Anomalous Commutators and the Triangle Diagram

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We consider matrix elements of the axial-vector current in spinor electrodynamics, and develop the change in the usual reduction formalism caused by the presence of the axial-vector-current–two-photon triangle diagram. When at most one photon is reduced in from the external states, we are able to characterize the anomalous behavior of the triangle diagram entirely in terms of a consistent set of anomalous field-current and current-current commutators.

IT has recently been shown[1] that the axial-vector current in spinor electrodynamics does not satisfy the usual divergence equation

$$\partial^\mu j_\mu{}^5(x)=2im_0 j^5(x)\,,$$

where

$$j_\mu{}^5(x)=\bar\psi(x)\gamma_\mu\gamma_5\psi(x)\,,\quad j^5(x)=\bar\psi(x)\gamma_5\psi(x)\,,\qquad(1)$$

expected from naive use of the equations of motion. Rather, because of the presence of the triangle diagram shown in Fig. 1, the axial-vector current satisfies the anomalous divergence condition

$$\partial^\mu j_\mu{}^5(x)=2im_0 j^5(x)+(\alpha_0/4\pi)F^{\xi\sigma}(x)F^{\tau\rho}(x)\epsilon_{\xi\sigma\tau\rho}\,,\qquad(2)$$

with $F^{\xi\sigma}$ the unrenormalized electromagnetic field-strength tensor. Because radiative corrections to the basic triangle diagram (Fig. 2) involve axial-vector loops with at least five vertices, and because these larger loops satisfy the usual axial-vector Ward identity, Eq. (2) is an *exact* equation, valid to all orders in perturbation theory.[2]

In the present paper we explore further consequences of the singular behavior of the triangle diagram in spinor electrodynamics. Although the anomalous divergence phenomenon appears in all matrix elements of the axial-vector current, we will consider explicitly only the axial-vector-current–two-photon matrix element $\langle 0|j_\mu{}^5|k_1,\epsilon_1;k_2,\epsilon_2\rangle$, which is described in lowest order by the graph of Fig. 1. (Here k_1, k_2 and ϵ_1, ϵ_2 denote, respectively, the four-momenta and polarizations of the two photons.) First, we develop the reduction formalism for the triangle graph. When one photon is reduced in, we are able to characterize the anomalous behavior of the triangle graph entirely in terms of anomalous commutators of the electromagnetic field with the axial-vector current ("seagulls") and of the electromagnetic current with the axial-vector current ("Schwinger terms"). We check that the various commutators which we obtain are consistent with each other, with the equations of motion, and with the electromagnetic-field canonical commutation relations. These formal considerations indicate that the equations obtained from explicit study of the matrix element $\langle 0|j_\mu{}^5|k_1,\epsilon_1;k_2,\epsilon_2\rangle$ can be applied unchanged to the matrix element $\langle A|j_\mu{}^5|B\rangle$, with A and B arbitrary, when at most one photon is reduced in from the external states. Using the anomalous commutation relations, we complete the heuristic verification that the quantity $\bar Q^5$ introduced in I is the chiral generator in massless electrodynamics. Finally, we show that when both photons are pulled in, one cannot represent the triangle graph by a reduction formula containing a time-ordered product with the usual properties.

To study the reduction formula for the triangle graph with one photon pulled in, we use the equation[3]

$$\begin{aligned}&\langle 0|j_\mu{}^5(0)|k_1,\epsilon_1;k_2,\epsilon_2\rangle[(2\pi)^3 2k_{10}(2\pi)^3 2k_{20}]^{1/2}\\&=-i\epsilon_1{}^\sigma\int d^4x\,e^{-ik_1\cdot x}\\&\qquad\times\Box_x\langle 0|T(j_\mu{}^5(0)A_\sigma(x))|k_2,\epsilon_2\rangle[(2\pi)^3 2k_{20}]^{1/2}\\&=-i\epsilon_1{}^\sigma\epsilon_2{}^\rho[e_0{}^2/(2\pi)^4]R_{\sigma\rho\mu}(k_1,k_2)\,,\end{aligned}\qquad(3)$$

where A_σ is the photon field and $R_{\sigma\rho\mu}(k_1,k_2)$ is the explicit expression for the lowest-order triangle graph given in Eqs. (17) and (18) of I. Bringing $\Box_x$ inside the time-ordered product (using the usual rules[4] for differentiating time-ordered products), we find

$$\int d^4x\,e^{-ik_1\cdot x}\Box_x\langle 0|T(j_\mu{}^5(0)A_\sigma(x))|k_2,\epsilon_2\rangle=A_{\mu\sigma}k_{10}+B_{\mu\sigma}+C_{\mu\sigma}(k_{10})\,,\quad(4)$$

[1] S. L. Adler, Phys. Rev. **177**, 2426 (1969). This paper will hereafter be referred to as I. See also J. Schwinger, *ibid.* **82**, 664 (1951), Sec. V; C. R. Hagen, *ibid.* **177**, 2622 (1969); R. Jackiw and K. Johnson, *ibid.* **182**, 1457 (1969); B. Zumino (unpublished). As in I, we use the notation and metric conventions of J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill Book Co., New York, 1965), pp. 377-390. In particular, we use $\epsilon_{0123}=\epsilon^{123}=1$.

[2] S. L. Adler and W. A. Bardeen, Phys. Rev. **182**, 1515 (1969). Note that the anomalous divergence term can be rewritten in terms of finite quantities as $(\alpha/4\pi)F_r{}^{\xi\sigma}F_r{}^{\tau\rho}\epsilon_{\xi\sigma\tau\rho}$, where $F_r{}^{\xi\sigma}$ is the renormalized electromagnetic field-strength tensor.

[3] Since in Eqs. (3)–(7) we work to lowest order only, we omit the wave-function renormalization factor from Eq. (3).

[4] S. L. Adler and R. F. Dashen, *Current Algebras* (W. A. Benjamin, Inc., New York, 1968), Eq. (2.7).

with[5]

$$
\begin{aligned}
A_{\mu\sigma}&=i\int d^4x\, e^{i\mathbf{k}_1\cdot\mathbf{x}}\delta(x_0)\langle 0|[A_\sigma(x),j_\mu{}^5(0)]|k_2,\epsilon_2\rangle\,,\\
B_{\mu\sigma}&=\int d^4x\, e^{i\mathbf{k}_1\cdot\mathbf{x}}\delta(x_0)\langle 0|[\dot{A}_\sigma(x),j_\mu{}^5(0)]|k_2,\epsilon_2\rangle\,,\\
C_{\mu\sigma}(k_{10})&=e_0\int d^4x\, e^{-ik_1\cdot x}\langle 0|T(j_\mu{}^5(0)j_\sigma(x))|k_2,\epsilon_2\rangle\,,\\
\dot{A}_\sigma(x)&\equiv\frac{\partial}{\partial x_0}A_\sigma(x)\,,\quad j_\sigma(x)\equiv\bar{\psi}(x)\gamma_\sigma\psi(x)\,.
\end{aligned}
\tag{5}
$$

Provided that the time-ordered product in $C_{\mu\nu}$ is not too singular, in the limit as $k_{10}\to\infty$, the function $C_{\mu\nu}(k_{10})$ has the Bjorken[6]-Johnson-Low[7] behavior

$$
C_{\mu\nu}(k_{10})=\frac{-ie_0}{k_{10}}\int d^4x\, e^{i\mathbf{k}_1\cdot\mathbf{x}}\delta(x_0)\times\langle 0|[j_\sigma(\mathbf{x}),j_\mu{}^5(0)]|k_2,\epsilon_2\rangle+O[(\ln k_{10})^\beta/k_{10}{}^2]\,,\tag{6}
$$

indicating that the equal-time commutators $[A_\sigma(x),j_\mu{}^5(0)]$, $[\dot{A}_\sigma(x),j_\mu{}^5(0)]$, and $[j_\sigma(x),j_\mu{}^5(0)]$ are to be identified, respectively, with the parts of $R_{\sigma\rho\mu}$ behaving like k_{10}, 1, and $k_{10}{}^{-1}$ as k_{10} becomes infinite. From Eqs. (17) and (18) of I, we find

$$
\begin{aligned}
\epsilon_2{}^\rho R_{\sigma\rho\mu}(k_1,k_2)
&=4\pi^2(k_2{}^\tau\epsilon_2{}^\rho-k_2{}^\rho\epsilon_2{}^\tau)\{\epsilon_{\tau\sigma\rho\mu}+g_{\sigma0}\epsilon_{0\tau\rho\mu}-g_{\rho0}\epsilon_{0\tau\sigma\mu}\\
&\quad+k_{10}{}^{-1}[\tfrac{1}{2}(1-g_{\sigma0})(k_{2\sigma}\epsilon_{0\tau\rho\mu}+k_{20}\epsilon_{\sigma\tau\rho\mu})\\
&\quad+g_{\sigma0}(1-g_{\eta0})k_1{}^\eta\epsilon_{\eta\tau\rho\mu}-g_{\rho0}(1-g_{\eta0})k_1{}^\eta\epsilon_{\eta\tau\sigma\mu}\\
&\quad+(\text{terms which vanish when } \sigma=0 \text{ or } \mu=0)]\}\\
&\quad+O[(\ln k_{10})^\beta/k_{10}{}^2]\,.
\end{aligned}
\tag{7}
$$

Comparing Eq. (7) with Eqs. (5) and (6), we find the equal-time commutation relations[8]

$$
\begin{aligned}
&[A_\sigma(x),j_\mu{}^5(y)]=[\dot{A}_0(x),j_\mu{}^5(y)]=0\,,\\
&[\dot{A}_r(x),j_0{}^5(y)]=(-2i\alpha_0/\pi)\delta^3(\mathbf{x}-\mathbf{y})B^r(y)\,,\\
&[\dot{A}_r(x),j_s{}^5(y)]=(i\alpha_0/\pi)\delta^3(\mathbf{x}-\mathbf{y})\epsilon^{rst}E^t(y)\,,\\
&[j_0(x),j_0{}^5(y)]=(-ie_0/2\pi^2)\mathbf{B}(y)\cdot\nabla_{\mathbf{x}}\delta^3(\mathbf{x}-\mathbf{y})\,,\\
&[j_r(x),j_0{}^5(y)]=(-ie_0/4\pi^2)[\mathbf{E}(x)\times\nabla_{\mathbf{y}}\delta^3(\mathbf{x}-\mathbf{y})]^r\,,\\
&[j_0(x),j_s{}^5(y)]=(ie_0/4\pi^2)[\mathbf{E}(y)\times\nabla_{\mathbf{x}}\delta^3(\mathbf{x}-\mathbf{y})]^s\,,
\end{aligned}
\tag{8}
$$

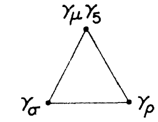

FIG. 1. Axial-vector triangle diagram which leads to the extra term in Eq. (2).

with

$$
\begin{aligned}
B^t(x)&=[\nabla\times\mathbf{A}(x)]^t=\epsilon^{rst}\frac{\partial}{\partial x^r}A^s(x)\,,\\
E^t(x)&=-\dot{A}^t(x)-\frac{\partial}{\partial x^t}A^0(x)\,,\\
\epsilon^{123}&=1\,.
\end{aligned}
\tag{9}
$$

We have only listed the current-current commutators containing at least one time component, since these are the only ones which appear when divergences with respect to the vector or axial-vector indices (σ or μ) are brought inside the time-ordered product in Eq. (5). All of the nonvanishing commutators in Eq. (8) are anomalous in the sense that if they are calculated by naive use of canonical commutation relations they vanish.

It is easy to check that the anomalous commutation relations of Eq. (8), together with the reduction formula of Eqs. (4) and (5), correctly reproduce the known divergence properties of the lowest-order triangle diagram. Consistent with our assumption that the time-ordered product $C_{\mu\nu}$ is not too singular, and obeys the Bjorken-Johnson-Low asymptotic formula, we use the usual formulas[4] for differentiation of the time-ordered product,

$$
\begin{aligned}
\partial_y{}^\mu T(j_\mu{}^5(y)j_\sigma(x))&=T(\partial_y{}^\mu j_\mu{}^5(y)j_\sigma(x))\\
&\quad+\delta(y^0-x^0)[j_0{}^5(y),j_\sigma(x)]\,,\\
\partial_x{}^\sigma T(j_\mu{}^5(y)j_\sigma(x))&=T(j_\mu{}^5(y)\partial_x{}^\sigma j_\sigma(x))\\
&\quad+\delta(x^0-y^0)[j_0(x),j_\mu{}^5(y)]\,.
\end{aligned}
\tag{10}
$$

To check gauge invariance for the photon which has been reduced in, we multiply Eq. (4) by $k_1{}^\sigma$. Using vector-current conservation ($\partial^\sigma j_\sigma=0$) and Eq. (10) to evaluate $k_1{}^\sigma C_{\mu\sigma}(k_{10})$, we find

$$
\begin{aligned}
&k_1{}^\sigma\int d^4x\, e^{-ik_1\cdot x}\Box_x\langle 0|T(j_\mu{}^5(0)A_\sigma(x))|k_2,\epsilon_2\rangle\\
&\quad=k_1{}^\sigma\int d^4x\, e^{i\mathbf{k}_1\cdot\mathbf{x}}\delta(x^0)\langle 0|[\dot{A}_\sigma(x),j_\mu{}^5(0)]|k_2,\epsilon_2\rangle\\
&\qquad-ie_0\int d^4x\, e^{i\mathbf{k}_1\cdot\mathbf{x}}\delta(x^0)\langle 0|[j_0(x),j_\mu{}^5(0)]|k_2,\epsilon_2\rangle\,.
\end{aligned}
\tag{11}
$$

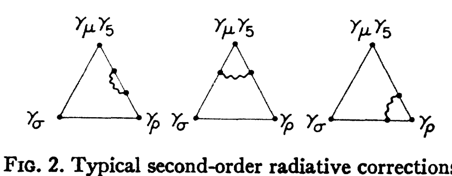

FIG. 2. Typical second-order radiative corrections to the triangle diagram.

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[5] We have suppressed the dependence of $A_{\mu\sigma},\ldots,C_{\mu\sigma}$ on $\mathbf{k}_1$ and k_2.

[6] J. D. Bjorken, Phys. Rev. **148**, 1467 (1968).

[7] K. Johnson and F. E. Low, Progr. Theoret. Phys. (Kyoto) Suppl. **37-38**, 74 (1966).

[8] We remind the reader that since we have deduced the commutators of Eq. (8) from the triangle graph alone, without considering other graphs, we have not yet ruled out the presence of additional terms in the field-current or current-current commutators of higher order than α_0 or e_0, respectively. However, the consistency argument of Eqs. (23)–(30) below suggests that such terms, if they occur at all, are at worst Schwinger terms and seagulls of the usual type, which cancel against each other when vector or axial-vector divergences are taken.

Using the commutators of Eq. (8), one can easily see that the right-hand side of Eq. (11) vanishes. To check the axial-vector divergence of the triangle, we multiply Eq. (4) by $-(k_1+k_2)^\mu$. Using the axial-vector-current divergence equation (2) and Eq. (10) to evaluate $(k_1+k_2)^\mu C_{\mu\sigma}(k_{10})$, we find

$$-(k_1+k_2)^\mu\int d^4x\, e^{-ik_1\cdot x}\Box_x\langle 0|T(j_\mu^5(0)A_\sigma(x))|k_2,\epsilon_2\rangle$$
$$=-ie_0\int d^4x\, e^{-ik_1\cdot x}\langle 0|T([2im_0j^5(0)+(\alpha_0/4\pi)F^{\xi\sigma}(0)F^{\tau\eta}(0)\epsilon_{\xi\sigma\tau\eta}]j_\sigma(x))|k_2,\epsilon_2\rangle$$
$$+\int d^4x\, e^{ik_1\cdot x}\delta(x_0)\times\{-(k_1+k_2)^\mu\langle 0|[A_\sigma(x),j_\mu^5(0)]|k_2,\epsilon_2\rangle+ie_0\langle 0|[j_\sigma(x),j_0^5(0)]|k_2,\epsilon_2\rangle\}. \quad (12a)$$

Since we are only working to lowest order (order e_0^2), the anomalous divergence term proportional to $e_0\alpha_0F^{\xi\sigma}F^{\tau\eta}\epsilon_{\xi\sigma\tau\eta}$ makes no contribution. However, the anomalous commutator terms in curly brackets may be evaluated from Eq. (8), and they give

$$\int d^4x\, e^{ik_1\cdot x}\delta(x_0)\{-(k_1+k_2)^\mu\langle 0|[A_\sigma(x),j_\mu^5(0)]|k_2,\epsilon_2\rangle+ie_0\langle 0|[j_\sigma(x),j_0^5(0)]|k_2,\epsilon_2\rangle\}[(2\pi)^3 2k_{20}]^{1/2}$$
$$=-\epsilon_2^\rho(e_0^2/2\pi^2)k_1^\xi k_2^\tau\epsilon_{\xi\sigma\tau\rho}. \quad (12b)$$

When multiplied by ϵ_1^σ, Eq. (12b) is identical with the matrix element $\langle 0|(\alpha_0/4\pi)F^{\xi\sigma}F^{\tau\rho}\epsilon_{\xi\sigma\tau\rho}|k_1,\epsilon_1;k_2,\epsilon_2\rangle\times[(2\pi)^3 2k_{10}(2\pi)^3 2k_{20}]^{1/2}$ which comes from the anomalous term in Eq. (2) if we calculate the divergence of $\langle 0|j_\mu^5|k_1,\epsilon_1;k_2,\epsilon_2\rangle$ directly, *before* reducing in one photon. We see then that the reduction formula of Eqs. (4) and (5), combined with the anomalous commutators of Eq. (8), correctly characterizes the anomalous axial-vector index divergence of the triangle diagram. As Jackiw and Johnson[1] have particularly emphasized, in the reduction formula the anomalous divergence term $k_1^\xi k_2^\tau\epsilon_{\xi\sigma\tau\rho}$ arises from the failure of the "Schwinger term" $[j_\sigma,j_0^5]$ and the "seagull" $[\dot{A}_\sigma,j_\mu^5]$ to cancel. {As a point of consistency, we note that the pseudoscalar–two-photon triangle $R_{\sigma\rho}$ [defined in Eq. (19) of I] has the asymptotic behavior $R_{\sigma\rho}(k_1,k_2)\to 0$ as $k_{10}\to\infty$. Thus the usual equal-time commutation relations

$$[A_\sigma(x),j^5(y)]=[\dot{A}_\sigma(x),j^5(y)]=0 \quad (13)$$

remain valid, and no extra seagull terms are picked up when the one-photon reduction formula is applied to the matrix element $\langle 0|2im_0j^5|k_1,\epsilon_1;k_2,\epsilon_2\rangle$.}

We proceed next to check whether the commutation relations of Eqs. (8) and (13) are formally consistent with each other, with the equations of motion, and with the usual electromagnetic-field canonical commutation relations. In the Feynman gauge, the electromagnetic-field equations of motion and commutation relations are

$$\Box A_\mu=\ddot{A}_\mu-\nabla^2A_\mu=e_0j_\mu,$$
$$[A^\lambda(x),A^\sigma(y)]|_{x^0=y^0}=[\dot{A}^\lambda(x),\dot{A}^\sigma(y)]|_{x^0=y^0}=0, \quad (14)$$
$$[A^\lambda(x),\dot{A}^\sigma(y)]|_{x^0=y^0}=-ig^{\lambda\sigma}\delta^3(\mathbf{x}-\mathbf{y}).$$

We also need the divergance equations satisfied by the currents $j_\mu(\mathbf{x},t)$ and $j_\mu^5(\mathbf{x},t)$,

$$\frac{\partial}{\partial t}j_0+\nabla\cdot\mathbf{j}=0,$$
$$\frac{\partial}{\partial t}j_0^5+\nabla\cdot\mathbf{j}^5=2im_0j^5+(2\alpha_0/\pi)\mathbf{E}\cdot\mathbf{B}, \quad (15)$$

with $\mathbf{E}$ and $\mathbf{B}$ given, of course, by Eq. (9). We proceed to combine Eqs. (14) and (15) with Eqs. (8) and (13). All the commutators which we write down are at equal time, with $x^0=y^0=t$.

(i) From $[A_\sigma(x),j_0^5(y)]=0$, we deduce[9]

$$[\dot{A}_\sigma(x),j_0^5(y)]+[A_\sigma(x),(\partial/\partial t)j_0^5(y)]=0. \quad (16)$$

On substituting Eq. (15) for $(\partial/\partial t)j_0^5(y)$ and using $[A_\sigma(x),j^5(y)]=[A_\sigma(x),\mathbf{j}^5(y)]=0$, we find

$$[\dot{A}_\sigma(x),j_0^5(y)]=-[A_\sigma(x),(2\alpha_0/\pi)\mathbf{E}(y)\cdot\mathbf{B}(y)]. \quad (17)$$

Using the canonical commutation relations of Eq. (14), we then get

$$[\dot{A}_0(x),j_0^5(y)]=0, \quad (18)$$
$$[\dot{A}_r(x),j_0^5(y)]=(-2i\alpha_0/\pi)\delta^3(\mathbf{x}-\mathbf{y})B^r(y), \quad (19)$$

in agreement with Eq. (8).

(ii) From $[\dot{A}_0(x),j_0^5(y)]=0$, we deduce

$$[\ddot{A}_0(x),j_0^5(y)]+[\dot{A}_0(x),(\partial/\partial t)j_0^5(y)]=0. \quad (20)$$

Substituting Eq. (15) for $(\partial/\partial t)j_0^5(y)$ and Eq. (14) for $\ddot{A}_0(x)$, and using the commutators $[A_0(x),j_0^5(y)]=[\dot{A}_0(x),j^5(y)]=[\dot{A}_0(x),\mathbf{j}^5(y)]=0$, we find

$$[e_0j_0(x),j_0^5(y)]=-[\dot{A}_0(x),(2\alpha_0/\pi)\mathbf{E}(y)\cdot\mathbf{B}(y)]$$
$$=(-2i\alpha_0/\pi)\mathbf{B}(y)\cdot\nabla_{\mathbf{x}}\delta^3(\mathbf{x}-\mathbf{y}), \quad (21)$$

that is,

$$[j_0(x),j_0^5(y)]=(-ie_0/2\pi^2)\mathbf{B}(y)\cdot\nabla_{\mathbf{x}}\delta^3(\mathbf{x}-\mathbf{y}), \quad (22)$$

in accord with Eq. (8).

(iii) From $[\dot{A}_r(x),j_0^5(y)]=-(2i\alpha_0/\pi)\delta^3(\mathbf{x}-\mathbf{y})B^r(y)$, we find

$$[\ddot{A}_r(x),j_0^5(y)]+[\dot{A}_r(x),(\partial/\partial t)j_0^5(y)]$$
$$=(-2i\alpha_0/\pi)\delta^3(\mathbf{x}-\mathbf{y})\dot{B}^r(y)$$
$$=(2i\alpha_0/\pi)\delta^3(\mathbf{x}-\mathbf{y})[\nabla_{\mathbf{y}}\times\mathbf{E}(y)]^r. \quad (23)$$

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[9] We use here the method of D. G. Boulware and L. S. Brown, Phys. Rev. **156**, 1724 (1967).

Substituting for $\ddot{A}_r(x)$ and $(\partial/\partial t)j_0^5(y)$ as before, we find

$$[e_0 j_r(x), j_0^5(y)] - [\dot{A}_r(x), \nabla_y \cdot \mathbf{j}^5(y)] = (2i\alpha_0/\pi)\delta^3(\mathbf{x}-\mathbf{y})[\nabla_y \times \mathbf{E}(y)]^r - [\dot{A}_r(x), (2\alpha_0/\pi)\mathbf{E}(y)\cdot\mathbf{B}(y)] = (-2i\alpha_0/\pi)[\mathbf{E}(x)\times\nabla_y\delta^3(\mathbf{x}-\mathbf{y})]^r. \quad (24)$$

Using Eq. (8) to evaluate $[e_0 j_r(x), j_0^5(y)]$ and $-[\dot{A}_r(x), \nabla\cdot\mathbf{j}^5(y)]$, we see that Eq. (24) is satisfied.

(iv) From $[j_0(x), j_0^5(y)] = -(ie_0/2\pi^2)\mathbf{B}(y)\cdot\nabla_x\delta^3(\mathbf{x}-\mathbf{y})$, we find

$$[(\partial/\partial t)j_0(x), j_0^5(y)] + [j_0(x), (\partial/\partial t)j_0^5(y)] = (-ie_0/2\pi^2)\dot{\mathbf{B}}(y)\cdot\nabla_x\delta^3(\mathbf{x}-\mathbf{y}) = (ie_0/2\pi^2)[\nabla_y\times\mathbf{E}(y)]\cdot\nabla_x\delta^3(\mathbf{x}-\mathbf{y}). \quad (25)$$

Substituting Eq. (15) for $(\partial/\partial t)j_0(x)$ and $(\partial/\partial t)j_0^5(x)$ gives[10]

$$-[\nabla_x\cdot\mathbf{j}(x), j_0^5(y)] - [j_0(x), \nabla_y\cdot\mathbf{j}^5(y)] = (ie_0/2\pi^2)[\nabla_y\times\mathbf{E}(y)]\cdot\nabla_x\delta^3(x-y). \quad (26)$$

Using Eq. (8) to calculate the commutators on the left-hand side, we find that Eq. (26) is satisfied.

(v) Finally, to check the consistency of quantization in the Feynman gauge, we must verify that

$$L \equiv \dot{A}_0 + \nabla\cdot\mathbf{A} \quad (27)$$

and $\dot{L}$ remain dynamically independent of the axial-vector current. That is, we must verify that

$$[L(x), j_\mu^5(y)] = 0 \quad (28)$$

and that

$$[\dot{L}(x), j_\mu^5(y)] = 0. \quad (29)$$

Equation (28) follows immediately from the first line of Eq. (8). To check Eq. (29), we substitute Eq. (14) for $\ddot{A}_0$ and use $[A_0(x), j_\mu^5(y)] = 0$, giving

$$[\dot{L}(x), j_\mu^5(y)] = [e_0 j_0(x), j_\mu^5(y)] + [\nabla_x\cdot\dot{\mathbf{A}}(x), j_\mu^5(y)]. \quad (30)$$

Substituting commutators from Eq. (8) then shows that the right-hand side of Eq. (30) vanishes.

We conclude that the commutation relations of Eq. (8), which were obtained from the triangle graph in lowest-order perturbation theory, are consistent with the equations of motion and canonical commutation relations of Eqs. (14) and (15). Moreover, the fact that Eq. (19) for $[\dot{A}_r(x), j_0^5(y)]$ and Eq. (22) for $[j_0(x), j_0^5(y)]$ were *deduced* from simpler, exact commutators[11] and equations of motion[2] suggests that Eqs. (19) and (22) are themselves exact to all orders of perturbation theory.[12] The values given in Eq. (8) for $[\dot{A}_r(x), j_s^5(y)]$, $[j_r(x), j_0^5(y)]$, and $[j_0(x), j_s^5(y)]$ cannot, on the other hand, be deduced from the consistency argument of Eqs. (23)–(30). To see this, we note that Eqs. (24), (26), and (30) [as well as the reduction formulas (11) and (12)] are all unchanged if we modify these commutators to read

$$[\dot{A}_r(x), j_s^5(y)] = \frac{i\alpha_0}{\pi}\delta^3(\mathbf{x}-\mathbf{y})\epsilon^{rst}E^t(y) - ie_0\delta^3(\mathbf{x}-\mathbf{y})S^{rs}(y),$$

$$[j_r(x), j_0^5(y)] = \frac{-ie_0}{4\pi^2}[\mathbf{E}(x)\times\nabla_y\delta^3(\mathbf{x}-\mathbf{y})]^r + i\frac{\partial}{\partial y^s}[\delta^3(\mathbf{x}-\mathbf{y})S^{rs}(y)], \quad (31)$$

$$[j_0(x), j_s^5(y)] = \frac{ie_0}{4\pi^2}[\mathbf{E}(y)\times\nabla_x\delta^3(\mathbf{x}-\mathbf{y})]^s - i\frac{\partial}{\partial x^r}[\delta^3(\mathbf{x}-\mathbf{y})S^{rs}(y)],$$

with $S^{rs}(y)$ a pseudotensor operator. In other words, the consistency check of Eqs. (23)–(30) does not rule out the possibility that higher orders of perturbation theory may modify Eq. (8) by adding Schwinger terms and seagulls of the usual type,[13] which cancel against each other [as in Eqs. (11) and (12)] when vector or axial-vector divergences are taken. It is expected[13] on general grounds that the commutator $[\dot{A}_r(x), j_s^5(y)]$ does not involve derivatives of the δ function and that the commutators $[j_r(x), j_0^5(y)]$ and $[j_0(x), j_s^5(y)]$ do

[10] We have used $[j_0(x), \mathbf{E}(y)\cdot\mathbf{B}(y)] = 0$, which follows from $[j_0(x), A_\sigma(y)] = [j_0(x), \dot{A}_\sigma(y)] = 0$. Note that $[j_0(x), \dot{A}_\sigma(y)] = 0$ can be derived from $[j_0(x), A_\sigma(y)] = 0$ and the divergence equation $\partial j_0(x)/\partial t + \nabla_x\cdot\mathbf{j} = 0$, in the same way that we derived Eq. (19).

[11] The commutation relations $[A_\sigma(x), j_\mu^5(y)] = [A_\sigma(x), j^5(y)] = [\dot{A}_0(x), j_\mu^5(y)] = [\dot{A}_0(x), j^5(y)] = 0$ can be proved to all orders in perturbation theory by the Bjorken-Johnson-Low method, using the Weinberg asymptotic rules discussed in Chap. 19 of Bjorken and Drell (Ref. 1). Let $T_{\sigma\mu 2fb}(k_1,\ldots)$ be an arbitrary amplitude involving an external photon of polarization σ and four-momentum k_1, an axial-vector current j_μ^5 with four-momentum $-k_1+\Delta$, $2f$ external fermions, and b additional external photons. Because of charge-conjugation invariance, we cannot have $2f = b = 0$. When $f > 0$ or $b > 1$, the *asymptotic coefficient* α associated with T, as $k_{10}\to\infty$, can never be greater than zero. When $f = 0$ and $b = 1$, the superficial asymptotic coefficient is 1 (the graph is linearly divergent), but gauge invariance implies that the photon b must couple through its field-strength tensor, and this reduces the effective α to zero. Thus α for T can never be greater than zero, and since T is arbitrary, this statement holds for all subgraphs of T as well. We conclude that $T_{\sigma\mu 2fb}(k_1,\ldots) \sim (\ln k_{10})^\beta$ as $k_{10}\to\infty$, and since $k_1^\sigma T_{\sigma\mu 2fb}(k_1,\ldots) = 0$ by gauge invariance, this means that $T_{0\mu 2fb}(k_1,\ldots) \sim k_{10}^{-1}(\ln k_{10})^\beta$. Comparing with Eq. (5), we conclude that $[A_\sigma(x), j_\mu^5(y)] = [\dot{A}_0(x), j_\mu^5(y)] = 0$. An identical argument holds with j_μ^5 replaced by j^5.

[12] We believe that Eqs. (19) and (22) are exact when sandwiched between normalizable states $\langle a|$ and $|b\rangle$. We make no claims about matrix elements involving non-normalizable states such as $j_\sigma(x)|a\rangle$ or $j_\mu^5(y)|a\rangle$ and, in particular, we do *not* demand that the commutators of Eq. (8) satisfy the Jacobi identity. (They do *not*.) For a discussion of Jacobi-identity breakdown, see Johnson and Low (Ref. 7).

[13] See Adler and Dashen (Ref. 4), Chap. 3; Boulware and Brown (Ref. 9); D. G. Boulware, Phys. Rev. **172**, 1625 (1968).

not involve derivatives of the δ function higher than the first. Under this assumption, Eq. (31) represents the most general form for these commutators consistent with Eqs. (14) and (15).

Using Eq. (19), we can easily complete the argument sketched in I that the operator

$$\bar{Q}^5=\int d^3x\,[j_0{}^5(x)+(\alpha_0/\pi)\mathbf{A}(x)\cdot\nabla_x\times\mathbf{A}(x)] \quad (32)$$

is the conserved generator of γ_5 transformations in massless electrodynamics. In I it was shown that

$$\frac{d}{dt}\bar{Q}^5=0\,,\quad [\bar{Q}^5,\psi(y)]=-i\gamma_5\psi(y)\,. \quad (33)$$

We now show that $\bar{Q}^5$ commutes with the photon field variables. From the first line of Eq. (8) we find

$$[\bar{Q}^5,A_\sigma(y)]=[\bar{Q}^5,\dot{A}_0(y)]=0\,, \quad (34a)$$

while from Eq. (19) we find[14]

$$[\bar{Q}^5,A_r(y)]=\left[\int d^3x\,j_0{}^5(x),A_r(y)\right]+\left[\int d^3x\,(\alpha_0/\pi)\mathbf{A}(x)\cdot\nabla_x\times\mathbf{A}(x),A_r(y)\right]=\frac{2i\alpha_0}{\pi}B^r(y)-\frac{2i\alpha_0}{\pi}B^r(y)=0\,, \quad (34b)$$

as promised.

Finally, we will show that when two photons are pulled in, the triangle graph cannot be represented by a reduction formula containing a time-ordered product with the usual properties. When two photons are pulled in, Eq. (3) is replaced by[15]

$$\int d^4x d^4y\, e^{-ik_1\cdot x}e^{-ik_2\cdot y}\times\Box_x\Box_y\langle 0|T(j_\mu{}^5(0)A_\sigma(x)A_\rho(y))|0\rangle=i[e_0{}^2/(2\pi)^4]R_{\sigma\rho\mu}(k_1,k_2)\,. \quad (35)$$

Bringing $\Box_x$ and $\Box_y$ inside the time-ordered product on the left-hand side of Eq. (35) gives[16]

$$\int d^4x d^4y\, e^{-ik_1\cdot x}e^{-ik_2\cdot y}\times\Box_x\Box_y\langle 0|T(j_\mu{}^5(0)A_\sigma(x)A_\rho(y))|0\rangle=C_{\mu\sigma\rho}(k_{10},k_{20})+S_{\mu\sigma\rho}(k_{10},k_{20})\,, \quad (36)$$

$$C_{\mu\sigma\rho}(k_{10},k_{20})=e_0{}^2\int d^4x d^4y\, e^{-ik_1\cdot x}e^{-ik_2\cdot y}\times\langle 0|T(j_\mu{}^5(0)j_\sigma(x)j_\rho(y))|0\rangle\,.$$

The "time-ordered product" $C_{\mu\sigma\rho}$ contains all of the dynamical singularities of the matrix element, but in addition there is a polynomial in k_1 and k_2, which we have labeled $S_{\mu\sigma\rho}$, arising from anomalous commutators of A and $\dot{A}$ with the currents. If the time-ordered product $C_{\mu\sigma\rho}$ were of the usual type, then it would have the Bjorken-Johnson-Low behavior in the limits as k_{10}, k_{20}, or $k_{10}-k_{20}$ become infinite. That is, we would have

$$C_{\mu\sigma\rho}(k_{10},k_{20})\xrightarrow[k_{10}\to\infty,\,k_{20}\text{ fixed}]{}\frac{-ie_0{}^2}{k_{10}}\int d^4xd^4y\times e^{ik_1\cdot x}\delta(x_0)e^{-ik_2\cdot y}\langle 0|T([j_\sigma(x),j_\mu{}^5(0)]j_\rho(y))|0\rangle+O((\ln k_{10})^\beta/k_{10}{}^2)\,,$$

$$C_{\mu\sigma\rho}(k_{10},k_{20})\xrightarrow[k_{20}\to\infty,\,k_{10}\text{ fixed}]{}\frac{-ie_0{}^2}{k_{20}}\int d^4xd^4y\times e^{-ik_1\cdot x}e^{ik_2\cdot y}\delta(y_0)\langle 0|T([j_\rho(y),j_\mu{}^5(0)]j_\sigma(x))|0\rangle+O((\ln k_{20})^\beta/k_{20}{}^2)\,, \quad (37)$$

$$C_{\mu\sigma\rho}(k_{10},k_{20})\xrightarrow[k_{10}-k_{20}\to\infty,\,k_{10}+k_{20}\text{ fixed}]{}\frac{-ie_0{}^2}{k_{10}-k_{20}}\int d^4xd^4y\times e^{-\frac{1}{2}i(k_1+k_2)\cdot(x+y)}e^{\frac{1}{2}i(\mathbf{k}_1-\mathbf{k}_2)\cdot(\mathbf{x}-\mathbf{y})}\delta(\tfrac{1}{2}(x_0-y_0))\times\langle 0|T([j_\sigma(x),j_\rho(y)]j_\mu{}^5(0))|0\rangle+O[(\ln(k_{10}-k_{20}))^\beta/(k_{10}-k_{20})^2]\,.$$

According to Eqs. (36) and (37), all terms in $R_{\sigma\rho\mu}$ which either approach constants or diverge linearly in the three limits must be contained entirely in the polynomial S. In Eq. (7) we saw that as $k_{10}\to\infty$, with k_{20} fixed, $R_{\sigma\rho\mu}$ approaches a nonzero finite limit and, by Bose symmetry, the same statement holds for the limit $k_{20}\to\infty$, with k_{10} fixed. In I it was shown that in the limit $k_{10}-k_{20}\to\infty$, with $k_{10}+k_{20}$ fixed, $R_{\sigma\rho\mu}$ diverges linearly (i.e., behaves as finite coefficient times $k_{10}-k_{20}$). Clearly, these three limiting behaviors *cannot* be described by a polynomial in k_{10} and k_{20}, which means that $C_{\mu\sigma\rho}$ cannot vanish in all three of the limits in Eq. (37). Thus, the time-ordered product appearing in the two-photon reduction formula is not of the usual type.

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[14] Equation (34) and Eqs. (16) and (19) may be combined into the simple observation that $[\bar{Q}^5,A_r]=0$ and $d\bar{Q}^5/dt=0$ implies $[\bar{Q}^5,\dot{A}_r]=0$.

[15] Again, we neglect the photon wave-function renormalization.

[16] We have suppressed the dependence of $C_{\mu\sigma\rho}$ and $S_{\mu\sigma\rho}$ on $\mathbf{k}_1$ and $\mathbf{k}_2$.