G_{A_1\rho\pi}{}^2\, m_\rho{}^3}{768\pi}(9+12\lambda_A+5\lambda_A{}^2)\,, \tag{19}$$

$$\Gamma_{A_1\rho\pi}{}^{(1)}=\frac{\sqrt{2}G_{A_1\rho\pi}{}^2\, m_\rho{}^3}{768\pi}\,\frac{G_{\rho\pi\pi}{}^2}{32}(2+\lambda_A)^2\,, \tag{20}$$

$$\Gamma_{A_1\sigma\pi}{}^{(0)}=\frac{m_\sigma}{192\pi}\,\frac{G_{A_1\sigma\pi}{}^2}{\sqrt{2}}\,, \tag{21}$$

$$\Gamma_{A_1\sigma\pi}{}^{(1)}=\frac{m_\sigma}{192\pi}\,\frac{G_{A_1\sigma\pi}{}^2}{\sqrt{2}}\,\frac{G_{\sigma\pi\pi}{}^2}{16m_\sigma{}^2}. \tag{22}$$

As seen in Eqs. (13) and (14), $\Gamma_{A_1\rho\pi}{}^{(0)}$, $\Gamma_{A_1\sigma\pi}{}^{(0)}$ are the partial widths $A_1\to\rho\pi$, $A_1\to\sigma\pi$ as calculated without interference, and $\Gamma_{A_1\rho\pi}{}^{(1)}$, $\Gamma_{A_1\sigma\pi}{}^{(1)}$ are the interference corrections.

Current algebra states that $G_{A_1\rho\pi}=(2F_\pi)^{-1}$. This folds with Eqs. (19) and (20) to give[6]

$$\Gamma_{A_1}=7.0[8+12\lambda_A+5\lambda_A{}^2+\tfrac{1}{32}G_{\rho\pi\pi}{}^2(2+\lambda_A)]^2 +\frac{G_{A_1\sigma\pi}{}^2}{4\pi}\left(1+\frac{G_{\sigma\pi\pi}{}^2}{16m_\sigma{}^2}\right)\times 11\text{ MeV}. \tag{23}$$

[5] See, e.g., G. Källén, *Elementary Particle Physics* (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1964). For the noninterference terms we use the canonical form $\int d\rho_{3\pi}\sim\int d\sigma \times q(\sigma)/\sqrt{\sigma}\int d\Omega\, k(s,\sigma)/2\sqrt{s}$, where $q(\sigma)$ and $\sqrt{\sigma}$ are the momentum and energy in the c.m. system of pions 1 and 2, and $k(s,\sqrt{\sigma})$ is the momentum of pion 3 in the A_1 c.m. system.

[6] We use $F_\pi\cos\theta_A=94$ MeV (where θ_A is the Cabbibo angle and $m_\rho=765$ MeV. Keeping additional terms in $m_\pi{}^2/m_\rho{}^2$ would tend to decrease the factor 7.0 in Eqs. (3) and (23).

With $G_{\rho\pi\pi}{}^2\simeq 28$ (giving $\Gamma_\rho=115$ MeV) and $G_{\sigma\pi\pi}{}^2\simeq 12m_\sigma{}^2$ (giving $\Gamma_\sigma\cong 260$ MeV), we have

$$\Gamma_{A_1}\cong 7.0[8+12\lambda_A+5\lambda_A{}^2+\tfrac{7}{8}(2+\lambda_A)^2] +(G_{A_1\sigma\pi}{}^2/4\pi)\times 18\text{ MeV}. \tag{24}$$

With $\lambda_A=0.3$, we have

$$\Gamma_{A_1}=117+(G_{A_1\sigma\pi}{}^2/4\pi)\times 18\text{ MeV}. \tag{25}$$

As stated earlier, the correction is 38% in the case of the ρ intermediate state. For the case of the σ, it is ~67%.

Finally, we turn to the contact term. By itself, this is estimated to give a small contribution to Γ_{A_1}, perhaps 15 MeV.[4] The question arises about its interference with the other terms. For this it is crucial to note that current algebra determines the phase of the Feynman amplitude for this "direct" decay mode to be pure imaginary.[4] A typical interference contribution will then take the form [see Eqs. (8) and (9) for the phase of the pole terms]

$$\int d\sigma_1 d\sigma_2\,\mathrm{Re}\left(\frac{ia}{m^2-\sigma_1-i\Gamma m}\right)^*[ib(\sigma_1,\sigma_2)] =\int d\sigma_1 d\sigma_2 (ab)(m^2-\sigma_1)(\pi/\Gamma m)\delta(m^2-\sigma_1) =0. \tag{26}$$

Hence there is no interference between the contact background and the tree diagrams in the present approximation.

II. $A_2\to 3\pi$

This case is similar to the last but considerably simpler, because (1) there is no σ intermediate state, (2) there is no current-algebraic prediction of a contact term, and (3) there is only one $A_2\rho\pi$ coupling constant, which we define by

$$\mathcal{L}(A_{2\mu\nu}{}^A(p+q)\to\rho_\lambda{}^e(p)+\pi^c(q)) =\epsilon^{Aec}G_{A_2\rho\pi}e_{\mu\nu}e_\lambda\epsilon^{\mu\kappa\lambda\sigma}p^\nu p_\kappa q_\sigma. \tag{27}$$

The Feynman amplitude for the decay $A_2{}^A(P)\to \pi^a(k_1)+\pi^b(k_2)+\pi^c(k_3)$ may be written as $e^{\mu\nu}M_{\mu\nu}$, where SU_2 invariance allows the decomposition [similar to Eq. (5)]

$$M_{\mu\nu}=A_{\mu\nu}\delta_{Aa}\delta_{bc}+B_{\mu\nu}\delta_{Ab}\delta_{ac}+C_{\mu\nu}\delta_{Ac}\delta_{ab}. \tag{28}$$

From the Appendix, the decay rate is

$$\Gamma_{A_2\to 3\pi}=\int d\rho_{3\pi}\,[\tfrac{3}{2}(A,A)+\mathrm{Re}(A,B)]\,, \tag{29}$$

where in this case $(X,Y)\equiv\frac{1}{5}X_{\alpha\beta}{}^*\mathcal{P}^{\alpha\beta;\gamma\delta}Y_{\gamma\delta}$, and $\mathcal{P}^{\alpha\beta;\gamma\delta}$ is the spin-2 projection operator

$$\mathcal{P}^{\alpha\beta;\gamma\delta}=-\tfrac{1}{3}\mathcal{P}^{\alpha\beta}\mathcal{P}^{\gamma\delta}+\tfrac{1}{2}(\mathcal{P}^{\alpha\gamma}\mathcal{P}^{\beta\delta}+\mathcal{P}^{\alpha\delta}\mathcal{P}^{\beta\gamma})\,, \tag{30}$$

with $\mathcal{P}^{\alpha\beta} \equiv -g^{\alpha\beta} + P^\alpha P^\beta / m_{A_2}{}^2$. Analogously to Eq. (7), we have

$$\begin{aligned} A_{\mu\nu} &= R_{\mu\nu}{}^{(2)} - R_{\mu\nu}{}^{(3)}, \\ B_{\mu\nu} &= R_{\mu\nu}{}^{(3)} - R_{\mu\nu}{}^{(1)}, \\ C_{\mu\nu} &= R_{\mu\nu}{}^{(1)} - R_{\mu\nu}{}^{(2)}, \end{aligned} \tag{31}$$

with

$$R_{\mu\nu}{}^{(1)} = 2G_{A_2\rho\pi} G_{\rho\pi\pi} k_{1\nu} \epsilon_{\mu\kappa\lambda\sigma} k_1{}^\kappa k_2{}^\lambda k_3{}^\sigma / (m_{A_2}{}^2 - \sigma_1). \tag{32}$$

$R^{(2)}$ and $R^{(3)}$ are defined by cyclic permutation of k_1, k_2, and k_3. As before, $\sigma_i = (P - k_i)^2$ ($i = 1, 2, 3$). Using the symmetries given in Eq. (13), we obtain

$$\Gamma_{A_2 \to 3\pi} = 2 \int d\rho_{3\pi} (R^{(1)}, R^{(1)}) - 2 \int d\rho_{3\pi} (R^{(1)}, R^{(2)}). \tag{33}$$

With the same δ-function approximations as in Sec. I (this time the intersection of the ρ bands is in the middle of the Dalitz plot), we obtain (neglecting all terms in $m_\pi{}^2/m_\rho{}^2$ and using $m_{A_2}{}^2 = 3m_\rho{}^2$)

$$\begin{aligned} \Gamma_{A_2 \to 3\pi} &= \Gamma_{A_2\rho\pi}{}^{(0)} + \Gamma_{A_2\rho\pi}{}^{(1)}, \\ \Gamma_{A_2\rho\pi}{}^{(0)} &= \frac{G_{A_2\rho\pi}{}^2}{3\pi} \frac{m_\rho{}^5}{60\sqrt{3}}, \\ \Gamma_{A_2\rho\pi}{}^{(1)} &= -\frac{G_{A_2\rho\pi}{}^2 G_{\rho\pi\pi}{}^2}{256\pi} \frac{m_\rho{}^5}{60\sqrt{3}}, \end{aligned}$$

or

$$\Gamma_{A_2 \to 3\pi} = \Gamma_{A_2\rho\pi}{}^{(0)} (1 - 3G_{\rho\pi\pi}{}^2/256),$$

where $\Gamma_{A_2\rho\pi}{}^{(0)}$ is the width calculated on the basis of $A_2 \to \rho\pi$ without interference. If $\Gamma_\rho = 115$ MeV, then $G_{\rho\pi\pi}{}^2 \simeq 28$ and the correction is $\sim 33\%$. What this means in practice is that the value of $G_{A_2\rho\pi}$ obtained by setting $\Gamma_{A_2} = \Gamma_{A_2\rho\pi}{}^{(0)}$ is suspected of being in error by about 16% on the low side. This may be of importance in the comparison with experiment of the results of superconvergent sum rules[7] and Veneziano-model relations.[8]

[7] F. Gilman and H. Harari, Phys. Rev. **165**, 1803 (1967); M. Ademollo, G. Veneziano, H. R. Rubenstein, and M. A. Viraroso, Phys. Rev. Letters **19**, 1402 (1967).

[8] H. Goldberg and Y. Srivastava, Phys. Rev. Letters **22**, 749 (1969); **22**, 1340 (1969).

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APPENDIX

The transition amplitude for an isovector particle (index A) into three π mesons (indices a, b, and c) can be written

$$M^{Aabc} = A\delta^{Aa}\delta^{bc} + B\delta^{Ab}\delta^{ac} + C\delta^{Ac}\delta^{ab}, \tag{A1}$$

where we have omitted the Lorentz indices on M, A, B, and C for typographical clarity. If we define the Feynman amplitudes for the decays

$$\begin{aligned} M^{00+-} &= {}_{\text{out}}\langle \pi^0(k_1)\pi^+(k_2)\pi^-(k_3) | A^0 \rangle_{\text{in}} \times (2M 2\omega_1 2\omega_2 2\omega_3)^{1/2}, \\ M^{0000} &= {}_{\text{out}}\langle \pi^0(k_1)\pi^0(k_2)\pi^0(k_3) | A^0 \rangle_{\text{in}} \times (2M 2\omega_1 2\omega_2 2\omega_3)^{1/2}, \end{aligned} \tag{A2}$$

then the decay rate of the A particle is

$$\Gamma_{A \to 3\pi} = \int d\rho_{3\pi} \left(|M^{00+-}|^2 + \frac{1}{3!} |M^{0000}|^2 \right), \tag{A3}$$

where the factor 3! is included because of the Bose identity of the three π^0's. The element of phase space $d\rho_{3\pi}$ was defined in Eq. (12). The notation $|M|^2$ includes an average over the spin of the A meson. In terms of the amplitude A, B, and C,

$$\begin{aligned} M^{00+-} &= A, \\ M^{0000} &= A + B + C. \end{aligned} \tag{A4}$$

There is enough symmetry in the problem [i.e., $\text{Re} \int d\rho_{3\pi}(A,B) = \text{Re} \int d\rho_{3\pi}(A,C) = \text{Re} \int d\rho_{3\pi}(C,A)$, $\int d\rho_{3\pi}(A,A) = \int d\rho_{3\pi}(B,B) = \int d\rho_{3\pi}(C,C)$] to allow (A3) and (A4) to combine to give

$$\Gamma_{A \to 3\pi} = \int d\rho_{3\pi} [\tfrac{3}{2}(A,A) + \text{Re}(A,B)]. \tag{A5}$$

The notation (A,B) denotes the spin-averaged scalar product

$$(A,B) \equiv (2J+1)^{-1} A_{\mu_1 \cdots \mu_J}{}^* \mathcal{P}^{\mu_1 \cdots \mu_J; \nu_1 \cdots \nu_J} B_{\nu_1 \cdots \nu_J}, \tag{A6}$$

where $\mathcal{P}$ is the spin-J projection operator.