3-to-3 Amplitude and the Triple-Regge Vertex*

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We develop a Reggeized model for the 3-to-3 amplitude, using group-theoretical variables. The triple-Regge vertex is defined by the asymptotic form of the amplitude. We show that the Veneziano model has the asympotic form predicted by our model.

I. INTRODUCTION

SINCE practical scattering experiments generally have only two particles in the initial state, theoretical physicists have focused most of their attention on such reactions. From a broad theoretical point of view, on the other hand, there is no reason to exclude processes with more than two particles in the initial state. Indeed, both crossing and unitarity imply that an understanding of processes with two particles in the initial state is intimately related to more general processes.

The usefulness of Regge-pole expansions for the description of the asymptotic behavior of the 2-to-n amplitude is well known. Expressions for the amplitude in terms of group-theoretical variables have been particularly convenient for the formulation of the Regge-pole hypothesis. Such variables were first introduced by Toller[1] for the 2–to-2 amplitude, and were extended to the general 2-to-n amplitude by Bali, Chew, and Pignotti.[2] The resulting expression for the 2-to-n amplitude can be schematically represented by a tree diagram. The diagram has $n+2$ external lines, $n-1$ internal lines, two vertices with two external lines and one internal line, and $n-2$ vertices with one external line and two internal lines. In the asymptotic region, each internal line corresponds to the exchange of a Regge pole, and a vertex function is associated with each vertex in the diagram.

For the general m-to-n amplitude, more complicated tree diagrams can be drawn; diagrams containing vertices with three internal lines are possible. Recently, Toller[3] has suggested a particular set of variables for an arbitrary tree diagram, his objective being an amplitude free of kinematic singularities and constraints.

In this paper, we extend the Regge-pole hypothesis to the tree diagram with one three-internal-line vertex for the 3-to-3 amplitude. This is the simplest tree diagram containing a vertex with three internal lines. In Sec. II we define a set of variables for the 3-to-3 amplitude; these variables are more similar to those used by Bali, Chew, and Pignotti[2] than to the new Toller variables.[3] We believe that the Regge analysis is more transparent in our variables. In Sec. III we relate our variables to the invariants, and in Sec. IV we define an asymptotic region of the variables and extend the Regge-pole hypothesis to the description of the amplitude in this region. The triple-Regge vertex is defined by the asymptotic behavior. In Sec. V we study the 3-to-3 amplitude in the narrow-resonance (Veneziano) approximation and find that it Reggeizes in the expected manner.

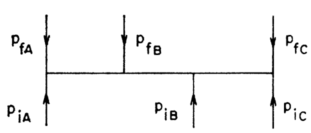

FIG. 1. Tree diagram with no three-internal-line vertex for the 3-to-3 amplitude.

II. DEFINITION OF VARIABLES FOR THE 3-to-3 AMPLITUDE

Let us consider the process $A_i+B_i+C_i \rightarrow A_f+B_f+C_f$. For this process there are two possible tree diagrams, which are shown in Figs. 1 and 2.

The analysis associated with Fig. 1 is very similar to the usual multi-Regge analysis for the 2-to-4 amplitude and is not expected to yield any essentially new information, whereas the analysis of Fig. 2 is more complicated and contains the concept of a triple-Regge vertex. Therefore, in the following we confine our attention to the tree diagram of Fig. 2.

For simplicity, we assume that all the particles are spinless and that they all have the same mass m. We adopt the convention that incoming particles have positive energies, whereas outgoing particles have negative energies.

We define Q_X by

$$Q_X = p_{iX} + p_{fX} \quad (X = A, B, C). \tag{2.1}$$

Energy-momentum conservation can be written as

$$Q_A + Q_B + Q_C = 0. \tag{2.2}$$

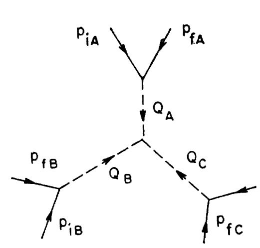

FIG. 2. Tree diagram with one three-internal-line vertex for the 3-to-3 amplitude.

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[1] M. Toller, Nuovo Cimento 37, 631 (1965).
[2] N. F. Bali, G. F. Chew, and A. Pignotti, Phys. Rev. 163, 1572 (1967).
[3] M. Toller, CERN Report No. Th. 975 (unpublished).

Since Q_A is a spacelike vector, there is a Lorentz frame in which Q_A points in the positive z direction. To specify this frame further, we require the three-vector $\mathbf{p}_{iA}$ to point in the z direction. Let this frame be called "frame a_p." Four-vectors in this frame have a superscript a_p. Equation (2.1) completely determines $p_{iA}{}^{a_p}$, $p_{fA}{}^{a_p}$, and $Q_A{}^{a_p}$:

$$p_{iA}{}^{a_p}=[(m^2-\tfrac{1}{4}t_A)^{1/2},\,0,\,0,\,\tfrac{1}{2}(-t_A)^{1/2}],$$
$$p_{fA}{}^{a_p}=[-(m^2-\tfrac{1}{4}t_A)^{1/2},\,0,\,0,\,\tfrac{1}{2}(-t_A)^{1/2}],\quad (2.3a)$$
$$Q_A{}^{a_p}=[0,\,0,\,0,\,(-t_A)^{1/2}].$$

We define frames b_p and c_p in an analogous manner. Thus,

$$p_{iB}{}^{b_p}=[(m^2-\tfrac{1}{4}t_B)^{1/2},\,0,\,0,\,\tfrac{1}{2}(-t_B)^{1/2}],$$
$$p_{fB}{}^{b_p}=[-(m^2-\tfrac{1}{4}t_B)^{1/2},\,0,\,0,\,\tfrac{1}{2}(-t_B)^{1/2}],\quad (2.3b)$$
$$Q_B{}^{b_p}=[0,\,0,\,0,\,(-t_B)^{1/2}],$$

and

$$p_{iC}{}^{c_p}=[(m^2-\tfrac{1}{4}t_C)^{1/2},\,0,\,0,\,\tfrac{1}{2}(-t_C)^{1/2}],$$
$$p_{fC}{}^{c_p}=[-(m^2-\tfrac{1}{4}t_C)^{1/2},\,0,\,0,\,\tfrac{1}{2}(-t_C)^{1/2}],\quad (2.3c)$$
$$Q_C{}^{c_p}=[0,\,0,\,0,\,(-t_C)^{1/2}].$$

Since Q_A is spacelike, there is a frame in which Q_A and Q_B are of the form

$$Q_A=[0,\,0,\,0,\,(-t_A)^{1/2}],$$
$$Q_B=[u,v,0,w],$$

where $u^2-v^2=t_B+w^2$ and $w=(-t_A)^{1/2}(-Q_A\cdot Q_B)$. Using (2.2), we can write

$$u^2-v^2=\tfrac{1}{4}(-t_A)^{-1}\lambda(t_A,t_B,t_C),$$

where

$$\lambda(x,y,z)=x^2+y^2+z^2-2xy-2xz-2yz.\quad (2.4)$$

Since $t_A<0$, $\lambda(t_A,t_B,t_C)$ and u^2-v^2 have the same sign.

If $u^2-v^2>0$, there is a frame, designated by a_r, in which Q_A points in the positive z direction, and only the z and t components of Q_B are nonzero. If $u^2-v^2<0$, there is a frame, designated by a_r', in which Q_A points in the positive z direction, only the x and z components of Q_B are nonzero, and the x component of Q_B is greater than zero.

Therefore, we have two cases to consider. The two cases are distinguished by the sign of $\lambda(t_A,t_B,t_C)$. We note that a different choice of variables, e.g., the new variables of Toller,[3] would allow us to unify the two cases. However, a judicious parametrization of Toller's variables is a nontrivial problem.

Case I: $\lambda(t_A,t_B,t_C)>0$. We have completely determined $Q_A{}^{a_r}$, and $Q_B{}^{a_r}$ is determined by (2.2) up to the sign of the t component. We have

$$Q_A{}^{a_r}=[0,\,0,\,0,\,(-t_A)^{1/2}],\quad (2.5a)$$
$$Q_B{}^{a_r}=\tfrac{1}{2}(-t_A)^{-1/2}[\pm\lambda^{1/2}(t_A,t_B,t_C),\,0,\,0,\,t_A+t_B-t_C],$$
$$Q^{a_r}=\tfrac{1}{2}(-t_A)^{-1/2}[\mp\lambda^{1/2}(t_A,t_B,t_C),\,0,\,0,\,t_A+t_C-t_B].$$

By application of a z boost of magnitude q_{ab}, where

$$\sinh q_{ab}=\mp(t_At_B)^{-1/2}\lambda^{1/2}(t_A,t_B,t_C),\quad (2.6a)$$

to the frame a_r, we arrive at a frame called b_r, in which

$$Q_A{}^{b_r}=\tfrac{1}{2}(-t_B)^{-1/2}[\mp\lambda^{1/2}(t_A,t_B,t_C)\,,0,\,0,\,t_A+t_B-t_C],$$
$$Q_B{}^{b_r}=[0,\,0,\,0,\,(-t_B)^{1/2}],$$
$$Q_C{}^{b_r}=\tfrac{1}{2}(-t_B)^{-1/2}[\pm\lambda^{1/2}(t_A,t_B,t_C),\,0,\,0,\,t_B+t_C-t_A].\quad (2.5b)$$

Similarly, by an application of a z boost of magnitude q_{bc}, where

$$\sinh q_{bc}=\mp\tfrac{1}{2}(t_Bt_C)^{-1/2}\lambda^{1/2}(t_A,t_B,t_C),\quad (2.6b)$$

to the frame b_r, we arrive at a frame which we call c_r. In this frame

$$Q_A{}^{c_r}=\tfrac{1}{2}(-t_C)^{-1/2}[\pm\lambda^{1/2}(t_A,t_B,t_C),\,0,\,0,\,t_A+t_C-t_B],$$
$$Q_B{}^{c_r}=\tfrac{1}{2}(-t_C)^{-1/2}[\mp\lambda^{1/2}(t_A,t_B,t_C),\,0,\,0,\,t_B+t_C-t_A],$$
$$Q_C{}^{c_r}=[0,\,0,\,0,\,(-t_C)^{1/2}].\quad (2.5c)$$

A z boost of magnitude q_{ca}, where

$$\sinh q_{ca}=\mp\tfrac{1}{2}(t_At_C)^{-1/2}\lambda^{1/2}(t_A,t_B,t_C),\quad (2.6c)$$

applied to frame c_r, takes us back to frame a_r.

Frame X_r is related to frame X_p by a Lorentz transformation g_X which preserves $Q_X{}^{X_p}=Q_X{}^{X_r}$, i.e., an element of the three-dimensional Lorentz group. We may parametrize g_X by a rotation through an angle μ_X around the z axis, a boost of magnitude ζ_X in the x direction, and a final rotation around the z axis through an angle ν_X. Therefore, we have

$$g_X=R_z(\nu_X)B_X(\zeta_X)R_r(\mu_X)\quad (X=a,\,b,\,c).\quad (2.7)$$

The set $\{t_A,\nu_A,\zeta_a,\mu_a,t_B,\nu_B,\zeta_b,\mu_b,t_C,\nu_C,\zeta_c,\mu_c\}$ is our set of Toller variables for the case in which $\lambda(t_A,t_B,t_C)>0$. Of course, the amplitude can depend upon only eight independent variables. We show below how to eliminate four of the above variables; but it will be convenient in Sec. IV to express the amplitude as a function of all 12 variables.

Frame a_p has been specified only up to an arbitrary rotation about the z axis. A redefinition of frame a_p by an arbitrary angle ϕ is equivalent to replacing μ_a by $\mu_a+\phi$. Therefore, the amplitude must be left invariant by the transformation $\mu_a\to\mu_a+\phi$, i.e., it is independent of μ_a (a kinematical dependence on μ_a would appear if particle A_i or particle A_f had spin). Similarly, the amplitude cannot depend upon μ_b or μ_C.

Frame a_r is also specified only up to an arbitrary z rotation, and redefinition of this frame by an arbitrary angle ϕ is equivalent to the following change of variables: $\nu_a\to\nu_a+\phi$, $\nu_b\to\nu_b+\phi$, and $\nu_c\to\nu_c+\phi$. This implies that the amplitude can depend upon ν_a, ν_b, and ν_c only in the combinations ω_{ab}, ω_{bc}, and ω_{ca}, where

$$\omega_{ab}=\nu_a-\nu_b,\text{ etc.}\quad (2.8)$$

Clearly, $\omega_{ab}+\omega_{bc}+\omega_{ca}=0$. Therefore, the amplitude depends upon only eight independent variables.

Case II: $\lambda(t_A,t_B,t_C)<0$. Equation (2.2) completely determines $Q_A{}^{a_r'}$, $Q_B{}^{a_r'}$, and $Q_C{}^{a_r'}$. We have

$$
\begin{aligned}
Q_A{}^{a_r'}&=[0,0,0,(-t_A)^{1/2}],\\
Q_B{}^{a_r'}&=\tfrac{1}{2}(-t_A)^{-1/2}\{0,[-\lambda(t_A,t_B,t_C)]^{1/2},0,t_A+t_B-t_C\},\\
Q_C{}^{a_r'}&=\tfrac{1}{2}(-t_A)^{-1/2}\{0,-[-\lambda(t_A,t_B,t_C)]^{1/2},0,t_A+t_C-t_B\}.
\end{aligned}\tag{2.9a}
$$

A rotation about the y axis through an angle θ_{ab}, where

$$
\begin{aligned}
\sin\theta_{ab}&=-\tfrac{1}{2}(t_At_B)^{-1/2}[-\lambda(t_A,t_B,t_C)]^{1/2},\\
\cos\theta_{ab}&=\tfrac{1}{2}(t_At_B)^{-1/2}(t_A+t_B-t_C),
\end{aligned}\tag{2.10a}
$$

carries us from frame a_r' to frame b_r'. A rotation about the y axis through an angle θ_{bc}, where

$$
\begin{aligned}
\sin\theta_{bc}&=-\tfrac{1}{2}(t_Bt_C)^{-1/2}[-\lambda(t_A,t_B,t_C)]^{1/2},\\
\cos_{bc}&=\tfrac{1}{2}(t_Bt_C)^{-1/2}(t_B+t_C-t_A),
\end{aligned}\tag{2.10b}
$$

carries us from frame b_r' to frame c_r'. A rotation about the y axis through an angle θ_{ca}, where

$$
\begin{aligned}
\sin\theta_{ca}&=-\tfrac{1}{2}(t_At_C)^{-1/2}[-\lambda(t_A,t_B,t_C)]^{1/2},\\
\cos\theta_{ca}&=\tfrac{1}{2}(t_At_C)^{-1/2}(t_A+t_C-t_B),
\end{aligned}\tag{2.10c}
$$

carries us from frame c_r' back to frame a_r'.

In frame b_r'

$$
\begin{aligned}
Q_A{}^{b_r'}&=\tfrac{1}{2}(-t_B)^{-1/2}\{0,-[-\lambda(t_A,t_B,t_C)]^{1/2},0,t_A+t_B-t_C\},\\
Q_B{}^{b_r'}&=[0,0,0,(-t_B)^{1/2}],\\
Q_C{}^{b_r},&=\tfrac{1}{2}(-t_B)^{-1/2}\{0,[-\lambda(t_A,t_B,t_C)]^{1/2},0,t_B+t_C-t_A\}.
\end{aligned}\tag{2.9b}
$$

In frame c_r',

$$
\begin{aligned}
Q_A{}^{c_r'}&=\tfrac{1}{2}(-t_C)^{-1/2}\{0,[-\lambda(t_A,t_B,t_C)]^{1/2},0,t_A+t_C-t_B\},\\
Q_B{}^{c_r'}&=\tfrac{1}{2}(-t_C)^{-1/2}\{0,-[-\lambda(t_A,t_B,t_C)]^{1/2},0,t_B+t_C-t_A\},\\
Q_C{}^{c_r'}&=[0,0,0,(-t_C)^{1/2}].
\end{aligned}\tag{2.9c}
$$

Frame X_r' is related to frame X_p by an element of the three-dimensional Lorentz group, denoted by g_X':

$$g_X'=R_z(\nu_X')B_x(\zeta_X')R_z(\mu_X')\quad(X=a,\,b,\,c).\tag{2.11}$$

The set $\{t_A,\nu_a',\zeta_a',\mu_a',t_B,\nu_b',\zeta_b',\mu_b',t_C,\nu_c',\zeta_c',\mu_c'\}$ is our set of variables for $\lambda(t_A,t_B,t_C)<0$. As in case I, the amplitude cannot depend upon μ_a', μ_b', or μ_c'. The removal of the fourth dependent variable is more complicated, however. It arises from the fact that frame a_r' is defined up to an arbitrary y boost. A redefinition of frame a_r' by a boost of magnitude η amounts to the following transformation of variables:

$$
\begin{aligned}
\cosh\zeta_X'&\to\cosh\zeta_X''=\cosh\zeta_X'\cosh\eta+\sinh\zeta_X'\sinh\eta\sin\nu_X',\\
\cos\nu_X'&\to\cos\nu_X''=\sinh\zeta_X'\cos\nu_X'/\sinh\zeta_X'',\\
\sin\nu_X'&\to\sin\nu_X''=(\cosh\zeta_X'\sinh\eta+\sinh\zeta_X'\cosh\eta\sin\nu_X')/\sinh\zeta_X'',\\
&(X=a,\,b,\,c).
\end{aligned}\tag{2.12}
$$

Therefore, the amplitude must be left invariant by the transformation (2.12).

The inelegance of (2.12) arises from our parametrization of g_X'. A different parametrization can lead to a simpler expression of the covariance condition. For example, the parametrization

$$g_X'=B_y(\eta_X)B_x(\gamma_X)R_z(\theta_X)\quad(X=a,\,b,\,c),\tag{2.13}$$

replaces (2.12) by the statement that the amplitude depends upon η_a, η_b, and η_c only in the combinations δ_{ab}, δ_{bc}, and $\delta_{ca}=-\delta_{ab}-\delta_{bc}$, where

$$\delta_{ij}=\eta_i-\eta_j.\tag{2.14}$$

However, the parametrization (2.11) is more suitable for the analysis of Sec. IV than is (2.13).

III. EXPRESSION OF THE INVARIANTS IN TERMS OF OUR VARIABLES

Case I: $\lambda(t_A,t_B,t_C)>0$. In frame a_p, Q_B is given by $Q_B{}^{a_p}=L(g_a{}^{-1})Q_B{}^{a_r}$. Using (2.3a), (2.5a), and (2.7), we can calculate $(p_{iA}+Q_B)^2$. The result is

$$
\begin{aligned}
(p_{iA}+Q_B)^2&=m^2+\tfrac{1}{2}(t_B+t_C-t_A)\\
&\pm(\tfrac{1}{4}-m^2/t_A)^{1/2}\lambda^{1/2}(t_A,t_B,t_C)\cosh\zeta_a.
\end{aligned}\tag{3.1}
$$

We can express $(p_{iB}+Q_C)^2$ and $(p_{iC}+Q_A)^2$ by cyclic permutations of (A,B,C) in Eq. (3.1).

In frame a_p, p_{iB} is given by $p_{iB}{}^{a_p}=L(g_a{}^{-1}q_{ab}{}^{-1}g_b)p_{iB}{}^{b_p}$. Using (2.3a), (2.3b), (2.6a), and (2.7), we can calculate $(p_{iA}+p_{iB})^2$. The result is

$$
\begin{aligned}
(p_{iA}+p_{iB})^2&=2m^2-\tfrac{1}{4}(t_A+t_B-t_C)\\
&\pm\tfrac{1}{2}\cosh\zeta_a\,(-t_A)^{-1/2}(m^2-\tfrac{1}{4}t_A)^{1/2}\lambda^{1/2}(t_A,t_B,t_C)\\
&\mp\tfrac{1}{2}\cosh\zeta_b\,(-t_B)^{-1/2}(m^2-\tfrac{1}{4}t_B)^{1/2}\lambda^{1/2}(t_A,t_B,t_C)\\
&+(t_At_B)^{-1/2}\cosh\zeta r\cosh\zeta_b\\
&\times(m^2-\tfrac{1}{4}t_A)^{1/2}(m^2-\tfrac{1}{4}t_B)^{1/2}(t_A+t_B-t_C)\\
&-2\sinh\zeta_a\sinh\zeta_b\cos\omega_{ab}\,(m^2-\tfrac{1}{4}t_A)^{1/2}(m^2-\tfrac{1}{4}t_B)^{1/2}.
\end{aligned}\tag{3.2}
$$

We can express $(p_{iA}+p_{fB})^2$ by changing the sign of $(m^2-\frac{1}{4}t_B)^{1/2}$ in (3.2); $(p_{fA}+p_{iB})^2$ by changing the sign of $(m^2-\frac{1}{4}t_A)^{1/2}$; and $(p_{fA}+p_{fB})^2$ by changing the signs of both $(m^2-\frac{1}{4}t_A)^{1/2}$ and $(m^2-\frac{1}{4}t_B)^{1/2}$. Expressions for the other two-particle invariants can be obtained by cyclic permutations of (A,B,C). All other invariants can easily be expressed in terms of the two-particle invariants.

We note that the invariants depend upon ν_a, ν_b, and ν_c only in the combinations ω_{ab}, ω_{bc}, and ω_{ca}, and that no invariant depends upon μ_a, μ_b, or μ_c.

Case II: $\lambda(t_A,t_B,t_C)<0$. The calculation of the invariants is similar to case I. The results are

$$
\begin{aligned}
(p_{iA}+Q_B)^2&=m^2+\tfrac{1}{2}(t_B+t_C-t_A)\\
&-\sinh\zeta_a'\cos\nu_a'(\tfrac{1}{4}-m^2/t_A)^{1/2}[-\lambda(t_A,t_B,t_C)]^{1/2}
\end{aligned}\tag{3.3}
$$

and

$$(p_{iA}+p_{iB})^2=2m^2-\tfrac{1}{4}(t_A+t_B-t_C)+2(m^2-\tfrac{1}{4}t_A)^{1/2}(m^2-\tfrac{1}{4}t_B)^{1/2}\{\cosh\zeta_a'\cosh\zeta_b'-\sinh\zeta_a'\sinh\zeta_b'[(\cos\nu_a'\cos\nu_b')\times\tfrac{1}{2}(t_At_B)^{-1/2}(t_A+t_B-t_C)+\sin\nu_a'\sin\nu_b')]\}+\tfrac{1}{2}[-\lambda(t_A,t_B,t_C)]^{1/2}\times[(-t_B)^{-1/2}(m^2-\tfrac{1}{4}t_B)^{1/2}\sinh\zeta_b'\cos\nu_b'-(-t_A)^{-1/2}(m^2-\tfrac{1}{4}t_A)^{1/2}\sinh\zeta_a'\cos\nu_a']. \quad (3.4)$$

Other invariants can be expressed by appropriate sign changes and permutations of (A,B,C) in (3.3) and (3.4).

We note that the invariants do not depend upon μ_a', μ_b', or μ_c' and that they are left unchanged by the transformation (2.12).

IV. ASYMPTOTIC BEHAVIOR OF THE AMPLITUDE AND DEFINITION OF THE TRIPLE-REGGE VERTEX

Case I: $\lambda(t_A,t_B,t_C)>0$. Let the amplitude be written as $f(t_A,g_a,t_B,g_b,t_C,g_c)$. We can expand its dependence on g_a in terms of its projection onto the unitary irreducible representations of the three-dimensional Lorentz group. We write this projection as

$$f_{mn}{}^l(t_A,t_B,g_b,t_C,g_c)=\int dg_a e^{-im\nu_a}d_{mn}{}^l(\zeta_a)e^{-in\mu_a}\times f(t_A,g_a,t_B,g_b,t_C,g_c), \quad (4.1)$$

where $e^{-im\nu}d_{mn}{}^l(\zeta)e^{-in\mu}$ is a unitary irreducible representation of the three-dimensional Lorentz group. The inverse formula for the amplitude is given by

$$f(t_A,g_a,t_B,g_b,t_C,g_c)=\sum_{m,n=-\infty}^{\infty}e^{im\nu_a}f_{mn}(t_A,\zeta_a,t_B,g_b,t_C,g_c)e^{in\mu_a}, \quad (4.2)$$

where

$$f_{mn}(t_A,\zeta_a,t_B,g_b,t_C,g_c)=i\int_{-1/2-i\infty}^{-1/2+i\infty}dl\,\frac{2l+1}{\tan\pi l}a_{mn}{}^{-l-1}(\zeta_a)\times f_{mn}{}^l(t_A,t_B,g_b,t_C,g_c)+\text{discrete terms}. \quad (4.3)$$

The functions $a_{mn}{}^l$ can easily be related to $d_{mn}{}^l$.[1]

If we assume that $f_{mn}{}^l$ is meromorphic in the l plane, the leading term in the asymptotic expansion of f_{mn} is controlled by the position $\alpha(t_A)$ and the residue of the leading pole in $f_{mn}{}^l$. If we assume that the residues are factorizable, we have

$$f_{mn}(t_A,\zeta_a,t_B,g_b,t_C,g_c)\underset{\zeta_a\to\infty}{\sim}\rho_n(t_A)(\cosh\zeta_a)^{\alpha(t_A)}\times\rho_m(t_A,t_B,g_b,t_C,g_c). \quad (4.4a)$$

The expression for the full amplitude is

$$f(t_A,g_a,t_B,g_b,t_C,g_c)\underset{\zeta_a\to\infty}{\sim}\phi(t_A,\mu_a)(\cosh\zeta_a)^{\alpha(t_A)}\times\phi(t_A,\nu_a,t_B,g_b,t_C,g_c), \quad (4.4b)$$

with

$$\phi(t_A,\mu_a)=\sum_{n=-\infty}^{\infty}e^{in\mu_a}\rho_n(t_A), \quad (4.5a)$$

and

$$\phi(t_A,\nu_a,t_B,g_b,t_C,g_c)=\sum_{m=-\infty}^{\infty}e^{im\nu_a}\rho_m(t_A,t_B,g_b,t_C,g_c). \quad (4.5b)$$

Since the amplitude cannot depend upon μ_a, we have $\rho_n(t_A)=\rho(t_A)\delta_{n0}$ and

$$f(t_A,g_a,t_B,g_b,t_C,g_c)\underset{\zeta_a\to\infty}{\sim}\rho(t_A)(\cosh\zeta_a)^{\alpha(t_A)}\times\phi(t_A,\nu_a,t_B,g_b,t_C,g_c). \quad (4.4c)$$

We repeat the above analysis for the dependence of $\phi(t_A,\nu_a,t_B,g_B,t_C,g_c)$ on g_b and g_c. The final result is

$$f(t_A,g_a,t_B,g_b,t_C,g_c)\underset{\zeta_a\zeta_b\zeta_c\to\infty}{\sim}\rho(t_A)\rho(t_B)\rho(t_C)\times(\cosh\zeta_a)^{\alpha(t_A)}(\cosh\zeta_b)^{\alpha(t_B)}(\cosh\zeta_c)^{\alpha(t_C)}\times\phi(t_A,\nu_a,t_B,\nu_b,t_C,\nu_c). \quad (4.6)$$

The triple-Regge vertex $\phi(t_A,\nu_a,t_B,\nu_b,t_C,\nu_c)$ for the case $\lambda(t_A,t_B,t_C)>0$ is defined by Eq. (4.6). Remembering the dependence of the amplitude on ν_a, ν_b, and ν_c, we can write

$$\phi(t_A,\nu_a,t_B,\nu_b,t_C,\nu_c)=\phi(t_A,t_B,t_C,\omega_{ab},\omega_{bc}). \quad (4.7)$$

In terms of invariants, Eq. (4.6) is of the form

$$f(t_A,t_B,t_C,s_a,s_b,s_c,s_{ab},s_{bc},s_{ca})\sim g(t_A)g(t_B)g(t_C)|s_a|^{\alpha(t_A)}|s_b|^{\alpha(t_B)}|s_c|^{\alpha(t_C)}\times V(t_A,t_B,t_C,\kappa_{ab},\kappa_{bc},\kappa_{ca}), \quad (4.8)$$

as $|s_a|$, $|s_b|$, $|s_c|\to\infty$ with κ_{ab}, κ_{bc}, κ_{ca} fixed, where

$$s_a=(p_{iA}+Q_B)^2,\quad s_b=(p_{iB}+Q_C)^2,\quad s_c=(p_{iC}+Q_A)^2, \quad (4.9)$$

$$s_{ab}=(p_{iA}+p_{fB})^2,\quad s_{bc}=(p_{iB}+p_{fC})^2,\quad s_{ca}=(p_{iC}+p_{fA})^2, \quad (4.10)$$

and

$$\kappa_{ab}=s_as_b/s_{ab},\quad \kappa_{bc}=s_bs_c/s_{bc},\quad \kappa_{ca}=s_cs_a/s_{ca}. \quad (4.11)$$

Case II: $\lambda(t_A,t_B,t_C)<0$. The analysis of $f(t_A,g_a',t_B,g_b',t_C,g_c')$ proceeds in the same way as in case I. The final result for the asymptotic behavior of the amplitude is

$$f(t_A,g_a',t_B,g_b',t_C,g_c')\underset{\zeta_a',\zeta_b',\zeta_c'\to\infty}{\sim}\rho(t_A)\rho(t_B)\rho(t_C)\times(\cosh\zeta_a')^{\alpha(t_A)}(\cosh\zeta_b')^{\alpha(t_B)}(\cosh\zeta_C')^{\alpha(t_C)}\times\phi'(t_A,\nu_a',t_B,\nu_b',t_C,\nu_c'). \quad (4.12)$$

Equation (4.12) defines the triple-Regge vertex $\phi'(t_A,\nu_a',t_B,\nu_b',t_C,\nu_c')$ for case II. Equation (4.12) must be left invariant by transformation (2.12). Asymptotically, (2.12) becomes

$$\cosh\zeta_X' \to \cosh\zeta_X''=\cosh\zeta_X'(\cosh\eta+\sinh\eta\,\sin\nu_X'),$$
$$\cos\nu_X' \to \cos\nu_X'' =\cos\nu_X'/\cosh\eta+\sinh\eta\,\sin\nu_X', \quad (4.13)$$
$$\sin\nu_X' \to \sin\nu_X'' =\sinh\eta+\cosh\eta\,\sin\nu_X'/\cosh\eta+\sinh\eta\,\sin\nu_X'.$$

This implies that the triple-Regge vertex must satisfy the condition

$$\phi'(t_A,\nu_a',t_B,\nu_b',t_C,\nu_c')=\cosh\eta+\sinh\eta\,\sin\nu_a')^{\alpha(t_A)} \times(\cosh\eta+\sinh\eta\,\sin\nu_b')^{\alpha(t_B)} \times(\cosh\eta+\sinh\eta\,\sin\nu_c')^{\alpha(t_C)} \times\phi'(t_A,\nu_a'',t_B,\nu_B'',t_C,\nu_c'') \quad (4.14)$$

for arbitrary η.

The covariance condition (4.14) of the triple-Regge vertex assumes a simpler form when we express our results in terms of the parametrization (2.13). For this purpose we need the relation between the two parametrizations:

$$\cosh\zeta_X'=\cosh\gamma_X\,\cosh\eta_X,$$
$$\cos\nu_X'=\sinh\gamma_X/\sinh\zeta_X', \quad (4.15)$$
$$\sin\nu_X'=\cosh\gamma_X\,\sinh\eta_X/\sinh\zeta_X'.$$

In the asymptotic region (4.15) becomes

$$\cosh\zeta_X'\sim\cosh\gamma_X\,\cosh\eta_X,$$
$$\cos\nu_X'\sim(\cosh\eta_X)^{-1}, \quad (4.16)$$
$$\sin\nu_X'\sim\tanh\eta_X.$$

The asymptotic behavior of the amplitude is given by

$$f(t_A,g_a',t_B,g_b',t_C,g_c') \underset{\gamma_a,\gamma_b,\gamma_c\to\infty}{\sim} \rho(t_A)\rho(t_B)\rho(t_C) \times(\cosh\gamma_a)^{\alpha(t_A)}(\cosh\gamma_b)^{\alpha(t_B)}(\cosh\gamma_c)^{\alpha(t_C)} \times\phi''(t_A,t_B,t_C,\eta_a,\eta_b,\eta_c), \quad (4.17)$$

where

$$\phi''(t_A,t_B,t_C,\eta_a,\eta_b,\eta_c)=\phi''(t_A,t_B,t_C,\delta_{ab},\delta_{bc}). \quad (4.18)$$

In terms of invariants, Eq. (4.12) is also of the form (4.8), except at the isolated points $\cos\nu_a'=0$, $\cos\nu_b'=0$, or $\cos\nu_c'=0$.

V. ASYMPTOTIC FORM OF THE AMPLITUDE IN THE NARROW-RESONANCE APPROXIMATION

An explicit expression for the six-line connected part in the narrow resonance approximation has been given by Chan.[4] In our notation, this expression is

$$f=\int_0^1 du_1du_2du_3\, u_1^{-1-\alpha(t_A)}u_2^{-1-\alpha(t_B)}u_3^{-1-\alpha(t_C)} \times(1-u_1)^{-1-\alpha(s_{ab})}(1-u_2)^{-1-\alpha(s_b)}(1-u_3)^{-1-\alpha(s_{bc})} \times[1-u_1(1-u_2)]^{\alpha(s_{ab})+\alpha(t_B)-\alpha(s_a)} \times[1-u_3(1-u_2)]^{\alpha(s_{bc})+\alpha(t_B)-\alpha(s_c)} \times[1-u_1u_3(1-u_2)]^{\alpha(s_a)+\alpha(s_c)-\alpha(s_{ca})-\alpha(t_B)}, \quad (5.1)$$

where $\alpha(s)=a+bs$.

The asymptotic form of this expression as $|s_a|$, $|s_b|$, $|s_c|\to\infty$ with κ_{ab}, κ_{bc}, κ_{ca} fixed, is given by

$$f\sim(-s_a)^{\alpha(t_A)}(-s_b)^{\alpha(t_B)}(-s_c)^{\alpha(t_C)} \times G(t_A,t_B,t_C,\kappa_{ab},\kappa_{bc},\kappa_{ca}), \quad (5.2)$$

where

$$G(t_A,t_B,t_C,\kappa_{ab},\kappa_{bc},\kappa_{ca})=b^{\alpha(t_A)+\alpha(t_B)+\alpha(t_C)} \times\int_0^\infty dv_1dv_2dv_3\, v_1^{-1-\alpha(t_A)}v_2^{-1-\alpha(t_B)}v_3^{-1-\alpha(t_C)} \times\exp\left[-v_1-v_2-v_3+\frac{1}{b}\left(\frac{v_1v_2}{\kappa_{ab}}+\frac{v_2v_3}{\kappa_{bc}}+\frac{v_3v_1}{\kappa_{ca}}\right)\right]. \quad (5.3)$$

Comparing (5.2) with (4.8), we see that the asymptotic behavior of the amplitude in the narrow-resonance approximation is correctly predicted by the group-theoretical arguments of the preceding sections.

VI. CONCLUSION

We have extended the Regge-pole hypothesis to the 3-to-3 amplitude. From the point of view discussed here, our hypothesis is as plausible as previous Regge-pole hypotheses. The concept of a triple-Regge vertex arises naturally in our considerations. We have seen that the narrow resonance approximation of the 3-to-3 amplitude Reggeizes in the predicted manner, lending credibility to our hypothesis. The considerations discussed here can evidently be extended to an arbitrary process.

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[4] Chan Hong-Mo, CERN Report No. Th. 963 (unpublished).