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Feynman-Like Diagrams Compatible with Duality. I. Planar Diagrams*

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We propose a perturbative approach in which the Veneziano representation plays the role of a Born term. We interpret Veneziano's formula as describing only the contribution of one-particle intermediate states. We then add to it the contribution of many-particle intermediate states by means of Feynman-like diagrams. The rules for writing the integrals corresponding to any planar diagram are given. Crossing symmetry, duality, and Reggeization are explicitly taken into account. We find the asymptotic behavior of each Feynman-like diagram. We sum them and prove that the whole amplitude has Regge behavior. The new trajectory, however, is no longer linear, and it incorporates correctly the elastic unitarity constraint. We argue that this approach will ultimately provide a framework in which generalized unitarity (in Cutkosky's sense) can be imposed.

I. INTRODUCTION

According to the recent phenomenological studies of hadronic reactions, most of the known hadrons lie on straight-line Regge trajectories and play a more or less equivalent role[1] in nature. The validity of duality[2] in hadronic reactions also becomes more and more convincing experimentally. Therefore, we try to make a dynamical theory of hadrons that incorporates Regge behavior and duality as first principles, and that treats all hadrons in an equivalent "democratic" way.[3]

In the usual field-theoretic perturbative approach, one ascribes a field for a hadron in the same way as for a lepton; one is then led to consider some hadrons as elementary while others (necessarily those with high spin) are composite. In that formulation one has to add the two separate Feynman diagrams for the s-channel and t-channel elementary-particle exchange, so that it is impossible to incorporate duality with this approach. A way out of this difficulty, within field theory, may be to consider a kind of field-theoretical quark model where the fundamental fields do not correspond to any observable particle. However, the present-day field theory is powerless to provide quantitative results with this model. Because of these difficulties, another opposite approach has been proposed: The analytic S matrix in which one tries to avoid field concepts and to formulate the theory entirely in terms of on-the-mass-shell matrix elements.[4]

Our approach lies between these two extremes. We try to build a perturbative series in which the Veneziano representation plays the role of a Born approximation.[5] In higher-order terms, we include the contribution of many-particle intermediate states in a way similar to the usual perturbation theory, i.e., we write Feynman-like diagrams in which closed loops are to be included. Nevertheless, in our approach the elementary entity that propagates off the mass shell is a tower of hadrons, as required to get superconvergence and duality.[6]

Our approach stems from the observation that Veneziano-like formulas violate unitarity in a way similar to that in which the Born approximation does in the usual Feynman-Dyson theory. Thus, we expect in this manner to be able to build a model such that, in addition to all the properties satisfied by Veneziano's representation, it complies also with unitarity.

Notice, however, the altogether different situation that we face here with respect to the usual field theory. We do not have any Lagrangian from which to derive the rules to write the scattering amplitude. On the other hand, we have the new principle of duality which has to be taken into account.

In this paper, we try to show how the multiparticle intermediate state can be described by Feynman-like amplitudes. We require the same duality property that appeared in the Veneziano formula for every four-point part of a diagram, and we show how crossing symmetry and Regge behavior can be kept. Although our ultimate purpose is to impose unitarity, we do not discuss it in general here. We calculate the asymptotic behavior of each diagram, then sum them and thus show that we recover Regge behavior. Now, however, the trajectory function has an imaginary part incorporating correctly quasielastic unitarity.

In Sec. II, we discuss the graphical interpretation of duality, which is indispensable in writing the scattering amplitude. In Sec. III, we discuss the method of writing the scattering amplitudes and establish the relevant rules. In Sec. IV, we investigate the high-energy behavior of the four-point amplitudes which are constructed

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[1] See, for instance, V. Barger and D. Cline, *Phenomenological Theories of High Energy Scattering* (W. A. Benjamin, Inc., New York, to be published).

[2] R. Dolen, D. Horn, and C. Schmid, Phys. Rev. **166**, 1768 (1968); C. Schmid, Phys. Rev. Letters **20**, 689 (1968); G. F. Chew and A. Pignotti, *ibid.* **20**, 1078 (1968); for experimental evidence of duality, see C. Schmid, CERN Report No. TH960; H. Harari, Ref. 19; V. Barger, Phys. Rev. **179**, 1371 (1969); V. Barger and D. Cline (unpublished).

[3] G. F. Chew, Comments Nucl. Particle Phys. **1**, 187 (1967).

[4] G. F. Chew, *The Analytic S-Matrix* (W. A. Benjamin, Inc., New York, 1966); Ref. 7.

[5] G. Veneziano, Nuovo Cimento **57A**, 190 (1968).

[6] H. R. Rubinstein, A. Schwimmer, G. Veneziano, and M. A. Virasoro, Phys. Rev. Letters **21**, 491 (1968); P. G. O. Freund, *ibid.* **20**, 235 (1968).

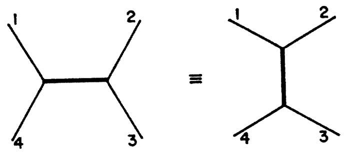


FIG. 1. Duality connected diagrams in the Veneziano amplitude.

using the rules. It will be shown that the sum of the leading planar graphs with loops has Regge behavior.

II. DUALITY AND ITS GRAPHICAL INTERPRETATION

In this paper, we restrict ourselves to the consideration of planar graphs, and for simplicity we take as an example the four-point function. The Born term is the Veneziano formula in which we keep only the s- and t-channel resonances in a dual way; i.e., summing over poles in the s channel reproduces the formula and is equivalent to the sum of t-channel poles (see Fig. 1). The formula is conveniently represented by the integral

$$\int_0^1 dx\, x^{-\alpha(s)-1}(1-x)^{-\alpha(t)-1}, \tag{2.1}$$

where $\alpha(s)$ is the Regge trajectory function, which is assumed to be a linear function of s. In this expression the poles in the s and t channels appear as a result of integration over x near $x\sim 0$ and $x\sim 1$, respectively.

In order to clarify the meaning of duality, we consider next the box diagram of Fig. 2 and apply duality to each internal line; i.e., we replace lines in one channel by the corresponding ones in the crossed channel. In this way we find a certain number of Feynman diagrams (Fig. 3) that are connected with the box of Fig. 2. In the usual approach one should add them. Here we want all of them to be described by a single expression: An integral where the different propagators of the different diagrams appear as a result of integration over a certain region just as in the Veneziano form (2.1).

Writing the so-called dual diagrams of the usual perturbation theory [for instance, Fig. 5(a) is the dual diagram of the box],[7] one can easily show that all diagrams connected by duality are described by different triangulations of a single skeleton: a deformable polygon defined by the external momenta as sides of the polygon and internal points which correspond to closed loops. We call duality diagrams those skeletons that are in one-to-one correspondence with the integrals to be added (Figs. 4 and 5). All possible propagators correspond to all possible lines that we can draw between any two points of the duality diagram. Two lines that cross correspond to channels where we cannot simultaneously find resonances; they have the same connection that the s-t channels have in the Veneziano formula. These lines can easily be seen in the duality diagram as diagonals of a quadrilateral which corresponds to a four-point-function part of the corresponding Feynman diagram. Feynman diagrams connected by duality can also be obtained from Fig. 6(a) through simple topological deformations, as shown in detail in Figs. 6(a)–6(d).[8] However, this description does not look so useful as the one of Fig. 4.

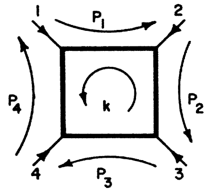


FIG. 2. Definitions of external momenta.

[7] For the definition of dual diagrams in usual perturbation theory, see R. J. Eden *et al.*, *The Analytic S-Matrix* (Cambridge University Press, New York, 1966).

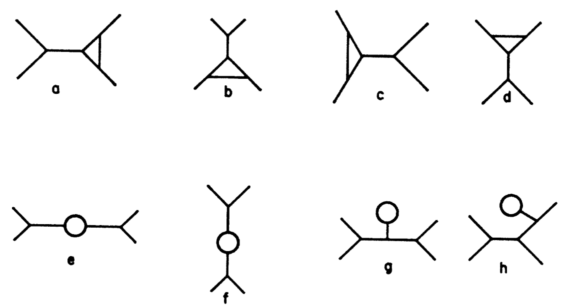


FIG. 3. Examples of Feynman diagrams connected by duality with the box of Fig. 2.

III. RULES FOR MATRIX ELEMENTS

We begin by writing the amplitude

$$\int \prod_{i=1}^{m} dx_i \prod_{j=1}^{k} [y_j(x_1\cdots x_m)]^{-1-\alpha(s_j)}. \tag{3.1}$$

The number m of independent variables is equal to the number of internal lines of a given Feynman diagram, or, equivalently, equal to the number of internal lines of a possible triangulation of the duality diagram.

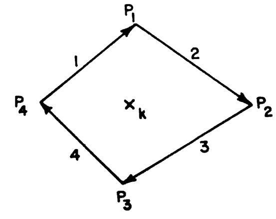


FIG. 4. Duality diagrams that are in one-to-one correspondence with the integrals to be added.

[8] H. Harari [Phys. Rev. Letters **22**, 562 (1968)] and J. L. Rosner [*ibid.* **22**, 689 (1969)] have considered similar diagrams, but where the quantity that propagates is an internal quantum number. We thank Dr. Rosner for interesting discussions concerning this point.

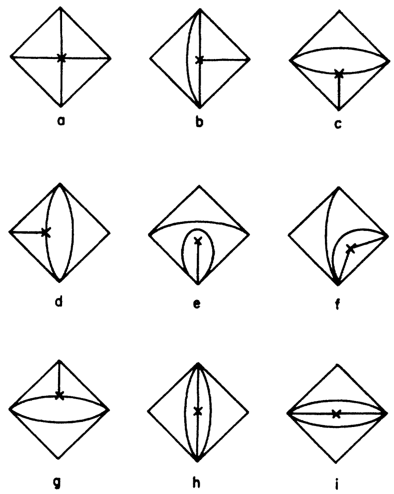


FIG. 5. Examples of dual diagrams for the one-loop diagram and its variations.

All k factors $y_j(x_1\cdots x_m)$ correspond to all possible propagators. Each y_i corresponds to a line in the duality diagram that joins pairs of points. Since there may be double poles in a certain variable [e.g., Fig. 3(f)], more than one y_i with the same exponent may appear. They correspond in the duality diagram to topologically inequivalent lines that join the same two points. Since a variable, say, y, corresponds to a line in the diagram, we simply call it the line y.

Let us consider a pair of lines, say, x and y, which are connected by duality. As explained in Sec. II, they are the diagonals of a quadrilateral (see Fig. 7). If all the sides of the quadrilateral correspond to external scalar particles on the mass shell, the relation of x and y should be reduced to Veneziano's: $y=1-x$. Now we make the natural hypothesis that even off the mass shell y is a function only of x and the variables a_1, a_2, a_3, a_4, which correspond to four sides of the rectangle $y=f(x;a_1,a_2,a_3,a_4)$.

One of the justifications of this hypothesis is due to the N-point generalization of Veneziano's formula.[9] In

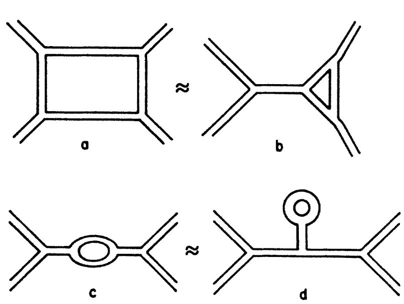
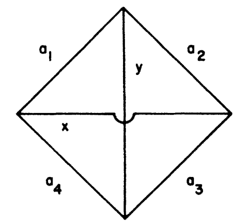


FIG. 6. Another way of getting the diagrams connected by duality.

[9] K. Bardakci and H. Ruegg, Phys. Letters **28B**, 342 (1968); M. A. Virasoro, Phys. Rev. Letters **22**, 37 (1969); C. J. Goebel and B. Sakita, *ibid.* **22**, 257 (1969); H. M. Chan, Phys. Letters **28B**, 485 (1969); S. T. Tsou, Z. Koba, and F. Nilssen, Niels Bohr Institute report (unpublished).

FIG. 7. A part of the duality diagram. Line y is determined by a_1, a_2, a_3, a_4, and x.



fact, as shown in Appendix A, the N-point function can be constructed once the function f is known. The function f must satisfy the requirements of duality that can be stated as the condition that y must go to 1 whenever any variable, say, z, that corresponds to a crossing line goes to 0. If z is not x, it must cross at least one side line, so that when $z\rightarrow 0$ some of the a's must go to 1. The eight-point function[9] provides us with an adequate expression for f. While the explicit expression for f may be obtained, we prefer to give it as an implicit function

$$y=\frac{1-x\alpha_2\alpha_3}{1-x\alpha_2\alpha_3a_1}\,\frac{1-x\alpha_2\alpha_3a_1a_4}{1-x\alpha_2\alpha_3a_4}\,, \tag{3.2}$$

where

$$a_2=\frac{1-\alpha_2}{1-\alpha_2a_1}\,\frac{1-\alpha_2a_1x}{1-\alpha_2x}\,,\qquad a_3=\frac{1-\alpha_3}{1-\alpha_3a_4}\,\frac{1-\alpha_3a_4x}{1-\alpha_3x}\,. \tag{3.3}$$

The following properties are proved in Appendix B:

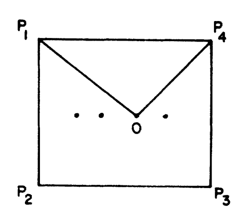
$$f(x;a_1a_2a_3a_4)=f(x;a_4a_3a_2a_1)=f(x;a_2a_1a_4a_3)\,, \tag{3.4}$$

$$y=f(x;a_1a_2a_3a_4)\Rightarrow x=f(y;a_1a_4a_3a_2)\,. \tag{3.5}$$

The same line may appear as a diagonal of different quadrilaterals. For the prescription to be consistent, all expressions obtained for the same line should be the same. In Appendix B, we prove that this is, in fact, the case.

We first choose n independent variables $x_1\cdots x_n$ which correspond to a triangulation of the duality diagram. Then we draw the dependent lines as diagonals of quadrilaterals and obtain the corresponding de-

FIG. 8. Diagram for the N-loop amplitude. Most internal lines are omitted here.



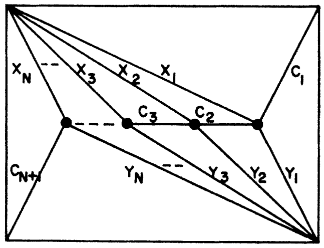


FIG. 9. A set of independent lines for the N-loop amplitude. This is transformed into Fig. 10 by $s \rightleftarrows t$ interchange.

pendent variables using (3.2). We now iterate the procedure, considering quadrilaterals such that one side is a dependent variable; the diagonal will then cut two independent lines, and so on. The game could go on until all possible topologically different lines have been drawn.

This number is infinite for all diagrams with more than one loop, because of the fact that a line may make an infinite number of circles around any pair of loops without crossing itself. However, it may not be necessary to include all the lines. To maintain duality, we must include all the lines needed to construct the triangulations corresponding to all Feynman diagrams connected by duality. Crossing symmetry also puts some restrictions on the number of lines. In our example —the four-point function with spin-zero identical external particles—crossing symmetry means $s \leftrightarrow t$ symmetry (we are considering only planar diagrams). This can be imposed most easily by requiring invariance of the integrand with respect to the transformation (see Fig. 8) $p_1 \leftrightarrows p_4$; $p_2 \leftrightarrows p_3$. Since in this transformation a line from 0 to 1 goes into a line from 0 to 4, we conclude that we need both of them to have crossing symmetry.

Finally, we expect unitarity to impose severe constraints. From the limited point of view of this paper—that is, with the idea of constructing a framework where unitarity can ultimately be imposed—we try at this point to leave as much freedom as possible. However, we will show in the next sections how the imposition of some kind of limited unitarity is possible and provides further restrictions on the number of lines to be included.

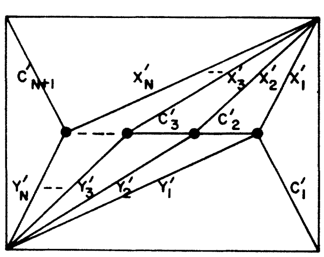


FIG. 10. Another set of independent lines for the N-loop function. This is transformed into Fig. 9 by $s \rightleftarrows t$ interchange.

As a final step towards giving a complete expression for the duality diagram with N loops, we must calculate the invariant volume elements of the integral to maintain the crossing symmetry. For the choice of variables shown in Fig. 9, the integral volume element should be invariant under the transformation $(x,y,c) \to (x',y',c')$, where (x',y',c') are indicated in Fig. 10. Since the transformation is self-invertible, i.e., $x_k' = g_k(x,y,c) \Rightarrow x_n = g_n(x',y',c')$, the invariant volume is given by the square root of the Jacobian. An explicit calculation shows that

$$\prod_{i=1}^{N+1} \{[1-c_i(x_i+x_{i-1})+x_ix_{i-1}]^2 - 4x_ix_{i-1}(1-c_i)^2\}^{-1/2} \times \{[1-c_i(y_i+y_{i-1})+y_iy_{i-1}]^2 - 4y_iy_{i-1}(1-c_i)^2\}^{-1/2} \quad (x_0 = x_{N+1} = y_0 = y_{N+1} = 0). \tag{3.6}$$

We can still multiply the whole integrand by an arbitrary function, provided this function is symmetric with respect to the previous transformation $(x,y,c) \to (x',y',c')$. However, once more we expect unitarity to make some restrictions on this arbitrariness, as will be borne out subsequently.

Example: box diagram. Here we apply the preceding rules to construct the box diagram. Let us first choose as independent variables the x_i shown in Fig. 11. The dependent lines z_i can be written at once from the prescription of the previous section:

$$z_i = f(x_i; x_{i-1}00x_{i+1}) = \frac{1-x_i}{1-x_ix_{i-1}} \frac{1-x_ix_{i-1}x_{i+1}}{1-x_ix_{i+1}}. \tag{3.7}$$

In the second step, the y_{ij} can be written, for instance,

$$y_{43} = f(x_4; x_2z_30x_1).$$

Replacing z_3, we obtain

$$y_{43} = \frac{1-x_3x_4}{1-x_3x_4x_1} \frac{1-x_3x_4x_1x_2}{1-x_3x_4x_2}. \tag{3.8}$$

Finally, the u_i lines can be considered. Notice that these lines correspond to no dynamical variable. It contributes a factor that is a function only of the x_i. The exponent is $-\alpha(0)-1$, where α is a trajectory with the quantum numbers of the vacuum. To compute it we have to extend the prescription, because only in a generalized sense can we consider this line to be a diagonal of a rectangle. Analogy suggests

$$u_1 = f(x_3; x_1z_2z_4x_1). \tag{3.9}$$

The invariant metric is

$$I = [(1-x_1x_2)(1-x_2x_3)(1-x_3x_4)(1-x_4x_1)]^{-1}, \tag{3.10}$$

and the arbitrary function may be, for instance, $(1-x_1x_2x_3x_4)^{-\lambda}$, because this is zero when all other variables are indeterminate.

IV. HIGH-ENERGY BEHAVIOR OF ELASTIC SCATTERING AMPLITUDE

In this section, we calculate the asymptotic behavior of the N-loop amplitude. Our purpose is to find out how the sum of all terms goes at high energy. We suppose that the leading asymptotic behavior of the sum will be given by summing the leading asymptotic behavior of each term.[10]

A. Integral Representation of N-Loop Function

To obtain the high-energy behavior, it is convenient to choose a set of independent variables (x_i,y_i,z_i) in the integral representation different from those of the preceding section (x_i,y_i,c_i). The new set of variables is shown in Fig. 12(a), and is obtained from the old set by the transformation

$$z_i = f(c_i;\, y_i y_{i-1} x_{i-1} x_i)\,, \tag{4.1}$$

with the Jacobian

$$\mathcal{J}_N = \prod_{i=1}^{N+1} \left| \frac{\partial z_i}{\partial c_i} \right|^{-1}, \tag{4.2}$$

where $x_0 = y_0 = x_{N+1} = y_{N+1} = 0$. The explicit form of the Jacobian is rather complicated, but the following asymptotic form is useful later:

$$\mathcal{J}_N \to \prod_{i=1}^{N} \frac{(1-x_i)^2(1-y_i)^2}{(1-x_iy_i)^2} \quad \text{as } z_i \to 0\,. \tag{4.3}$$

The Feynman diagram which should be associated with Fig. 12(a) is shown in Fig. 12(b), which is of course equivalent to the N-ladder diagram or its various variations, as discussed before.

In the following, we assume that all external particles are scalar, with a common mass m. In terms of p_1,p_2,p_3,p_4, the invariant variables s and t can be expressed as

$$s = (p_2 - p_4)^2\,, \quad t = (p_1 - p_3)^2\,.$$

For convenience we choose $p_4 = 0$. To write the integral, it is convenient to introduce a notation for the products of all variables that correspond to the same Regge trajectory (i.e., those that have the same exponent). Let us denote by A_J, $\bar{A}_J$, B_J, $\bar{B}_J$, and X_{JK} $(J,K = 1\cdots N)$ the product of all variables, dependent as well as independent, that correspond to lines connecting the loop momentum k_J with the points of momentum p_2, p_4, p_1, p_3, and k_K, respectively. Let us further denote by C_N the products of those between p_2 and p_4, and by D_N the products of those between p_1 and p_3.

Then the integral representation of the N-loop amplitude I_N is given by

$$\begin{aligned} I_N = (-ig^2)^{N+1} \int \prod_{J=1}^{N} d^4k_J \int_0^1 dX\,dY\,dZ\, \mathcal{J}_N \\ \times G_N(XYZ) D_N{}^{-\alpha_{13}((p_1-p_3)^2)-1} \\ \times \Big(\prod_J A_J{}^{-\alpha_2{}^{(J)}((k_J-p_2)^2)-1}\Big)\Big(\prod_J \bar{A}_J{}^{-\alpha_4{}^{(J)}((k_J-p_4)^2)-1}\Big) \\ \times \Big(\prod_J B_J{}^{-\alpha_1{}^J((k_J-p_1)^2)-1}\Big)\Big(\prod_J \bar{B}_J{}^{-\alpha_3{}^J((k_J-p_3)^2)-1}\Big) \\ \times C_N{}^{-\alpha_{24}((p_2-p_4)^2)-1}\Big(\prod_{J<K} X_{JK}{}^{-\alpha^{(JK)}((k_J-k_K)^2)-1}\Big), \end{aligned} \tag{4.4}$$

where α_{ij}, $\alpha^{(JK)}$, and $\alpha_i{}^{(J)}$ are Regge trajectories in $(p_i-p_j)^2$, $(k_J-k_K)^2$, and $(k_J-p_i)^2$, respectively. X, Y,

[10] Compare with the calculations done with ladder diagrams, Ref. 7.

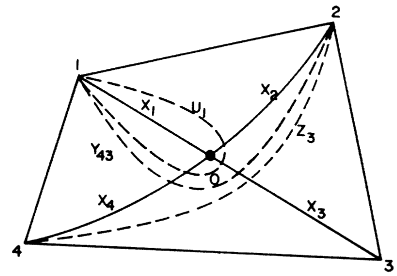


FIG. 11. Diagram for the box diagram; y_{43} is the self-energy correction to an external particle, and u_1 is a tadpole line.

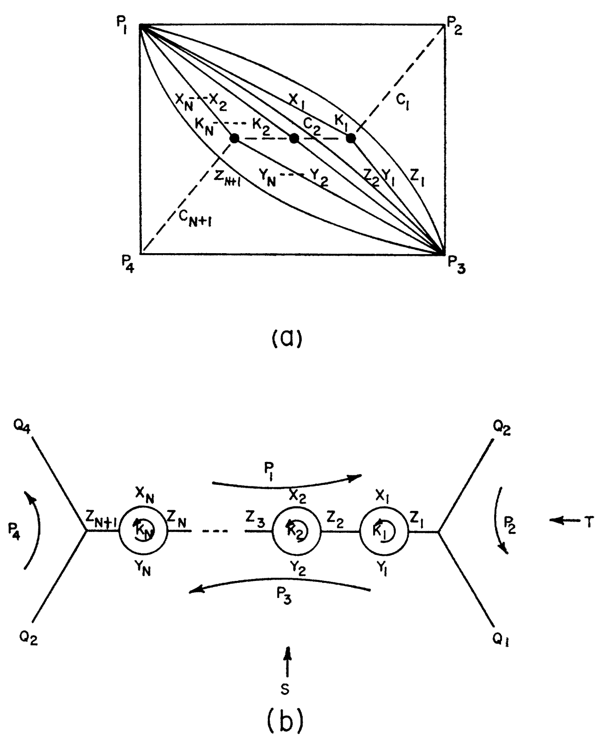


FIG. 12. (a) A set of independent lines (x_i,y_i,z_i) for the N-loop function. (b) Feynman diagram associated with (a).

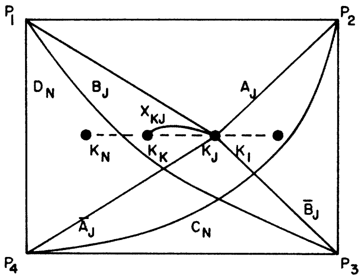


FIG. 13. Lines included in the N-loop amplitude. $A_J, B_J \cdots$ represent products of all possible lines from k_J to $p_2, p_1 \cdots$.

and Z stand for sets of variables $x_1 \cdots x_N$, $y_1 \cdots y_N$, and $z_1 \cdots z_{N+1}$. The function G_N is a regular function at $z_i \sim 0$. The invariant volume V_N obtained in Sec. III and dependent variables which correspond to the self-energy diagrams are implicitly included in G_N.[11]

To make integrations with respect to k_J, we rewrite the integrand of (4.4) in an exponential form (we take $d\alpha/dt=1$):

$$\tilde{G}_N D_N^{-\alpha_{13}(t)-1} \exp(\tilde{F}), \tag{4.5}$$

where

$$\tilde{F} = -\sum_J \left[(k_J-p_2)^2 \ln A_J + (k_J-p_4)^2 \ln \bar{A}_J + \sum_K (k_J-k_K)^2 \ln X_{JK} + (k_J-p_1)^2 \ln B_J + (k_J-p_3)^2 \ln \bar{B}_J\right] - (p_2-p_4)^2 \ln C_N, \tag{4.6}$$

$$\tilde{G}_N = (-ig^2)^{N+1} \mathcal{J}_N \times G_N\Big(\prod_J B_J^{-\alpha_{13}(t)-1}\Big)\Big(\prod_J \bar{B}_J^{-\alpha_3^J(0)-1}\Big) \times \Big(\prod_{J<K} X_{JK}^{-\alpha^{(JK)}(0)-1}\Big)\Big(\prod_J A_J^{-\alpha_2^J(0)-1}\Big) \times \Big(\prod_J \bar{A}_J^{-\alpha_4^J(0)-1}\Big) C_N^{-\alpha_{24}(0)-1}. \tag{4.7}$$

The expression (4.6) is further rearranged to a compact form:

$$\begin{aligned}\tilde{F} &= -(\mathbf{k}, \mathbf{A}\mathbf{k}) + 2(\mathbf{a}, \mathbf{k}) - p_2^2(b_N + c_N) - p_1^2 \ln\Big(\prod_J B_J\Big) - p_3^2 \ln\Big(\prod_J \bar{B}_J\Big) \\ &= -\sum_J \lambda_J \bar{k}_J^2 + (\mathbf{a}, \mathbf{A}^{-1}\mathbf{a}) - s(b_N + c_N) - p_1^2 \ln\Big(\prod_J B_J\Big) - p_3^2 \ln\Big(\prod_J \bar{B}_J\Big),\end{aligned} \tag{4.8}$$

where we have introduced a Hermitian $N \times N$ matrix $\mathbf{A}$ and two N-dimensional vectors $\mathbf{k}$ and $\mathbf{a}$:

$$\mathbf{A} = (a_{JK}), \tag{4.9}$$

$$a_{JJ} = \ln\Big(A_J \bar{A}_J B_J \bar{B}_J \prod_{K \neq J}^{N} X_{JK}\Big), \tag{4.10}$$

$$a_{JK} = -\ln X_{JK}, \quad K \neq J$$

$$\mathbf{k} = \{k_J\}, \quad J = 1, \cdots, N$$

$$\mathbf{a} = \{-(p_2 \ln A_J + p_1 \ln B_J + p_3 \ln \bar{B}_J)\}, \quad J = 1, \cdots, N \tag{4.11}$$

and

$$b_N = \ln(A_1 \cdots A_N), \quad c_N = \ln C_N. \tag{4.12}$$

In going from the first line to the second in (4.8), we have diagonalized $\mathbf{A}$; λ_J are the eigenvalues of $\mathbf{A}$, and $\{\bar{k}_J\}$ represents the stationary vector of the bilinear form $\tilde{F}$.

Substituting (4.8) into (4.5), and performing the integration over $\bar{k}_J$, we obtain

$$I_N = \left(\frac{i}{2\pi^2}\right)^N \int_0^1 dX dY dZ \times D_N^{-\alpha_{13}(t)-1} \tilde{G}_N \left(\frac{1}{\det|\mathbf{A}|}\right)^2 \times \exp[-sF(XYZ) - H(XYZ)], \tag{4.13}$$

where

$$F(XYZ) = \frac{-1}{\det|\mathbf{A}|}\Big\{\sum_{J,K} \tfrac{1}{2} C_{JK}\big[\ln A_J \ln(A_K B_K \bar{B}_K) + \ln A_K \ln(A_J B_J \bar{B}_J)\big] - \big[\ln(A_1 \cdots A_N) + \ln C_N\big] \det|\mathbf{A}|\Big\}, \tag{4.14a}$$

$$H(XYZ) = \frac{t}{\det|\mathbf{A}|} \sum_{J,K} C_{KJ} \ln B_J \ln \bar{B}_K - m^2\left[\frac{C_{JK} \ln(B_J \bar{B}_J) \ln(B_K \bar{B}_K)}{\det|\mathbf{A}|} - \ln\Big(\prod_J B_J \bar{B}_J\Big)\right] \to \sum_J t \frac{\ln B_J \ln \bar{B}_J}{\ln(B_J \bar{B}_J)}, \quad \text{as } z_i \to 0 \tag{4.14b}$$

and C_{JK} is the cofactor of the matrix $\mathbf{A}$. Notice that to integrate over $\bar{k}_J$ we first perform a Wick rotation so that $\bar{k}_J^2$ becomes negative definite.[12]

[11] The constant g^2 is the normalization factor of I_0 (Veneziano amplitude). The normalization of I_N is determined in such a way that it reduces to $(I_0)^{N+1}$ when lines x_J and y_J are put on the mass shell of the lowest recurrence of α_i^J.

[12] We have performed this rotation formally without taking into account any possible contribution from $|k_0| = \infty$. One of us (M.A.V.) thanks S. Fubini for interesting discussions concerning this point.

Equation (4.14a) can be further simplified:

$$F(X,Y,Z)=\frac{-1}{\det|\mathbf{A}|}\{\sum_{L,K,J} C_{JK}a_{LK}\ln A_J-\sum_{J,K}\ln A_J C_{JK}\ln A_K-[\ln(A_1\cdots A_N)+\ln C_N]\det|\mathbf{A}|\}$$

$$=\frac{1}{\det|\mathbf{A}|}(\sum_{J,K}\ln A_J C_{JK}\ln A_K+\ln C_N\det|\mathbf{A}|),\quad (4.14c)$$

where we have used

$$\ln(A_K B_K \bar{B}_K)=\sum_L a_{LK}-\ln\bar{A}_K.$$

A careful investigation of (4.14b) leads us to the following graphical rules that allow us to write at once the expression for the coefficients of the different invariants in the exponent.[13]

We consider the diagram drawn in Fig. 13. To write $\det|\mathbf{A}|$, we proceed in the following way: We consider a subset of lines of Fig. 13 such that (i) Either among themselves or when considered together with the external momenta, no closed loop can be formed; and (ii) it is a maximal set with property (i)—that is, if we add any new line to the set we violate (i).

Then each subset so constructed contributes to $\det|\mathbf{A}|$ a term equal to the product of all logarithms of the variables corresponding to lines included in the subset. The complete expression for $\det|\mathbf{A}|$ arises from adding all possible products corresponding to all possible sets of lines.

To construct the numerator in the exponent, we proceed similarly: (i) We consider a set of internal lines such that when considered together with some external lines they form a closed loop. (ii) We then add to this set all possible lines such that no new loop is formed, and (iii) such that if any new line is added, then (ii) is violated.

Each set so constructed contributes to the coefficient of the square of the sum of external momenta which were needed to form the loop. Its contribution equals, as before, the product of the logarithms of all lines included in the set. The sign of this contribution is always negative, so that an over-all minus sign is present in the exponent.

This construction still does not include the contribution of the lines corresponding to poles in s, t, or any other invariant that is not a function of the loop momenta.

[13] These rules are similar to the Symanzik rules for Feynman diagrams [Progr. Theoret. Phys. (Kyoto) **20**, 690 (1958)]; C. S. Lam and J. P. Lebrun, Nuovo Cimento **59A**, 397 (1969).

B. Asymptotic Form of N-Loop Diagram

Let us now consider the limit $s\to-\infty$ in (4.13). We know the integral in terms of the $A_J\bar{A}_J B_J\bar{B}_J$ and X_{JK} because of the rules of Sec. IV A. We must now express these variables in terms of the Z,X,Y. From the explicit expression (4.14a) for $F(X,Y,Z)$, or from the rules we have given, it is obvious at once that F is a negative definite quantity. Therefore the behavior of the integral at $s\to-\infty$ will be given by that region of integration where $F(X,Y,Z)=0$. In particular, when one of the z_i variables is equal to zero, because of duality at least one of the variables included in any set of lines that go from p_2 to p_4 is 1, i.e., its logarithm is zero. Since $F(X,Y,Z)$ is equal to the sum of products of logarithms of the variables that complete at least one uncut path from p_2 to p_4, $F(X,Y,Z)$ is equal to zero when any $z_i=0$. That is, we have

$$F(X,Y,Z)=-z_1\cdots z_{N+1}f_N(X,Y)+\cdots.\quad (4.15)$$

Therefore the high-energy contribution comes from the region where at least one of the z is zero. We expand the whole integrand around that point and keep only the first term.[14] Then we have

$$F(X,Y,Z)=-(z_1\cdots z_{N+1})\prod_{J=1}^{N} f(x_J,y_J)+\cdots,\quad (4.16a)$$

where

$$f(x,y)=\frac{1}{(1-xy)^2}\left[x(1-y)^2+y(1-x)^2-\frac{(1-x)^2(1-y)^2}{\ln(xy)}\right].\quad (4.16b)$$

The proof of (4.16a) and (4.16b) is given in Appendix C.

We are now ready to get the final formula for I_N in the high-energy region. Substitution of (4.16) into (4.13) leads us to

$$I_N=\frac{g^{2N+2}}{(2\pi)^N}\int_0^1\prod_{J=1}^{N}\frac{(1-x_J)^2(1-y_J)^2}{(1-x_Jy_J)^2\ln^2(x_Jy_J)}\frac{dx_J}{x_J}\frac{dy_J}{y_J}\times\prod_{i=1}^{N+1}dz_i\,G_N(X,Y)\,(z_1\cdots z_{N+1})^{-\alpha_{13}(t)-1}$$
$$\times\exp\left[s(z_1\cdots z_{N+1})\prod_J f(x_J,y_J)-\sum_J\left(t\frac{\ln x_J\ln y_J}{\ln(x_Jy_J)}+\alpha_1{}^{(J)}(0)\ln x_J+\alpha_3{}^{(J)}(0)\ln y_J\right)\right].\quad (4.17)$$

[14] The z_i lines are not the only ones joining p_1 and p_3. If we take another set of n lines we can consider the contribution to the integral from the region where these lines are zero. However, we have been able to show that these new contributions can be consistently regarded as nonleading contributions multiplying the integrand by an appropriate function. It is an open question whether this will be the case when unitarity is imposed.

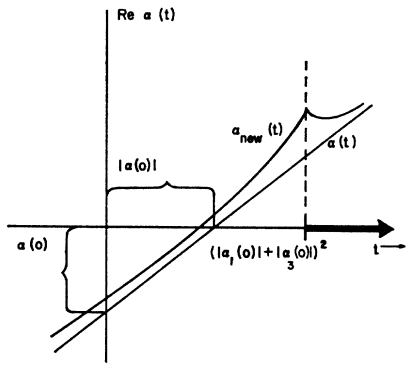


FIG. 14. Chew-Frautschi plots; the old $\alpha(t)$ and $\alpha_{\rm new}(t)$ are illustrated.

In obtaining (4.17), we have used (4.3) and set z_i equal to zero in the nonsingular factors in the integrand. Then G_N turns out to be the product of the invariant volume factor and the self-energy factor of x_J and y_J lines, which can be taken as

$$G_N \to \prod_{J=1}^{N} v(x_J,y_J)\,, \quad \text{as } z_i \to 0 \tag{4.18}$$

with

$$v(x,y)=\frac{(1-xy)^{-\lambda-1}}{(1-x)^2(1-y)^2}\left(\frac{(1-x)(1-y)}{(1-xy)^2}\right)^{-\alpha(0)-1}. \tag{4.19}$$

Integration (4.17) over z_i is easily performed by the transformation from $(z_i\cdots z_{N+1})$ to

$$\begin{aligned}\xi_0&=z_1z_2\cdots z_{N+1}\,,\\ \xi_1&=z_2\cdots z_{N+1}\,,\\ &\vdots\\ \xi_N&=z_{N+1}\,.\end{aligned} \tag{4.20}$$

Using the well-known formula

$$\int_0^1 g'(\xi_N)d\xi_N\int_0^{\xi_N}\cdots\int_0^{\xi_2} g'(\xi_1)d\xi_1\int_0^{\xi_1}\mathfrak{F}(\xi_0)d\xi_0 =\frac{1}{N!}\int_0^1[g(1)-g(\xi)]^N\mathfrak{F}(\xi)d\xi\,,$$

we get

$$\begin{aligned}I_N=\frac{g^{2N+2}}{N!(2\pi^2)^N}\int_0^1\prod_J J(x_J,y_J)\frac{dx_Jdy_J}{x_Jy_J}
&\times\exp\left[-\sum_J\left(t\frac{\ln x_J\ln y_J}{\ln(x_Jy_J)}+\alpha_1{}^{(J)}(0)\ln x_J+\alpha_3{}^{(J)}(0)\ln y_J\right)\right]\\
&\times\int_0^1 d\xi(-\ln\xi)^N\xi^{-\alpha_{13}(t)-1}\\
&\times\exp[s\xi\prod_J f(x_J,y_J)]\,,\end{aligned} \tag{4.21}$$

where

$$J(x,y)=\frac{(1-x)^2(1-y)^2v(x,y)}{(1-xy)^2\ln^2(xy)}\,. \tag{4.22}$$

The highest-power term of ln's in I_N is

$$I_N\to\Gamma(-\alpha_{13}(t))\frac{g^{2N+2}}{N!}[\Sigma(t)\ln s]^N s^{\alpha_{13}(t)}\,, \tag{4.23}$$

with

$$\Sigma(t)=\left(\frac{1}{2\pi^2}\right)\int_0^1\int_0^1 dxdy\frac{J(x,y)}{xy}[f(x,y)]^{\alpha_{13}(t)} \times\exp\left(-t\frac{\ln x\ln y}{\ln(xy)}-\alpha_1(0)\ln x-\alpha_3(0)\ln y\right). \tag{4.24}$$

In (4.23) and (4.24) we have assumed that $\alpha_1{}^{(J)}(0)=\alpha_1(0)$ and $\alpha_3{}^{(J)}(0)=\alpha_3(0)$, for all J.

If $\alpha(0)$ and λ in (4.19) are negative enough, the expression (4.24) converges. For t going to $-\infty$, we obtain (from $x\approx1$)

$$\begin{aligned}\Sigma(t)&\to\frac{1}{2\pi^2}\int_0^1 dxdy\frac{(1-y)^{\alpha(0)-\lambda-2}}{\ln^2 y}y^{-\alpha(0)-1}\times(1-x)^{-\alpha(0)-1}\exp[-2t(x-1)]\\
&=\frac{1}{2\pi^2}\Gamma(-\alpha(0))\times\left(\int_0^1\frac{(1-y)^{\alpha(0)-2-\lambda}}{\ln^2 y}y^{-\alpha(0)-1}dy\right)(-t)^{\alpha(0)}.\end{aligned} \tag{4.25}$$

C. Total Amplitude

The summation over the number of loops can easily be taken in (4.23) to give the total amplitude

$$T=C\Gamma(-\alpha_{13}(t))s^{\alpha_{13}(t)+g^2\Sigma(t)}\,, \tag{4.26}$$

where C is an over-all normalization constant.

The most interesting point about (4.26) is the Regge behavior with a new trajectory

$$\alpha_{\rm new}(t)=\alpha_{13}(t)+g^2\Sigma(t)\,, \tag{4.27}$$

where $\Sigma(t)$ is given by (4.24). The analytic properties of the trajectory $\alpha(t)$ can be seen from (4.24). Making the transformation

$$\ln x=-u^{-1}\,,\quad \ln y=-v^{-1}\,, \tag{4.28}$$

we obtain

$$\Sigma(t)=\frac{1}{2\pi^2}\int_0^\infty\int_0^\infty dudv\frac{J(u,v)}{u^2v^2}[f(u,v)]^{\alpha_{13}(t)} \times\exp\left(\frac{t}{u+v}+\frac{v\alpha_1(0)+u\alpha_3(0)}{uv}\right). \tag{4.29}$$

The Landau method tells us that the location of singularities is at

$$t=[\sqrt{(|\alpha_1(0)|)}\pm\sqrt{(|\alpha_3(0)|)}]^2 \qquad (4.30)$$

which are the threshold and the pseudothreshold of two-particle states with the mass $\sqrt{(|\alpha_1(0)|)}$ and $\sqrt{(|\alpha_3(0)|)}$, the firs tintercepts of the trajectories (Fig. 14). (Note that we have taken $d\alpha/dt=1$.)

The new trajectory $\alpha_{\text{new}}(t)$ approaches the straight line $\alpha(t)$ asymptotically as $|t|\to\infty$ because of Eq. (4.25), if $\alpha(0)$ is negative.

V. DISCUSSION OF RESULTS

In this paper we have tried to build a framework in which unitarity might be superimposed on an amplitude which has the correct analyticity and crossing-symmetry properties. As originally stressed by Mandelstam[15] (and more recently by Veneziano[16]), the idea of building a bootstrap of the hadrons on superconvergence in the narrow-width approximation[15] implicitly assumes that unitarity corrections are small and that they will be provided by a scheme of approximations where the narrow-width solution will correspond to a zeroth-order approximation. However, the way to impose unitarity remained an open question. An on-the-mass-shell approach has been tried by Veneziano.[17] This method, nevertheless, is not manifestly crossing-symmetric. Therefore we cannot know *a priori* whether in imposing unitarity we are not losing crossing symmetry. It is difficult to imagine a crossing-symmetric on-the-mass-shell approach.

From a different viewpoint, i.e., trying to keep crossing symmetry and duality in each step of approximation, we regard Veneziano's formula as a Born term and generate the higher-order approximations by means of an off-the-mass-shell approach similar to the Feynman-Dyson expression. However, we stress that this is a rather unorthodox interpretation of Veneziano's formula. In fact, in our approach all the experimental successes of Veneziano's formula should be reproduced only if the corrections (after renormalization) turn out to be small. This is not evident *a priori*, given the fact that the effective coupling constant is still the strong coupling constant.

In particular, we now have a background, and only if this background turns out to be small will we have "duality" in the experimental sense that resonance contributions dominate in the FESR.

[15] S. Mandelstam, Phys. Rev. **166**, 1539 (1968). The same kind of approach (but without discussing unitarity) was proposed independently by M. Ademollo, H. R. Rubinstein, G. Veneziano, and M. A. Virasoro, Phys. Rev. Letters **19**, 1402 (1967); Phys. Rev. **176**, 1904 (1968); and references therein.

[16] G. Veneziano, in Proceedings of the Sixth Coral Gables Conference on Symmetry Principles at High Energy (unpublished).

[17] G. Veneziano, M.I.T. report (unpublished); see also P. G. O. Freund, Phys. Rev. Letters **22**, 565 (1969). Notice that we are not ruling out the possibility of an on-the-mass-shell calculation once we know that a crossing-symmetric theory compatible with duality and unitarity exists.

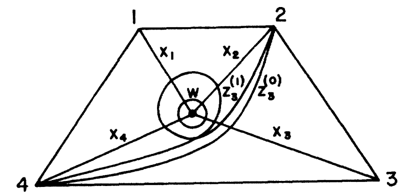


FIG. 15. Box diagram showing examples of lines to be added.

Although there is no experimental evidence as yet in favor of our approach, we have the following rather impressive theoretical evidence. We find that the model is Reggeized, and furthermore that the output Regge trajectory has the correct threshold behavior and incorporates crossed-channel semielastic unitarity, at infinite energy. The fact that it does not include multiresonance unitarity effects is just a result of keeping only the leading contribution in s for each order. This is equivalent to keeping all orders of $g^2 \ln s$, neglecting lower orders in g^2 alone. For this reason, we did not get any correction to the Regge residue either.

Another feature of this model is the apparent absence of cuts in the angular momentum plane. Although our proof is not rigorous, a rough consideration of all possible contributions to the asymptotic behavior seems to indicate that they will essentially give more corrections to the Regge-pole behavior. On the other hand, we expect cuts from nonplanar graphs. The complete Veneziano representation [including the (s,u) term] has fixed poles at wrong-signature points. In principle, these fixed poles could begin to move (as happens with weak-interaction fixed poles). However, Mandelstam[18] has recently given convincing arguments against such a possibility. As an immediate consequence, cuts will appear imposed by unitarity.

An outstanding feature of the model is that there is no place for the Pomeranchukon. The number of Regge trajectories is "conserved" in the process of unitarizing the amplitude. Consistently with Harari's[19] theory, we have not included the Pomeranchukon in the Born term; therefore we have not found any output contribution corresponding to it. We may conjecture that the Pomeranchukon is a cut in the angular momentum plane and so will appear when nonplanar graphs are included.

Finally, we mention that internal quantum numbers may be ascribed to the lines of Fig. 6. Then a very interesting connection with Harari and Rosner's[8] graphical analysis of duality appears. In fact, it is our feeling that if unitarity corrections turn out to be small, this model will possibly become a relativistic justification of the quark model.

Note. When this work was in an advanced stage of preparation, we learned from Fubini and Veneziano

[18] S. Mandelstam, Berkeley report (unpublished).

[19] P. G. O. Freund, Ref. 6; H. Harari, Phys. Rev. Letters **20**, 1385 (1968).

that they were trying a program similar to the one proposed here. They have written an integral expression for the box diagram essentially equivalent to the one written in Sec. III, and have proved several properties about it. One of us (M. A. V.) thanks them for interesting discussions.

Note added in proof. In collaboration with Veneziano we have imposed factorization of the residues of resonances that appear in the Feynman-like diagrams. We assume that the same resonances necessitated to ensure factorization for the N-point Born term[20] must appear as intermediate states and with the same couplings.[21] For the case of the one-loop diagrams, our conclusions are the following:

(a) The lines that we have previously included are given correctly by our prescription.

(b) More lines have to be added corresponding to lines crossing themselves, as shown in Fig. 15.

A line that turns around the loop k times is obtained from the one that does not turn around by multiplying the product of the x_i that appear in the rational expression by $\eta^k=(\prod_1^n x_i)^k$. Furthermore, the exponent is equal to the exponent of the original line minus kn.

Example. In the example in Sec. III we have taken

$$\left(\frac{1-x_3}{1-x_3x_2}\,\frac{1-x_3x_2x_4}{1-x_3x_4}\right)^{-\alpha_{24}-1}$$

for the Z_3 line, and now we multiply this by the following factor:

$$\left(\frac{1-x_3\eta}{1-x_3x_2\eta}\,\frac{1-x_3x_2x_4\eta}{1-x_3x_4\eta}\right)^{-\alpha_{24}-4-1}.$$

It can easily be proven that these lines are never zero in the range of integration. They do not correspond to any propagator. That is the reason why we could not discover them by duality considerations.

(c) Finally, a line that may be depicted by a circle around the loop must be included (see Fig. 15, line w). Its expression is

$$w=\prod_{i=1}^{\infty}(1-\eta^i)^{-4}.$$

As is the case with the N-point function, no arbitrary invariant function is allowed if we do not modify all Born terms correspondingly.

With the inclusion of these lines, unitarity is presumably satisfied (to second order in the coupling constant). Unfortunately the integrals turn out to be divergent. We have not yet succeeded in renormalizing out the divergences.

[20] S. Fubini and G. Veneziano, Nuovo Cimento (to be published).

[21] This problem has also been discussed independently by K. Bardakci, M. B. Halpern, and J. A. Shapiro, Phys. Rev. (to be published). They take the Ward-like identity into account.

ACKNOWLEDGMENT

We thank Dr. C. Michael for a critical reading of the manuscript.

APPENDIX A: N-POINT GENERALIZATION OF VENEZIANO FORMULA

In this appendix we prove that the prescription of Sec. III, when it is applied to the N-point tree diagram, gives the usual expression.

As indicated in the corresponding dual diagram (Fig. 16), we choose the lines $y_{0,i}=x_i$ as the independent variables. The lines that cross only one independent

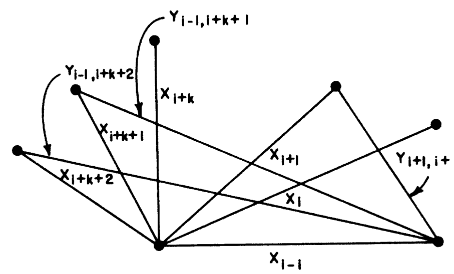


FIG. 16. Independent lines x_i and dependent lines y_i.

line, say, x_i, can be written by the prescription of setting $a_1=x_{i-1}$, $a_2=x_{i+1}$, and $a_3=a_4=0$. We then have

$$y_{i-1,i+1}=\frac{1-x_i}{1-x_ix_{i-1}}\,\frac{1-x_ix_{i-1}x_{i+1}}{1-x_ix_{i+1}}, \qquad \text{(A1)}$$

where we define $x_0=x_N=0$. Now we prove by mathematical induction that a line that crosses $x_i\cdots x_{i+k}$ can be expressed by

$$y_{i+k+1,i-1}=\frac{1-x_i\cdots x_{i+k}}{1-x_i\cdots x_{i+k}x_{i-1}}\times\frac{1-x_{i-1}x_i\cdots x_{i+k}x_{i+k+1}}{1-x_i\cdots x_{i+k}x_{i+k+1}}. \qquad \text{(A2)}$$

We suppose this to be true for k and we prove it for $k+1$. For this purpose we consider the rectangle formed by x_{i-1}, $y_{i+k+1,i-1}$, $y_{i+k+1,i+k+2}$ $(\equiv 0)$, x_{i+k+2}. Then the prescription implies

$$y_{i-1,i+k+2}=\frac{1-x_{k+i+1}\alpha}{1-x_{k+i+1}\alpha x_{i-1}}\,\frac{1-x_{k+i+1}\alpha x_{i-1}x_{i+k+2}}{1-x_{k+i+1}\alpha x_{i+k+2}}, \qquad \text{(A3)}$$

where α is defined implicitly by

$$y_{i-1,i+k+1}=\frac{1-\alpha}{1-\alpha x_{i-1}}\,\frac{1-\alpha x_{i-1}x_{i+k+1}}{1-\alpha x_{i+k+1}}. \qquad \text{(A4)}$$

From (A2) and (A4) we obtain

$$\alpha = x_i \cdots x_{i+k}.$$

When this replacement is made in (A3), the theorem follows. The formulas (A1) and (A2) agree with the solution obtained in Ref. 9.

APPENDIX B: UNIQUENESS OF EXPRESSION OBTAINED BY PRESCRIPTION OF SEC. III

Let us suppose that a line can be regarded as the diagonal of two different quadrilaterals. In Fig. 17, y can be suggested as the diagonal of $a_1a_2a_3a_4$, dual to x, or as the diagonal of $a_1'a_2'a_3'a_4'$, dual to x'. In turn, a_i' and x' may be determined in terms of the same variables that a_i and x can depend on. Then we want the same expression for y irrespective of the manner of construction. To prove this, we first notice that the two quadrilaterals, together with all the lines needed to express y as a function of certain variables, are isomorphic to a corresponding set of lines appearing in a certain N-point function. This is due to the fact that the prescription of Sec. III specifies that the quadrilateral cannot enclose any loop. Thus, the loops being external to the relevant part of the diagram, the latter is topologically equivalent to a certain part of an N-point function. Now we prove that the expressions (A2) imply the prescription of Sec. III for any quadrilaterals that one can imagine. Referring to Fig. 18, we obtain the formula

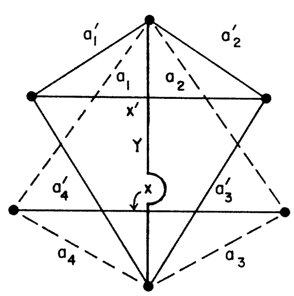


FIG. 17. Two possible ways to determine the line y.

$$y = \frac{1-xx_1\cdots x_n}{1-xx_1\cdots x_n a_1}\,\frac{1-xx_1\cdots x_n a_1 a_2}{1-xx_1\cdots x_n a_2}, \tag{B1}$$

$$a_3 = \frac{1-x_1\cdots x_k}{1-x_1\cdots x_k a_2}\,\frac{1-x_1\cdots x_k a_2 x}{1-x_1\cdots x_k x},$$

$$a_4 = \frac{1-x_{k+1}\cdots x_n}{1-x_{k+1}\cdots x_n a_1}\,\frac{1-x_{k+1}\cdots x_n x a_1}{1-x_{k+1}\cdots x_n x}.$$

Identifying $x_i\cdots x_k$ with α_3 and $x_{k+1}\cdots x_u$ with α_4, we recover the formula of Sec. III.

Finally, the symmetry properties [Eqs. (3.4) and (3.5)] are trivially proved from the symmetry properties of the eight-point function.

APPENDIX C: PROOF OF (4.16)

Throughout this Appendix we assume that all z_J are small, so that in the following equations higher-order terms of z's are always disregarded. We first prove the following four lemmas:

Lemma 1. The shortest line[22] from the point k_J to k_{J-1} is given by

$$X_{J,J-1} = 1 - \frac{(1-x_{J-1})(1-y_{J-1})}{1-x_{J-1}y_{J-1}}\,\frac{(1-x_J)(1-y_J)}{1-x_J y_J} z_J, \qquad J=1,2,\cdots,N+1 \tag{C1}$$

where

$$X_{1,0} = A_1, \quad X_{N+1,N} = \bar{A}_N,$$
$$x_0 = y_0 = x_{N+1} = y_{N+1} = 0.$$

Proof. The line $X_{J,J-1}$ is determined from x_J, y_J, x_{J-1}, y_{J-1}, and z_J (Fig. 19) by the use of the duality function f introduced in (3.2):

$$X_{J,J-1} = f(z_J; x_J, y_J, y_{J-1}, x_{J-1}). \tag{C2}$$

The power-series expansion of (C2) with respect to z_J leads us to (C1).

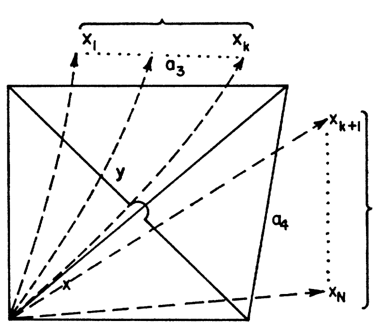


FIG. 18. The line y as a function of a_1, a_2, a_3, a_4, and x.

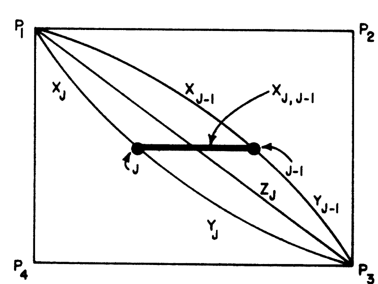


FIG. 19. The shortest line $X_{J,J-1}$ as a function of x_J, y_J, x_{J-1}, y_{J-1}, and z_J.

[22] "Shortest line" is any line that does not cut any z_i line twice.

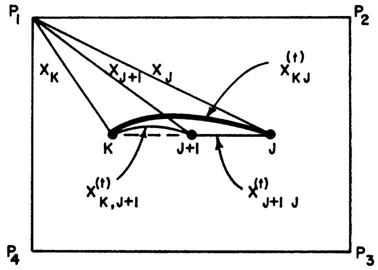


FIG. 20. A nodeless line from K to J; $X_{KJ}^{(+)}$.

Lemma 2. The line from K to J passing above all points L $(K>L>J)$, i.e., the line crossing x_L $(K>L>J)$ (Fig. 20), is given by

$$X_{K,J}^{(+)}=1-\frac{(1-x_K)(1-y_K)}{1-x_Ky_K}\left(\prod_{i=J+1}^{K}\frac{x_i(1-y_i)^2}{(1-x_iy_i)^2}\right)\times\frac{(1-x_J)(1-y_J)}{1-x_Jy_J}z_{J+1}\cdots z_K. \quad \text{(C3)}$$

Proof. This is proved by mathematical induction. The formula (C2) is true for $K=J+1$ because of Lemma 1. Let us assume (C2) to hold when J is $J+1$. Then $X_{K,J}^{(+)}$ is expressed in terms of $X_{J+1,J}$, x_J, x_K, x_{J+1}, and $X_{K,J+1}^{(+)}$ (Fig. 20) as

$$X_{K,J}^{(+)}=f(x_{J+1};X_{J+1,J},x_J,x_K,X_{K,J+1}^{(+)}). \quad \text{(C4)}$$

The right-hand side of (C4) is expanded in powers around $X_{J+1,J}=1$ and $X_{K,J+1}^{(+)}=1$ to be

$$1-\frac{\alpha\beta x_{J+1}}{(1-\alpha\beta x_{J+1})^2}(1-X_{J+1,J})(1-X_{K,J+1}^{(+)}), \quad \text{(C5)}$$

where α and β are given by

$$x_J=\frac{1-\alpha}{1-\alpha x_{J+1}}\,\frac{1-\alpha x_{J+1}X_{J+1,J}}{1-\alpha X_{J+1,J}}, \quad \text{(C6)}$$

$$x_K=\frac{1-\beta}{1-\beta x_{J+1}}\,\frac{1-\beta x_{J+1}X_{K,J+1}^{(+)}}{1-\beta X_{K,J+1}^{(+)}}. \quad \text{(C7)}$$

Since $(1-X_{J+1,J})(1-X_{K,J+1}^{(+)})=O(Z_{J+1}\cdots z_K)$, we take the zeroth order in α and β of (C5). In (C6), because $X_{J+1,J}\to 1$ as $z_{J+1}\to 0$ and $x_J\neq 1$, α should approach unity. The same is true for β. Therefore the substitution of $X_{J+1,J}$ into (C5) gives

$$X_{K,J}^{(+)}=1-\frac{x_{J+1}}{(1-x_{J+1})^2}(1-X_{K,J+1}^{(+)})\times\frac{(1-x_{J+1})(1-y_{J+1})}{1-x_{J+1}y_{J+1}}\,\frac{(1-x_J)(1-y_J)}{1-x_Jy_J}z_{J+1}.$$

This proves (C3).

Lemma 3. The line from K to J passing under all points L $(K>L>J)$, i.e., the line crossing y_L (for all L of $K>L>J$), is given by

$$X_{K,L}^{(-)}=\{\text{interchange of } x_i\leftrightarrow y_i \text{ in (C3)}\}. \quad \text{(C8)}$$

Proof. From the symmetry of the dual diagram and Lemma 2, this is obvious.

Lemma 4. The shortest line[22] from K to J crossing x_L and $y_{L'}$ $(K>L, L'>J$, and $L\neq L')$ is given by

$$X_{K,J}^{(C,C')}=1-\frac{(1-x_K)(1-y_K)}{1-x_Ky_K}\left(\prod_{i\in C}\frac{x_i(1-y_i)^2}{1-x_iy_i}\right)\times\left(\prod_{j\in C'}\frac{y_j(1-x_j)^2}{1-x_jy_j}\right)\frac{(1-x_J)(1-y_J)}{1-x_Jy_J}z_{J+1}\cdots z_K, \quad \text{(C9)}$$

where C (C') are a class of lines x_i (y_j) which are crossed by the line $X_{K,J}^{(C,C')}$.

Proof. Equation (C9) is true for arbitrary K and J if either the set C or C' is "empty" due to Lemmas 2 and 3. Let us call such a line the "nodeless" line. The line which crosses some of $X_{L,L-1}(K\geqq L>J)$ n times is called the line with n nodes. We assume that (C9) is true for the line with $n-1$ nodes or less and for arbitrary K and J (Fig. 21). Then we prove (C9) for the line with n nodes.

Let us consider a line $X_{K,J}^{(n)}$ with n nodes, which has the nth node between $L+1$ and L $(K>L>J)$ (bold line in Fig. 21). We want to determine $X_{K,J}^{(n)}$ from $a_1a_2a_3a_4$ and $X_{L+1,L}$ in Fig. 21. The lines a_1 and a_2 are nodeless. The line a_3 (a_4) goes from K to L $(L+1)$, crossing the same lines (x and y) that $X_{K,J}^{(n)}$ does. Therefore, a_3 (a_4) has $n-1$ nodes ($n-1$ or $n-2$). Our problem is to prove that

$$X_{K,J}^{(n)}=1-A\frac{x_{L+1}(1-y_{L+1})^2}{(1-x_{L+1}y_{L+1})^2}\times\frac{y_L(1-x_L)^2}{(1-x_Ly_L)^2}Bz_{J+1}\cdots z_K \quad \text{(C10)}$$

under the following assumptions:

$$a_1=1-B\frac{y_L(1-x_L)^2}{(1-x_Ly_L)^2}\times\frac{(1-x_{L+1})(1-y_{L+1})}{1-x_{L+1}y_{L+1}}z_{J+1}\cdots z_{L+1}, \quad \text{(C11)}$$

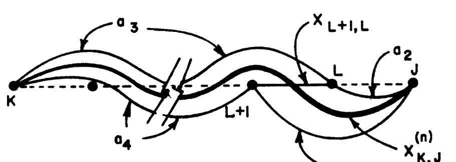


FIG. 21. A shortest line with n nodes $X_{K,J}^{(n)}$.

$$a_2=1-B\frac{(1-x_L)(1-y_L)}{1-x_Ly_L}z_{J+1}\cdots z_L\,, \tag{C12}$$

$$a_3=1-A\frac{x_{L+1}(1-y_{L+1})^2}{(1-x_{L+1}y_{L+1})^2}\times\frac{(1-x_L)(1-y_L)}{1-x_Ly_L}z_{L+1}\cdots z_K\,, \tag{C13}$$

$$a_4=1-A\frac{(1-x_{L+1})(1-y_{L+1})}{1-x_{L+1}y_{L+1}}z_{L+2}\cdots z_K\,, \tag{C14}$$

and

$$X_{L+1,L}=1-\frac{(1-x_{L+1})(1-y_{L+1})}{1-x_{L+1}y_{L+1}}\times\frac{(1-x_L)(1-y_L)}{1-x_Ly_L}z_{L+1}. \tag{C15}$$

Here A and B are known functions of x_i and y_i from (C9) for the $n-1$ nodes or less, factors which correspond to lines from K to $L+2$ and from $L-1$ to J, respectively. Using the duality function f, we write $X_{K,J}{}^{(n)}$ in terms of a_i and $X_{J+1,J}$, and expand it in powers of $(1-a_1)$ and $(1-a_4)$:

$$X_{K,J}{}^{(n)}=f(X_{J+1,J};a_1,a_2,a_3,a_4)\sim 1-\frac{\alpha\beta}{(1-\alpha\beta)^2}(1-a_1)(1-a_4). \tag{C16}$$

Since the factor $(1-a_1)(1-a_4)$ is of order $z_{J+1}\cdots z_K$, we may take the limit as the z's$\rightarrow 0$ for $\alpha\beta/(1-\alpha\beta)^2$. The explicit formulas of α and β are

$$\alpha=2\xi_3/\{2\xi_3+(\xi_4\delta-\xi_4\xi_3-\delta\xi_3)+[4\xi_3\xi_4\delta+(\xi_4\delta-\xi_4\xi_3-\delta\xi_3)^2-4\xi_4\delta\xi_3{}^2]^{1/2}\}\,, \tag{C17}$$

$$\beta=(\xi_3\leftrightarrow\xi_2\,,\ \xi_4\leftrightarrow\xi_1)\,, \tag{C18}$$

where

$$\xi_i=1-a_i \quad\text{and}\quad \delta=1-X_{J+1,J}.$$

By careful consideration of the order of magnitude of ξ_i and δ, $O(\xi_3)\sim O(\xi_4\delta)$, etc., we can show that

$$\alpha\rightarrow x_{L+1}\,,\quad \beta\rightarrow 1\,, \tag{C19}$$

as the z's$\rightarrow 0$.

Substitution of (C19), (C11), and (C14) into (C16) leads us to (C10), which guarantees Lemma 4 for the line with n nodes. Q.E.D.

Proof of (4.16). To prove (4.16), we first substitute (C9) into (4.14a). Lines other than the shortest,[22] which cross a certain z_i twice or more, can be disregarded in the evaluation, because the shortest line is the higher order in z_i. Note that such a line has a structure like $1-O(z_K\cdots z_i{}^2\cdots z_{J+1})$. The second term in (4.14a) is

$$-\ln C_N=z_1\cdots z_{N+1}\prod_{i=1}^{N}\left(\frac{x_i(1-y_i)^2+y_i(1-x_i)^2}{(1-x_iy_i)^2}\right). \tag{C20}$$

The first term in (4.14a) is the sum of the following terms:

$$\frac{\ln X_{N+1,L_m}\ln X_{L_m,L_{m-1}}\cdots\ln X_{L_1,0}}{\ln(B_{L_m}\bar{B}_{L_m})\ln(B_{L_{m-1}}\bar{B}_{L_{m-1}})\cdots\ln(B_{L_1}\bar{B}_{L_1})}. \tag{C21}$$

The summation is taken over all possible L_i's and m such that $N+1>L_m>L_{m-1}>\cdots>L_1>0$ for $m=1,\ldots,N$. Substituting (C9) into (C21), and replacing $\ln(B_L\bar{B}_L)$ by $\ln(x_Ly_L)$, we write (4.14a) in terms of x_L, y_L, and z_L.

Next we compare the result so obtained with

$$\prod_{J=1}^{N}\left(\frac{x_J(1-y_J)^2+y_J(1-x_J)^2}{(1-x_Jy_J)^2}-\frac{(1-x_J)^2(1-y_J)^2}{\ln(x_Jy_J)}\right)z_1\cdots z_{N+1}\,,$$

term by term. This proves (4.16). Q.E.D.