External Spin Continuation in a Three-Channel ρ Bootstrap*

M. L. THIEBAUX
Department of Physics and Astronomy, University of Massachusetts, Amherst, Massachusetts 01002
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External spin continuation is proposed in a three-channel self-consistent ρ bootstrap in which the channels are composed of pions and first pion recurrences. Emphasis is placed on presenting those aspects of the problem that may survive generalization to other multichannel bootstrap situations. The relationship between the existence of recurrences and the statement of unitarity is studied in an endeavor to treat internal and external particles more uniformly than in a previously proposed one-channel problem. Unconventional amplitudes are constructed which, unlike the conventional helicity amplitudes, possess the property that kinematic zeros are uniformly factorizable under external spin continuation. The set of spin-dependent coupling constants introduced into the problem satisfies a number of constraints due to the nonexistence of certain types of couplings, to Bose-Einstein statistics, and to redundancies in the channel definitions. Crossing relations, unitarity, and a unitarization procedure for the unconventional amplitudes are formulated.

I. INTRODUCTION

THERE has been proposed[1] a ρ-meson bootstrap model in $\pi\pi$ scattering in which the bootstrap conditions and the $\pi\pi$ p-wave phase shift generate an implicit relation between the analytically continued spin J and mass M of an external pion. This view of the ρ bootstrap provides us with a particular framework in which to study the possibility of the Regge continuation of an external particle.

The connection between recent bootstrap investigations on the one hand, and the questions we hope to answer in the program started in I and continued in the present work on the other hand, is explained below.

Several external scattering systems have been studied as models in which to test the idea of a self-consistent ρ, or more generally, a self-consistent set of particles.[2,3] The statement of self-consistency itself can assume a variety of forms. For example, one could demand strictly self-consistent properties of the ρ (input mass equals output mass, etc.) and then, if the model does in fact determine such properties, compare them with the physical values. Or, as was done in a Reggeized ρ bootstrap,[4] one could search for input ρ Regge trajectory parameters, including setting the input mass equal to the physical ρ mass, that yield an amplitude best satisfying crossing symmetry and bypass the specific question of the existence of an output resonance.

Thus there is no single definition of self-consistency, nor is there general agreement on what finite systems of external particles may best demonstrate the self-consistency hypothesis of a given set of internal particles, with the exception that multichannel systems with their inherent inelastic effects are recognized to have a better record of achieving self-consistency than single-channel systems.[5]

Since an ultimate bootstrap theory of hadronic matter, assuming one is ever realized, must very likely encompass an infinite number of channels, the obvious enormity of the task of constructing such a theory suggests turning away our attention to more immediately fruitful problems. However, any conventional single-channel or multichannel scattering system is extendible to a system of greater dimensionality through the addition of channels in which the original external particles are replaced by their Regge recurrences. While these recurrence channels, extended, say, from only a few original channels, may account for only a minor part of the totality of channels in an ultimate theory, there is an obvious computational advantage in carrying out such an extension and it is, therefore, one that deserves exploitation. The advantage is that an infinite number of channels are introduced at once if the recurrence channels stemming from the Regge recurrences of a particular external particle are handled in a uniform way, i.e., if the spin of the particle is considered as a continuous variable. Accordingly, we call the infinite set of channels labeled by a continuous spin variable a *continuous* channel when it is necessary to distinguish it from a conventional finite set of *discrete* channels.

The complexity that a bootstrap system would assume, even in the simplest example of this type of extension to an infinite-channel system, correspondingly admits a larger variety of definitions of self-consistency. Thus, in a suitably chosen model, we could again demand strict self-consistency for a set of internal particles (input masses equal output masses, etc.) for a fixed physical set of Regge parameters describing the external particles and then compare the internal self-consistent properties with the physical ones. Alternatively, we could demand, as well, that input masses equal output masses equal physical masses, etc., and satisfy these equalities by adjusting variable parameters

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[1] M. L. Thiebaux, Phys. Rev. 170, 1244 (1968), hereafter referred to as I; 173, 1806(E) (1968).

[2] F. Zachariasen and C. Zemach, Phys. Rev. 128, 849 (1962).

[3] L. A. P. Balázs and S. M. Vaidya, Phys. Rev. 140, B1025 (1965); J. Boguta and H. W. Wyld, Jr., *ibid.* 164, 1996 (1967).

[4] W. J. Abbe, P. Kaus, P. Nath, and Y. N. Srivastava, Phys. Rev. 154, 1515 (1967).

[5] J. R. Fulco, G. L. Shaw, and D. Y. Wong, Phys. Rev. 137, B1242 (1965).

describing the external particles and then compare the external parameters, so determined, with physical values. It is clear that as tests of self-consistency these two procedures are the same in content. The proposed ρ-bootstrap model in I is essentially a realization of the latter procedure (in retrospect the former procedure may actually be easier to work out) and is the simplest such extension of the original ρ bootstrap[2] in $\pi\pi$ scattering to an infinite-channel system. The long-range emphasis in this model is not necessarily to establish an improved ρ bootstrap or to determine a realistic external pion trajectory, but to discover and explore features that seem to be characteristic of the general problem of external spin continuation.

Should self-consistency be actually achieved in the proposed model, a more immediate connection with other current work is possible. A feasible version of the model in which the ρ is more properly treated as a Regge pole would involve Reggeon-Reggeon–physical-particle couplings, and would therefore be closely related to the successful multi-Reggeon exchange model of production processes.[6]

The specific questions we wish to answer in the immediate context of the model are: (i) Can crossing symmetry approximated in a hadron scattering process (specifically $\pi\pi$ scattering) in the form of a self-consistent bootstrap determine a mass-spin relation for an external particle, and if so, (ii) does this relation bear any resemblance to a conventional Regge trajectory? In question (ii), particular attention might be paid to the analyticity of the trajectory and to its slope where it passes through the physical pion. An affirmative answer to question (i) was tentatively established in I, but a number of difficulties arising in that study were inadequately treated. Therefore, it is the intent of this paper to reexamine some of these difficulties in their own right, especially those concerning the relationship between unitarity and recurrences, before attempting a numerical evaluation of trajectory parameters. Furthermore, we wish to confine the content of this paper largely to the formal development of generalizable aspects of external spin continuation in a multichannel problem, and defer for following work the more specialized aspects, such as the formulation and solution of the ρ-bootstrap equations.

The key steps in I are briefly reviewed in this paragraph, since their logical sequence is essentially the same as the sequence of steps in the present calculation: (i) The helicity amplitudes for $\pi^*\pi \to \pi\pi$ via intermediate ρ formation and decay are computed where π^* has spin J, odd normality, mass M, and otherwise carries the quantum numbers of the pion, while the ρ is treated as an elementary vector meson of known mass and width. The amplitudes contain two coupling constants which presumably have an *a priori* unknown dependence on J and M. (ii) The ρ-exchange crossed amplitudes are constructed and the helicity p waves are projected out. (iii) The partial waves thus obtained are made to satisfy elastic unitarity by a phase N/D method, based on the p-wave phase-shift input. (iv) The unitarized ρ-exchange amplitudes are then matched to the direct amplitudes computed in step (i) with a sufficient number of conditions so that the mass-spin relation is determined. A certain amount of information about the coupling constants is incidentally determined as well.

In Sec. II, a three-channel generalization of the bootstrap model studied in I is formulated in which the channels are composed of pions and first pion recurrences and where again one external particle (in the initial state, say) is continued in spin. In this way, the recurrences are explicitly introduced into the unitarity sum and we thus approach a more uniform treatment of internal and external particles. Most of the results of this section [with the exception of the explicit structure of the Q and $Q(J)$ matrices derived in Sec. II D] are readily extended to more channels and, therefore, constitute the beginnings of a general model-independent formalism for studying external spin continuation. Much of the effort in this section is directed toward the construction of linear combinations of partial-wave amplitudes such that under the continuation of an external spin, they are *uniformly* free of kinematic zeroes. These combinations are the U amplitudes of Sec. II E where the meaning of the preceding statement will be clarified. Partial-wave ρ-exchange amplitudes are constructed in a standard way in Sec. III and the corresponding U amplitudes are found. A statement of unitarity for the U amplitudes and the outline of a tentative unitarization procedure are formulated.

Further progress in the three-channel bootstrap is considerably more model-dependent and involves specialized approximations that will be the subject of later work.

II. THREE-CHANNEL AMPLITUDES

A. Unitarity Sum

We require some preliminary consideration of the statement of unitarity here. For the moment we do not need to assume any simplifying reduction of the number of intermediate states in the unitarity sum. Where one of the intermediate particles is a pion, the question arises of whether or not to continue the amplitude in the spin of this pion simultaneously with that of the external pion. Furthermore, the question may equally well be asked of any pion recurrences belonging to the intermediate states.

The question is plausibly answered in I, but perhaps not in a truly convincing way. There it is argued that if the model is to admit Regge recurrences as well as π's in the intermediate states of the unitarity sum, the spins of these recurrences as well as of the π's should

[6] N. F. Bali, G. F. Chew, and A. Pignotti, Phys. Rev. Letters **19**, 614 (1967); Phys. Rev. **163**, 1572 (1967).

be regarded as discrete quantities always fixed at their physical values in contrast to the continuum of spin values of the external π^*. This is made plausible by showing that there appears to be no sensible way to continue analytically in the spins carried by intermediate states, despite the temptation to say that this somehow ought to be done, since it seems inconsistent to treat the external and internal π^*'s in unlike manner.

Here, we adopt a more positive stand by stating that the correct way to treat the recurrences in the unitarity sum follows from the observations that (i) for any of the discrete physical values J of an external spin, an invariant set of physical intermediate states, with or without recurrences, occurs in the unitarity sum independently of J, and (ii) the analytic continuation in external spin is nothing more than a type of smooth interpolation of the amplitude between the physically meaningful values. The first observation is equivalent to the fact that any channel which communicates with a two-particle channel $A+B$ also communicates with channel $A+B^*$, where particles B and B^* differ by an even number of units of spin, but otherwise are identical. The obvious smooth interpolation of observation (i) to arbitrary values of J is then that the set of intermediate states *remains* invariant with respect to arbitrary changes in J.

It would clearly be desirable to construct a model of sufficient flexibility so that pion recurrences are a possibility and are introduced, at least tentatively, into the unitarity sum. This feature complicates the problem in that it must then be solved by many-channel techniques. For example, if only two-particle intermediate states consisting of pions and the first n pion recurrences are allowed in the unitarity sum, the model then has $\frac{1}{2}(n+1)(n+2)$ *discrete* channels which may be thought of as special cases of the $n+1$ *continuous* channels consisting of a π^* of continuous spin J and a pion or one of the recurrences. In the case of the ρ-exchange model, there would be $(n+1)(3n+2)$ *a priori* unknown J-dependent coupling constants.

B. Three-Channel Problem

We consider the case $n=1$ with three *discrete* channels, which are special cases of two *continuous* channels, and ten J-dependent coupling constants in the ρ-exchange model. Let channel index $i=1, 2, 3$ denote discrete channels $\pi\pi$, $\pi'\pi$, and $\pi'\pi'$, respectively, where π' is the first recurrence; let index $\xi=\alpha$, β denote continuous channels $\pi^*\pi$, $\pi^*\pi'$, respectively.

It is of interest at this point to count up the number of independent helicity states within these channels that communicate with the ρ. We assume that all three coupling particles have isotopic spin$=1$, in which case there is only one way that the isotopic spins couple, and hence the isotopic spin degree of freedom does not contribute to the multiplicity of states.

For $i=1$, there is only one state communicating with the ρ. In channel 2, of the five helicity states, angular momentum and parity conservation reduce these to only two states communicating with the ρ, while for channel 3, angular momentum and parity conservation and Bose-Einstein statistics reduce the number of independent states to five. Hence, there are a total of eight independent states in the discrete channels.

In channel α there are two independent helicity states, while in channel β there are eight. There are then a total of ten states in the continuous channels and therefore a maximum of ten J-dependent coupling constants when each state is coupled to the ρ.

We note that channels 1 and 2 may be considered as special cases of channel α, while channel 3 may be considered as a special case of channel β. Hence, the three coupling constants of channels 1 and 2 are special cases of the two J-dependent channel-α coupling constants, while the five channel-3 coupling constants are special cases of the eight J-dependent channel-β coupling constants. Also, channel 2 is a special case of channel β. These cross relations between the discrete and continuous channels contribute to the existence of certain constraints among the ten J-dependent coupling constants. These constraints are discussed more fully in Sec. II D 4.

C. Resonant Scattering Amplitudes

Let us suppose that an arbitrary two-particle helicity partial-wave scattering amplitude may be written as the sum of *n resonant amplitudes*, i.e., amplitudes each of which represents the formation and subsequent decay of some resonant state or unstable particle between external two-particle states of definite helicities and angular momentum. A scattering amplitude then has the form of an n-term dyad,

$$F(J_3\sigma_3,J_4\sigma_4;J_1\sigma_1,J_2\sigma_2)=\sum_{i=1}^{n} P_i\langle i;\sigma_3-\sigma_4|T_i\|J_3\sigma_3\rangle \times R|J_4\sigma_4\rangle\rangle\langle i;\sigma_1-\sigma_2|T_i\|J_1\sigma_1\rangle R|J_2\sigma_2\rangle\rangle, \quad (1)$$

where P_i contains the pole and form factor parts of the propagator of resonant state i, T_i is the transition operator connecting an external two-particle state to i and includes the angular momentum projection operator, $\langle i;\sigma|$ is the state describing resonance i at rest with spin projection σ along the z axis, $|J_k\sigma_k\rangle$ is a plane-wave state of particle of spin J_k, helicity σ_k moving in the positive z direction, and R is a rotation by π about the y axis.

D. ρ-Formation Amplitudes

1. Notation

The scattering amplitude representing the formation and decay of an intermediate ρ is a special case of Eq.

(1) in which $n=1$ and the formation or inverse decay amplitude is

$$\langle\rho;\sigma_1-\sigma_2|T\|J_1\sigma_1\rangle R|J_2\sigma_2\rangle\rangle.$$

We now introduce some compact notations for these formation amplitudes as further specialized to the channels described in Sec. II B. We define

$$\Gamma_{\sigma 0}{}^{\alpha}(J)=\langle\rho;\sigma|T\|J\sigma\rangle R|00\rangle\rangle \tag{2}$$

and

$$\Gamma_{\sigma\tau}{}^{\beta}(J)=(-)^{\tau}\langle\rho;\sigma-\tau|T\|J\sigma\rangle R|2\tau\rangle\rangle \tag{3}$$

to be the ρ-formation amplitudes in channels α and β, respectively. It is always understood that the spin-J π^* has variable mass M, the spin-2 π' has unknown mass μ, and the spin-0 π has mass 1, and hence no notation for the particle masses is included in (2) and (3). The factor $(-)^{\tau}$ in Eq. (3) ensures that the spin-2 π' is particle (2) in the sense of Jacob and Wick.[7] Also suppressed in Eqs. (2) and (3) is a notation for the isotopic spin part of the state vectors. In the absence of that notation, which would explicitly show the isotopic spin-exchange antisymmetry, we must remark that Bose-Einstein statistics require

$$\|a\rangle R|b\rangle\rangle=-|R|b\rangle|a\rangle\rangle. \tag{4}$$

For ρ formation in the discrete channels, we define

$$\Gamma_{\sigma\tau}{}^{i}=\Gamma_{\sigma\tau}{}^{\xi_i}(J_i)\,, \tag{5}$$

where $i=1, 2, 3$ is the discrete-channel index, $\xi_1=\xi_2=\alpha$, $\xi_3=\beta$, $J_1=0$, and $J_2=J_3=2$.

Conservation of parity requires

$$\Gamma_{\sigma,\tau}{}^{\xi}=\Gamma_{-\tau,-\sigma}{}^{\xi}\,, \tag{6}$$

while conservation of angular momentum requires $|\sigma-\tau|\leq 1$. Hence, a choice of independent helicity subscript pairs $(\sigma\tau)$ is (00), (01), (10), (11), (12), (21), (22), and (32). Accordingly, we introduce another, more uniform notation $\Gamma_k(J)$, $k=1,\cdots$, 10 for the 10 independent continuous-channel formation amplitudes by the following definitions:

$$\begin{aligned}
&\Gamma_1(J)=\Gamma_{00}{}^{\alpha}(J)\,, \quad \Gamma_2(J)=\Gamma_{10}{}^{\alpha}(J)\,,\\
&\Gamma_3(J)=\Gamma_{00}{}^{\beta}(J)\,, \quad \Gamma_4(J)=\Gamma_{01}{}^{\beta}(J)\,,\\
&\Gamma_5(J)=\Gamma_{10}{}^{\beta}(J)\,, \quad \Gamma_6(J)=\Gamma_{11}{}^{\beta}(J)\,,\\
&\Gamma_7(J)=\Gamma_{12}{}^{\beta}(J)\,, \quad \Gamma_8(J)=\Gamma_{21}{}^{\beta}(J)\,,\\
&\Gamma_9(J)=\Gamma_{22}{}^{\beta}(J)\,, \quad \Gamma_{10}(J)=\Gamma_{32}{}^{\beta}(J)\,,
\end{aligned} \tag{7}$$

and a notation Γ_j, $j=1,\cdots,8$ for the eight independent discrete-channel formation amplitudes:

$$\begin{aligned}
&\Gamma_1=\Gamma_1(0)\,, \quad \Gamma_2=\Gamma_1(2)\,,\\
&\Gamma_3=\Gamma_2(2)\,, \quad \Gamma_4=\Gamma_3(2)\,,\\
&\Gamma_5=\Gamma_4(2)\,, \quad \Gamma_6=\Gamma_6(2)\,,\\
&\Gamma_7=\Gamma_7(2)\,, \quad \Gamma_8=\Gamma_9(2).
\end{aligned} \tag{8}$$

[7] M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) 7, 404 (1959).

2. *Channel* α

We now construct the channel-α formation amplitudes $\Gamma_{\sigma 0}{}^{\alpha}(J)$ in terms of a pair of covariant coupling constants g_1 and g_2. In the covariant tensor representation, the spin state of any one of the particles under consideration is a Clebsch-Gordon product of basic spin-1 vector states.[8] Thus, in the rest frame of the π^*, the π^* spin state is the purely spacelike rank J tensor $(\Phi_{\sigma}{}^{J})^{\alpha\cdots\beta}$, when J has integral value. The π spin state is the scalar $\Phi_0{}^0=1$, while in the c.m. or ρ rest frame, the on-shell ρ spin state is the purely spacelike rank-1 tensor $(\Phi_{\sigma}{}^1)^{\alpha}$.

Introducing two provisional coupling constants g_1' and g_2', we can write

$$\Gamma_{\sigma 0}{}^{\alpha}(J)=(g_1'f_{\alpha}f_{\beta}+g_2'g_{\alpha\beta})f_{\gamma}\cdots f_{\delta}(\Phi_{\sigma}{}^{1*})^{\alpha}(\Phi_{\sigma}{}^{J})^{\beta\gamma\cdots\delta}, \tag{9}$$

where the four-vector f is some linear combination of the π^* four-momentum p^* and the π four-momentum p, namely, $f=ap^*-bp$. We carry out the reduction of Eq. (9), use the notation defined in Eq. (7), and obtain

$$\Gamma_i(J)=\sum_{j=1}^{2}Q_{ij}(J)g_j(J)\,,\quad i=1,2 \tag{10}$$

where the elements of the matrix $Q(J)$ factorize into

$$Q_{ij}(J)=C_{ij}(J)k_j(J)\,. \tag{11}$$

The elements of the nonsingular matrix $C(J)$ are

$$C_{11}(J)=s^{-1/2},\quad C_{12}(J)=[\tfrac{1}{2}(J+1)]^{-1/2}\gamma(S,M^2;1)\,,$$
$$C_{22}(J)=J^{-1/2},\quad C_{21}(J)=0\,,$$

and the elements of the diagonal matrix $k(J)$ are

$$k_1(J)=S^{J+1}(M^2,1)\,,$$
$$k_2(J)=S^{J-1}(M^2,1)\,,$$

where

$$s=(p^*+p)^2\,,$$
$$S(x,y)=[s^2-2s(x+y)+(x-y)^2]^{1/2}\,,$$
$$\gamma(x,y;z)=\tfrac{1}{2}(xy)^{-1/2}(x+y-z)\,.$$

We have replaced the provisional coupling constants by new ones according to the definitions

$$g_1(J)=-\tfrac{1}{2}[J!/(2J-1)!!]^{1/2}(a+b)b^J(2M)^{-J}g_1'$$

and

$$g_2(J)=[(J+1)!/2(2J-1)!!]^{1/2}b^{J-1}(2M)^{J-1}g_2'.$$

3. *Channel* β

In similar fashion, we construct the channel-β formation amplitudes $\Gamma_{\sigma\tau}{}^{\beta}(J)$ in terms of eight covariant coupling constants $g_3,\cdots,g_{10}$. Here, the π' has four-momentum p and is described in its rest frame by the purely spacelike rank-2 tensor $(\Phi_{-\tau}{}^2)^{\alpha\beta}$. In terms of a

[8] C. Zemach, Phys. Rev. 140, B97 (1965).

set of provisional coupling constants $g_3', \cdots, g_{10}'$, we have

$$\Gamma_{\sigma\tau}{}^{\beta}(J) = (g_3' f_\alpha f_\beta f_\gamma f_\delta f_\lambda f_\mu + g_4' g_{\alpha\beta} f_\gamma f_\delta f_\lambda f_\mu + g_5' g_{\alpha\delta} f_\beta f_\gamma f_\lambda f_\mu + g_6' f_\alpha f_\beta g_{\gamma\delta} f_\lambda f_\mu + g_7' g_{\alpha\beta} g_{\gamma\delta} f_\lambda f_\mu + g_8' g_{\alpha\delta} g_{\beta\lambda} f_\gamma f_\mu + g_9' f_\alpha g_{\beta\delta} g_{\gamma\lambda} f_\mu + g_{10}' g_{\alpha\delta} g_{\beta\lambda} g_{\gamma\mu}) \times f_\nu \cdots f_\epsilon (\Phi_{\sigma-\tau}{}^1)^{\alpha} (\Phi_{-\tau}{}^2)^{\beta\gamma} (\Phi_\sigma{}^J)^{\delta\lambda\mu\nu\cdots\epsilon}. \quad (12)$$

The reduction of Eq. (12) yields

$$\Gamma_i(J) = \sum_{j=3}^{10} Q_{ij}(J) g_j(J), \quad i=3,\cdots,10, \quad (13)$$

where the elements of the matrix $Q(J)$ again factorize into

$$Q_{ij}(J) = C_{ij}(J) k_j(J). \quad (14)$$

In turn, the nonzero elements of the nonsingular matrix $C(J)$ are

$$C_{33}(J) = s^{-1/2},$$
$$C_{34}(J) = (\sqrt{\tfrac{2}{3}})\gamma_2,$$
$$C_{35}(J) = [\tfrac{1}{2}(J+1)]^{-1/2}\gamma_1,$$
$$C_{36}(J) = -4[\tfrac{3}{2}(J+1)]^{-1/2}\gamma_3 s^{-1/2},$$
$$C_{37}(J) = -[\tfrac{3}{2}(J+1)]^{-1/2}\gamma_2\gamma_3,$$
$$C_{38}(J) = -2[\tfrac{3}{2}(J+1)(J+2)]^{-1/2}\gamma_1\gamma_3,$$
$$C_{39}(J) = -[\tfrac{3}{2}(J+1)(J+2)]^{-1/2} s^{-1/2}(\tfrac{1}{2}+\gamma_3^2),$$
$$C_{3,10}(J) = [\tfrac{3}{2}(J+1)(J+2)(J+3)]^{-1/2}\gamma_1(\tfrac{1}{2}+\gamma_3^2),$$
$$C_{44}(J) = \sqrt{\tfrac{1}{2}},$$
$$C_{47}(J) = -[2(J+1)]^{-1/2}\gamma_3,$$
$$C_{48}(J) = [2(J+1)(J+2)]^{-1/2},$$
$$C_{4,10}(J) = -[2(J+1)(J+2)(J+3)]^{-1/2}\gamma_3,$$
$$C_{55}(J) = J^{-1/2},$$
$$C_{57}(J) = (12J)^{-1/2},$$
$$C_{58}(J) = -2[3J(J+2)]^{-1/2}\gamma_3,$$
$$C_{5,10}(J) = [3J(J+2)(J+3)]^{-1/2}(\tfrac{1}{2}+\gamma_3^2),$$
$$C_{66}(J) = (Js)^{-1/2},$$
$$C_{67}(J) = \tfrac{1}{2}J^{-1/2}\gamma_2,$$
$$C_{68}(J) = [J(J+2)]^{-1/2}\gamma_1,$$
$$C_{69}(J) = [J(J+2)]^{-1/2}s^{-1/2}\gamma_3,$$
$$C_{6,10}(J) = [J(J+2)(J+3)]^{-1/2}\gamma_1\gamma_3,$$
$$C_{77}(J) = (2J)^{-1/2},$$
$$C_{7,10}(J) = [8J(J+2)(J+3)]^{-1/2},$$
$$C_{88}(J) = [2J(J-1)]^{-1/2},$$
$$C_{8,10}(J) = -[2J(J-1)(J+3)]^{-1/2}\gamma_3,$$
$$C_{99}(J) = -[4J(J-1)]^{-1/2}s^{-1/2},$$
$$C_{9,10}(J) = [4J(J-1)(J+3)]^{-1/2}\gamma_3,$$
$$C_{10,10}(J) = [8J(J-1)(J-2)]^{-1/2},$$

and the elements of the diagonal matrix $k(J)$ are

$$k_3(J) = S^{J+3}(M^2,\mu^2),$$
$$k_4(J) = k_5(J) = k_6(J) = S^{J+1}(M^2,\mu^2),$$
$$k_7(J) = k_8(J) = k_9(J) = S^{J-1}(M^2,\mu^2),$$
$$k_{10}(J) = S^{J-3}(M^2,\mu^2).$$

In these expressions we have used the abbreviated notations

$$\gamma_1 = \gamma(s,M^2;\mu^2),$$
$$\gamma_2 = \gamma(s,\mu^2;M^2),$$
$$\gamma_3 = -\gamma(M^2,\mu^2;s).$$

Again, the provisional coupling constants are replaced by new ones, appearing in Eq. (13), defined by

$$g_3(J) = -\tfrac{1}{2}[2J!/3(2J-1)!!]^{1/2} \times (a+b)a^2 b^J (2\mu)^{-2}(2M)^{-J} g_{3'},$$
$$g_4(J) = [J!/(2J-1)!!]^{1/2} a b^J (2\mu)^{-1}(2M)^{-J} g_{4'},$$
$$g_5(J) = [(J+1)!/3(2J-1)!!]^{1/2} a^2 b^{J-1}(2\mu)^{-2}(2M)^{1-J} g_{5'},$$
$$g_6(J) = -\tfrac{1}{8}[(J+1)!/(2J-1)!!]^{1/2} \times (a+b) a b^{J-1}(2\mu)^{-1}(2M)^{1-J} g_{6'},$$
$$g_7(J) = [(J+1)!/(2J-1)!!]^{1/2} b^{J-1}(2M)^{1-J} g_{7'},$$
$$g_8(J) = \tfrac{1}{2}[(J+2)!/(2J-1)!!]^{1/2} a b^{J-2}(2\mu)^{-1}(2M)^{2-J} g_{8'},$$
$$g_9(J) = \tfrac{1}{2}[(J+2)!/(2J-1)!!]^{1/2}(a+b) b^{J-2}(2M)^{2-J} g_{9'},$$
$$g_{10}(J) = [(J+3)!/(2J-1)!!]^{1/2} b^{J-3}(2M)^{3-J} g_{10'}.$$

4. *Coupling-Constant Constraints*

When the continuous-channel formation amplitudes are specialized to the discrete channels, a number of constraints are found to exist among the coupling constants. These arise from (a) the nonexistence of certain types of couplings when $J=0$, 2, (b) Bose-Einstein statistics in channel β at $J=2$, and (c) the redundancy between channel α at $J=2$ and channel β at $J=0$.

Type (a) constraints are readily derived by inspection of Eqs. (9) and (12). These are

$$0 = g_2(0) = g_5(0) = g_6(0) = g_7(0) = g_8(0) = g_9(0) = g_{10}(0) = g_{10}(2). \quad (15)$$

Type (b) constraints are found by first applying rule (4) to Eq. (3) at $J=2$ to obtain

$$\Gamma_{\sigma\tau}{}^{\beta}(2) = \Gamma_{\tau\sigma}{}^{\beta}(2). \quad (16)$$

From definitions (7) the only nontrivial applications of this result are $\Gamma_4(2) = \Gamma_5(2)$ and $\Gamma_7(2) = \Gamma_8(2)$, or equivalently in terms of elements of the matrix $Q(J)$ evaluated at $J=2$, $M=\mu$,

$$Q_{44}(2)g_4(2) + Q_{47}(2)g_7(2) + Q_{48}(2)g_8(2) = Q_{55}(2)g_5(2) + Q_{57}(2)g_7(2) + Q_{58}(2)g_8(2)$$

and

$$Q_{77}(2)g_7(2) = Q_{88}(2)g_8(2).$$

We then find that these equations are identically satisfied at all energies if and only if

$$g_4(2)=g_5(2) \quad \text{and} \quad g_7(2)=g_8(2). \tag{17}$$

Type (c) constraints can be derived by applying rule (4) to Eq. (3) at $J=0$ to obtain

$$\Gamma_{0\sigma}{}^{\beta}(0)=\Gamma_{\sigma 0}{}^{\alpha}(2), \tag{18}$$

which, by definitions (7), becomes $\Gamma_3(0)=\Gamma_1(2)$ and $\Gamma_4(0)=\Gamma_2(2)$. Again, in terms of elements of $Q(J)$, these are

$$Q_{33}(0)g_3(0)+Q_{34}(0)g_4(0)=Q_{11}(2)g_1(2)+Q_{12}(2)g_2(2)$$

and

$$Q_{44}(0)g_4(0)=Q_{22}(2)g_2(2),$$

both of which are identically satisfied at all energies if and only if

$$g_3(0)=g_1(2) \quad \text{and} \quad g_4(0)=g_2(2). \tag{19}$$

Equations (15), (17), and (19) together constitute 12 constraints among the coupling constants. That these are very simple in form is attributed to the way certain J-dependent factors were absorbed into the provisional coupling constants g_k'.

The way in which the coupling-constant constraints are to be incorporated into a three-channel bootstrap model will be seen in later work. In the one-channel model of I, where there is only one constraint [of type (a)] it is found that this constraint selects one of the two possible solutions to the bootstrap equations. Also in I, Bose-Einstein statistics do not emerge as an additional requirement, since in the simple system under consideration it merely happens to be consistent with the conservation laws of parity, isotopic spin, and angular momentum. Hence, no constraints of type (b) exist in the system, nor of type (c) since there is only the one continuous channel α.

5. Discrete Channels

As we have already observed, the discrete channels may be considered as special cases of the continuous channels. Hence we may specialize Eq. (13) to the discrete channels by using rule (5) and definitions (8). The results so obtained can be further reduced, using the coupling-constant constraints and properties of $Q(J)$, to the form

$$\Gamma_i=\sum_{j=1}^{8} Q_{ij}g_j \tag{20}$$

in which

$$\begin{aligned} g_1&=g_1(0), & g_2&=g_1(2), \\ g_3&=g_2(2), & g_4&=g_3(2), \\ g_5&=g_4(2), & g_6&=g_6(2), \\ g_7&=g_7(2), & g_8&=g_9(2), \end{aligned} \tag{21}$$

and the elements of Q factorize into

$$Q_{ij}=C_{ij}k_j, \tag{22}$$

where the nonzero elements of the 8×8 matrix C are

$$\begin{aligned} C_{11}&=C_{11}(0), & C_{22}&=C_{11}(2), \\ C_{23}&=C_{12}(2), & C_{33}&=C_{22}(2), \\ C_{44}&=C_{33}(2), & C_{45}&=2C_{34}(2), \\ C_{46}&=C_{36}(2), & C_{47}&=2C_{37}(2), \\ C_{48}&=C_{39}(2), & C_{55}&=C_{44}(2), \\ C_{57}&=C_{47}(2)+C_{48}(2), & C_{66}&=C_{66}(2), \\ C_{67}&=C_{67}(2)+C_{68}(2), & C_{68}&=C_{69}(2), \\ C_{77}&=C_{77}(2), & C_{88}&=C_{99}(2), \end{aligned}$$

and the elements of the diagonal matrix k are

$$\begin{aligned} k_1&=S(1,1), \quad k_2=S^3(\mu^2,1), \\ k_3&=S(\mu^2,1), \quad k_4=S^5(\mu^2,\mu^2), \\ k_5&=k_6=S^3(\mu^2,\mu^2), \\ k_7&=k_8=S(\mu^2,\mu^2). \end{aligned}$$

E. *U* Amplitudes

According to Eq. (1), and from Eqs. (13) and (20), the resonant scattering amplitude representing the formation of the ρ from a continuous channel and its subsequent decay into a discrete channel is

$$\begin{aligned} F_{ij}(J)&=P_\rho\Gamma_i\Gamma_j(J) \\ &=\sum_{k,l} P_\rho Q_{ik}g_kQ_{jl}(J)g_l(J), \end{aligned}$$

or, in an equivalent matrix notation,

$$F(J)=QU(J)\tilde{Q}(J), \tag{23}$$

where

$$U_{kl}(J)=g_kP_\rho g_l(J).$$

While amplitudes $F(J)$ have factorizable kinematic singularities, they do not all have kinematic *zeros* which are factorizable in a uniform way as J varies. This is easily seen from the type (a) constraints listed in Eqs. (15) and the structure of matrix $k(J)$ which carries the kinematic singularities and zeros. For example, in the neighborhood of the channel thresholds and pseudothresholds, $\Gamma_3(J)\sim S^{J-3}$ for $J\geq 4$, while $\Gamma_3(2)\sim\Gamma_3(0)\sim S$. In fact, all amplitudes except those containing $\Gamma_2(J)$ and $\Gamma_{10}(J)$ exhibit this nonuniform behavior of kinematic zeros. This is a serious defect carried by $F(J)$, since it is clearly desirable to have scattering amplitudes which are free of kinematic zeros as well as singularities, and furthermore, we are especially concerned here with analyticity in J.

A solution to this difficulty presents itself in the form of the U amplitudes introduced above. The undesirable features of $F(J)$ are actually carried only in $Q(J)$, which may be factored out according to

$$U(J)=Q^{-1}F(J)\tilde{Q}^{-1}(J)\,, \tag{24}$$

which equation we now take as the definition of $U(J)$ corresponding to any amplitude $F(J)$. This procedure obviously works satisfactorily for the n-term dyad form of Eq. (1) as well as for the pure ρ resonant amplitudes because $Q(J)$ depends only on the external states. Then, insofar as a partial-wave amplitude $F(J)$ derived from *any* dynamical model and possessing the standard threshold and pseudothreshold kinematic singularities can be expressed in the form (1), the corresponding $U(J)$ defined by Eq. (24) will be free of kinematic zeros for all values of J.

Unrelated to the problem of kinematic zeros, another feature in favor of $U(J)$ over $F(J)$ is that the coupling-constant constraints apply directly to elements of $U(J)$. Hence, we have included in Eq. (24) the factor Q^{-1}, whereas the factor k^{-1} would have sufficed to remove the fixed kinematic singularities of the discrete channels.

We can also define U amplitudes in the case of discrete channel-discrete channel scattering. A typical resonant scattering amplitude representing the formation of the ρ from a discrete channel and its subsequent decay into a discrete channel is

$$\begin{aligned}F_{ij}&=P_\rho\Gamma_i\Gamma_j\\&=\sum_{k,l}P_\rho Q_{ik}g_kQ_{jl}g_l\,,\end{aligned}$$

or, in the matrix notation,

$$F=QU\tilde{Q}\,, \tag{25}$$

where

$$U_{kl}=g_kP_\rho g_l\,.$$

We then define

$$U=Q^{-1}F\tilde{Q}^{-1} \tag{26}$$

to be the U amplitude corresponding to *any* partial-wave amplitude F describing the scattering between discrete external channels.

III. ρ-EXCHANGE AMPLITUDES

A. Crossing Relations for U Amplitudes

The full (angle-dependent) s-channel resonant scattering amplitude, for intermediate ρ formation between an initial continuous channel composed of particles of spins, J, J' and helicities σ, σ' and a final discrete channel composed of particles of spins L, L' and helicities τ, τ', is

$$F(s,\theta_s;L\tau,L'\tau';J\sigma,J'\sigma')=d_{\sigma-\sigma',\tau-\tau'}{}^1(\theta_s)F_{ij}(s,J)\,. \tag{27}$$

The s-channel scattering angle θ_s is the angle in the c.m. system between particles of spin J and L. We insert the previously suppressed energy variable s in the partial-wave amplitude $F_{ij}(s,J)$.

The functional relation between the subscripts ij and the spins and helicities is given by $i=i(L\tau,L'\tau')$ and $j=j(\sigma,J'\sigma')$, where

$$\begin{aligned}&1=i(00,00)\,,\quad 2=i(20,00)\,,\\&3=i(21,00)\,,\quad 4=i(20,20)\,,\\&5=i(20,21)\,,\quad 6=i(21,21)\,,\\&7=i(21,22)\,,\quad 8=i(22,22)\,,\end{aligned} \tag{28}$$

and

$$\begin{aligned}&1=j(0,00)\,,\quad 2=j(1,00)\,,\\&3=j(0,20)\,,\quad 4=j(0,21)\,,\\&5=j(1,20)\,,\quad 6=j(1,21)\,,\\&7=j(1,22)\,,\quad 8=j(2,21)\,,\\&9=j(2,22)\,,\quad 10=j(3,22)\,.\end{aligned} \tag{29}$$

For negative helicity values we have

$$i(L\tau,L'\tau')=i(L|\tau|,L'|\tau'|)$$

and

$$j(\sigma,J'\sigma')=j(|\sigma|,J'|\sigma'|)\,, \tag{30}$$

which follow from Eqs. (6), (16), and (18). The inverse relations

$$\begin{aligned}\tau=\tau(i)\,,\quad \tau'=\tau'(i)\,,\quad L=L(i)\,,\quad L'=L'(i)\,,\\ \sigma=\sigma(j)\,,\quad \sigma'=\sigma'(j)\,,\quad J'=J'(j)\,,\end{aligned} \tag{31}$$

are implicit in Eqs. (28) and (29).

Let B' and B'' denote the s-channel helicity amplitudes corresponding to ρ formation in the t and u channels, respectively. The helicity crossing relations[9] are then

$$\begin{aligned}&B'(s,\theta_s;L\tau,L'\tau';J\sigma,J'\sigma')\\&\quad=\sum_{\lambda,\lambda',\nu,\nu'}(-)^{\sigma+\lambda'+\nu+\tau'}d_{\lambda\sigma}{}^J(\psi_1)d_{\lambda'\sigma'}{}^{J'}(\psi_2)d_{\nu\tau}{}^L(\psi_3)\\&\qquad\times d_{\nu'\tau'}{}^{L'}(\psi_4)F(t,\theta_t;J'\lambda',L'\nu';J\lambda,L\nu)\end{aligned} \tag{32}$$

and

$$\begin{aligned}&B''(s,\theta_s;L\tau,L'\tau';J\sigma,J'\sigma')\\&\quad=\sum_{\lambda,\lambda',\nu,\nu'}(-)^{\lambda+\sigma'+\lambda'+\tau}d_{\lambda\sigma}{}^J(\chi_1)d_{\lambda'\sigma'}{}^{J'}(\chi_2)d_{\nu\tau}{}^L(\chi_3)\\&\qquad\times d_{\nu'\tau'}{}^{L'}(\chi_4)F(u,\theta_u;L\nu,J'\nu';J\nu,L'\lambda')\,.\end{aligned} \tag{33}$$

The crossing angles are defined in terms of an angle $\psi=\psi(x,y;M_1,M_2,M_3,M_4)$ such that

$$\cos\psi=\frac{(x+M_1^2-M_2^2)(y+M_1^2-M_3^2)-2M_1^2(M_1^2-M_2^2-M_3^2+M_4^2)}{[x^2-2x(M_1^2+M_3^2)+(M_1^2-M_3^2)^2]^{1/2}[y^2-2y(M_1^2+M_2^2)+(M_1^2-M_2^2)^2]^{1/2}} \tag{34}$$

[9] T. L. Trueman and G. C. Wick, Ann. Phys. (N. Y.) 26, 322 (1964).

and $\sin\psi \geq 0$. Then with $M(J)$, $M(J')$, etc., denoting the masses of particles of spin J, J', etc., we have

$$\begin{aligned} \psi_1 &= \psi(s,t; M(J),M(J'),M(L),M(L')),\\ \psi_2 &= \psi(s,t; M(J'),M(J),M(L'),M(L)),\\ \psi_3 &= \psi(s,t; M(L),M(L'),M(J),M(J')),\\ \psi_4 &= \psi(s,t; M(L'),M(L),M(J'),M(J)),\\ \chi_1 &= \psi(s,u; M(J),M(J'),M(L'),M(L)),\\ \chi_2 &= \psi(s,u; M(J'),M(J),M(L),M(L')),\\ \chi_3 &= \psi(s,u; M(L),M(L'),M(J'),M(J)),\\ \chi_4 &= \psi(s,u; M(L'),M(L),M(J),M(J')). \end{aligned} \tag{35}$$

Particle (1) and (2) assignments are in the same order in B', B'', and F: e.g., $B'(s,\theta_s; (1),(2); (1),(2))$. These assignments serve (a) not only to define helicity states, but also (b) to define the scattering angle, which in all channels is the c.m. angle between particles of the same assignment, and (c) to define the isotopic spin states. The pair of particles in each channel, crossed or uncrossed, has total isotopic spin $=1$. The total isotopic spin state is, therefore, conveniently constructed in the same definite way for each channel by introducing a label that distinguishes between the particles which are otherwise identical in isotopic spin. All three above considerations are incorporated into the helicity crossing relations.

The complete ρ-exchange amplitude is then

$$B = \tfrac{1}{2}(B' + B''),$$

and the angular-momentum-1 partial-wave projection of this, reverted to the ij subscript notation, is

$$\bar{B}_{ij}(s,J) = \tfrac{3}{2}\int d(\cos\theta_s) d_{\delta(j),\epsilon(i)}{}^1(\theta_s) \times B(s,\theta_s; L(i)\tau(i), L'(i)\tau'(i); J\sigma(j), J'(j)\sigma'(j)), \tag{36}$$

where

$$\delta(j) = \sigma(j) - \sigma'(j)$$

and

$$\epsilon(i) = \tau(i) - \tau'(i).$$

The U amplitudes corresponding to $\bar{B}_{ij}(s,J)$ are obtained by using Eq. (23) in (27), carrying out the appropriate $s \leftrightarrow t$ and $s \leftrightarrow u$ exchanges, and inserting the results into Eqs. (32) and (33), which in turn are fed into Eq. (36). Then, Eq. (24) is applied to (36), and we find

$$\bar{U}_{mn}(s,J) = \sum_{kl} \beta_{mnkl}(s,J) g_k g_l(J), \tag{37}$$

where

$$\beta_{mnkl}(s,J) = \tfrac{3}{4}\sum_{ij} Q^{-1}{}_{mi}(s) Q^{-1}{}_{nj}(s,J) \times \int d(\cos\theta_s) d_{\delta(j)\epsilon(i)}{}^1(\theta_s) \times [P_\rho(t) X_{ijkl}(t,J) + P_\rho(u) X_{ijkl}(u,J)], \tag{38}$$

$$X_{ijkl}(t,J) = \sum_{\lambda\lambda'\nu\nu'} (-)^{\sigma(j)+\lambda'+\nu+\tau'(i)} d_{\lambda\sigma(j)}{}^J(\psi_{1ij}) \times d_{\lambda'\sigma'(j)}{}^{J'(j)}(\psi_{2ij}) d_{\nu\tau(i)}{}^{L(i)}(\psi_{3ij}) \times d_{\nu'\tau'(i)}{}^{L'(i)}(\psi_{4ij}) d_{\lambda-\nu,\lambda'-\nu'}{}^1(\theta_t) \times Q_{i'k}(t) Q_{j'l}(t,J), \tag{39}$$

$$X_{ijkl}(u,J) = \sum_{\lambda\lambda'\nu\nu'} (-)^{\lambda+\sigma'(j)+\lambda'+\tau(i)} d_{\lambda\sigma(j)}{}^J(\chi_{1ij}) \times d_{\lambda'\sigma'(j)}{}^{J'(j)}(\chi_{2ij}) d_{\nu\tau(i)}{}^{L(i)}(\chi_{3ij}) \times d_{\nu'\tau'(i)}{}^{L'(i)}(\chi_{4ij}) d_{\lambda-\nu',\nu-\lambda'}{}^1(\theta_u) \times Q_{i''k}(u) Q_{j''l}(u,J). \tag{40}$$

Here, we shorten the crossing-angle notation to

$$\psi_{1ij} = \psi(s,t; M(J), M(J'(j)), M(L(i)), M(L'(i))), \text{ etc.}, \tag{41}$$

and define

$$\begin{aligned} i' &= i(J'(j)\lambda', L'(i)\nu'),\\ j' &= j(\lambda, L(i)\nu),\\ i'' &= i(L(i)\nu, J'(j)\lambda'),\\ j'' &= j(\lambda, L'(i)\nu'). \end{aligned} \tag{42}$$

The appropriate variables s, t, u, are introduced as necessary into Q, $Q(J)$, and P_ρ.

In a similar fashion the U amplitude corresponding to $\bar{B}_{ij}(s)$, the partial-wave projection of the complete ρ-exchange amplitude in discrete-channel–discrete-channel scattering, is found to be

$$\bar{U}_{mn}(s) = \sum_{kl} \beta_{mnkl}(s) g_k g_l, \tag{43}$$

where

$$\beta_{mnkl}(s) = \tfrac{3}{4}\sum_{ij} Q^{-1}{}_{mi}(s) Q^{-1}{}_{nj} \int d(\cos\theta_s) d_{\epsilon(j)\epsilon(i)}{}^1(\theta_s) \times [P_\rho(t) X_{ijkl}(t) + P_\rho(u) X_{ijkl}(u)]. \tag{44}$$

The quantities $X_{ijkl}(t)$ and $X_{ijkl}(u)$ are the same as the quantities defined in Eqs. (39) and (40), respectively, with the following replacements: in Eqs. (39) and (40), $\sigma(j) \to \tau(j)$; in Eqs. (39)–(41), $J \to L(j)$, $J'(j) \to L'(j)$; in Eq. (39), $Q(t,J) \to Q(t)$; in Eq. (40), $Q(u,J) \to Q(u)$; and in Eq. (42), $i' \to i(L'(j)\lambda', L'(i)\nu')$, $j' \to i(L(j)\lambda, L(i)\nu)$, $i'' \to i(L(i)\nu, L'(j)\lambda')$, $j'' \to i(L(j)\lambda, L'(i)\nu')$.

B. Unitarity for U Amplitudes

The statement of unitarity for an 8×8 helicity partial-wave amplitude describing scattering between discrete channels, is

$$\mathrm{Im}\, A_{ij}(s) = \sum_{k=1}^{8} A_{ik}{}^*(s) \rho_k(s) \theta(s - s_k) A_{kj}(s), \tag{45}$$

where s_k is the threshold value of s in the channel corresponding to index k, θ is the unit step function, ρ_k is an element of the diagonal matrix $\rho=q^3$, and q in turn is the diagonal momentum matrix

$$\begin{bmatrix} q_1 & 0 & 0 \\ 0 & q_2 I_2 & 0 \\ 0 & 0 & q_3 I_5 \end{bmatrix}$$

with q_i= c.m. momentum in channel i and $I_n=n\times n$ identity matrix. The matrix A is such that qAq is the helicity partial-wave amplitude matrix which may be, for example, the resonant amplitude $F(s)$ or a unitarized version of the ρ-exchange amplitude $\bar{B}(s)$. Thus A is the helicity partial-wave amplitude with kinematic singularities and zeros removed in the form of powers of the initial and final c.m. momenta. The U amplitudes are ordinarily of no necessity here since all spins are fixed. The various pure helicity amplitudes within any pair of initial and final channels all have threshold and pseudothreshold singularities corresponding to orbital angular momentum = 1.

We now take the crucial step, in accordance with the discussion in Sec. II A, of reinterpreting the index j in Eq. (45) to denote one of the continuous-channel helicity states. Hence, j now runs over the values $1,\cdots,10$. In the customary way, we distinguish the continuous-channel amplitude matrix by inserting the additional argument J, suppress the variable s, and write the new statement of unitarity in matrix form

$$\mathrm{Im}A(J)=A^*\rho\theta A(J)\,. \tag{46}$$

The diagonal matrix $\theta=\theta(s)$ has typical element $\theta(s-s_k)$, and the matrix $A(J)$ is such that $qA(J)q(J)$ is the helicity partial-wave amplitude describing scattering from an initial state in a continuous channel to a final state in a discrete channel. The 10×10 diagonal matrix $q(J)$ is

$$\begin{pmatrix} (q_\alpha)^{J-1}I_2 & 0 \\ 0 & (q_\beta)^{J-3}I_8 \end{pmatrix},$$

where q_α and q_β are the c.m. momenta in continuous channels α and β, respectively.

We emphasize that the matrix $A(J)$ has no kinematic singularities, but it does develop kinematic zeros when $J=0$, 2. For this reason, the helicity amplitudes are unsatisfactory and the U amplitudes were introduced. Therefore, we now determine the statement of unitarity for the U amplitudes.

From Eqs. (11), (14), and (24) we have

$$A(J)=q^{-1}CkU_A(J)k(J)\tilde{C}(J)q^{-1}(J)\,, \tag{47}$$

where $U_A(J)$ is the U amplitude corresponding to $A(J)$. We may now write

$$\mathrm{Im}A(J)=Cq^{-1}k\,\mathrm{Im}U_A(J)k(J)q^{-1}(J)\tilde{C}(J) \tag{48}$$

and

$$A^*=Cq^{-1}kU_A^*kq^{-1}\tilde{C} \tag{49}$$

which follow from the observations that $q^{-1}C=Cq^{-1}$ and $q^{-1}(J)\tilde{C}(J)=\tilde{C}(J)q^{-1}(J)$ and that $q^{-1}k$ and $k(J)q^{-1}(J)$ have no kinematic singularities. Substituting (47)–(49) into (46), we obtain

$$\mathrm{Im}U_A(J)=U_A^*\rho_U\theta U_A(J)\,, \tag{50}$$

where

$$\rho_U=\tilde{Q}Qq\,. \tag{51}$$

In the case of scattering between initial- and final-discrete channels, Eq. (46) becomes $\mathrm{Im}A=A^*\rho\theta A$ and leads to

$$\mathrm{Im}U_A=U_A^*\rho_U\theta U_A\,. \tag{52}$$

C. Unitarization Procedure

In this section we outline a dispersion relation method, generalized from the corresponding method in I, for converting the input amplitudes $\bar{U}$ and $\bar{U}(J)$, constructed in Sec. III A, into amplitudes U and $U(J)$ satisfying unitarity.

If we write $U=D^{-1}N$, where D has only a right-hand cut R in s and N has only a left-hand cut L, then Eq. (52) is satisfied if $\mathrm{Im}D=-N\rho_U\theta$. To determine N and D we make the customary dynamical assumption that $\mathrm{Im}U(s)=\mathrm{Im}\bar{U}(s)$ on L, and find

$$U(s)=\bar{U}(s)-\frac{1}{\pi}D^{-1}(s)\int_R\frac{\mathrm{Im}\,D(s')\bar{U}(s')ds'}{s'-s}, \tag{53}$$

where D is formally defined as the solution of the integral equation

$$D(s)=1+\frac{1}{\pi}\int_L ds'D(s')G(s,s') \tag{54}$$

with

$$G(s,s')=\frac{1}{\pi}\,\mathrm{Im}\bar{U}(s')\int_R\frac{\rho_U(s'')\theta(s'')ds''}{(s''-s)(s''-s')}. \tag{55}$$

Furthermore, it is easily seen that

$$U(s,J)=\bar{U}(s,J)-\frac{1}{\pi}D^{-1}(s)\int_R\frac{\mathrm{Im}D(s')\bar{U}(s',J)ds'}{(s'-s)} \tag{56}$$

satisfies the dynamical assumption $\mathrm{Im}U(s,J)=\mathrm{Im}\bar{U}(s,J)$ on L and the J-dependent statement of unitarity Eq. (50).

The above procedure is actually heuristic in the sense that we have ignored the question of convergence of the integrals. As a practical matter, it may be more convenient not to attempt to solve the integral equation for D, as would be done in a conventional bootstrap calculation and which as experience tells us would very likely not yield a satisfactory ρ resonance,[2,5] but instead to construct D directly from presumed properties of the ρ in much the same spirit as the $\pi\pi$ p-wave phase shift is used to construct the D function in I. While N and D may be chosen such that $U=D^{-1}N$ satisfies Eq. (52), Eq. (53) may in turn not be identically satisfied for all s; i.e. the reduction of $\bar{U}(s,J)$ to $\bar{U}(s)$ when $J \to 0$, 2 would not automatically guarantee the corresponding reduction of $U(s,J)$ to $U(s)$. This type of failure of unitarity is partially corrected in I by adding some terms to the input amplitudes so that unitarity is exactly satisfied when $J \to 0$ only at the resonant energy. Further discussion of this technical point is unwarranted here and rightly belongs to the subject matter of latter work, since it clearly involves highly specialized approximations.

IV. CONCLUSION

We have established the basic formalism for the three-channel generalization of a ρ-bootstrap model in $\pi\pi$ scattering proposed for investigating external spin continuation. The three-channel model was motivated as a means of obtaining a statement of unitarity that was more compatible with the infinite set of conventional channels introduced by external spin continuation. The formalism hinges around the set of amplitudes constructed so as to be free of kinematic zeros throughout the spin continuation, and concerns the formulation of constraints, crossing relations, and unitarity for these amplitudes.

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Interference Effects and Corrections in $A_1 \to 3\pi$ and $A_2 \to 3\pi$ Decays

HYMAN GOLDBERG
Department of Physics, Northeastern University, Boston, Massachusetts 02115

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Corrections due to overlapping ρ and σ bands in the decay $A_1 \to 3\pi$, and ρ bands in $A_2 \to 3\pi$, are evaluated to first order in the ρ and σ widths. The corrections in the case of the A_1 indicate the use of a smaller anomalous magnetic moment λ_A (or δ) than was previously needed to fit the width of the A_1.

I. $A_1 \to 3\pi$

RECENT current-algebraic "hard-pion" calculations[1] have established a correlation between the decay rates for $\rho \to \pi\pi$ and $A_1 \to 3\pi$, namely,

$$\Gamma_\rho = 141(1-\tfrac{1}{4}\lambda_A)^2 \text{ MeV}, \quad (1)$$

$$\Gamma_{A_1} = \Gamma_{A_1\to\rho\pi} + \Gamma_{A_1\to\sigma\pi} + \Gamma_C, \quad (2)$$

where $\Gamma_{A_1\to\rho\pi}$ is the decay rate into $\rho\pi$,

$$\Gamma_{A_1\to\rho\pi} = 7.0(8+12\lambda_A+5\lambda_A^2) \text{ MeV}, \quad (3)$$

$\Gamma_{A_1\to\sigma\pi}$ is the decay rate into $\sigma\pi$ (undetermined by current algebra, Γ_C is the nonresonant "seagull" contribution, and λ_A is the anomalous magnetic moment of the charged A_1 particle. The experimental value $\Gamma_\rho = 111 \pm 17$ MeV is obtained by choosing

$$\lambda_A = 0.4 \pm 0.3. \quad (4)$$

Equation (4), combined with Eqs. (2) and (3), provides for minimum values of Γ_{A_1}. For example, if $\Gamma_\rho = 120$ MeV, then $\lambda_A = 0.3$ and $\Gamma_{A_1} \geq 78$ MeV. This is to be compared with the most recent experimental compilation,[2] in which $\Gamma_{A_1} = 80 \pm 35$ MeV. The implication is either that the other modes are small or that significant interference effects occur, so that the overall width stays within experimental limits. It is to these interference effects that we address ourselves in this section.

The effect of finite widths may influence current-algebra results in essentially two different ways: (1) The inclusion of a spread in the two-point spectral functions will alter the longitudinal constraints on the vertex functions due to the generalized Ward identities,[3,4] and (2) the use of the altered two-point functions (propagators) will affect the calculation of the four-point tree diagrams, such as $A_1 \to 3\pi$ via ρ mesons. The first type of correction is constrained by minimal-coupling principles to a replacement of the mass m^2 in the inverse σ and ρ propagators by $m^2 - i\Gamma m$.[3] Since quantities such as $\Delta^{-1}(p^2) - \Delta^{-1}(q^2)$ enter the Ward identities, this type of replacement will have no effect within the minimal-coupling framework.

The second type of correction will have quite discernible effects. We will calculate these in the approxi-

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[3] Schnitzer and Weinberg (Ref. 1).

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