Current Algebra, K_{l3}^+ Form Factors, and Radiative K_{l3}^+ Decay*

E. FISCHBACH† AND J. SMITH
Institute for Theoretical Physics, State University of New York at Stony Brook, Stony Brook, New York 11790

(Received 19 March 1969)

The complete gauge-invariant matrix element for the decays $K^+ \to \pi^0 l^+ \nu\gamma$ is derived up to and including terms linear in the photon energy k. The contributions of order k^{-1} and k^0 are evaluated through Low's theorem, while the contributions linear in k, which represent the most important structure-dependent terms, are related to the vector and axial-vector couplings in $K^+ \to l^+\nu\gamma$ via the hypothesis of partially conserved axial-vector current (PCAC). It is shown that the spectra and rates from the radiative decay provide an independent means of measuring the K_{l3} parameters $f_+(0)$, ξ, and $\lambda_\pm$. Results are given for the rates and photon spectra in both $K^+ \to \pi^0 e^+\nu\gamma$ and $K^+ \to \pi^0\mu^+\nu\gamma$ decays for representative values of the K_{l3} parameters. With improved experimental data, it is then possible to check the predictions of various theoretical models both for the t dependence of the K_{l3} form factors and for the size of the structure-dependent matrix elements.

I. INTRODUCTION

SEMILEPTONIC K-meson decay modes have been the subject of many experimental[1] and theoretical[2] papers. Although the general picture of these decays agrees with $V-A$ theory, μ-e universality, and the $|\Delta \mathbf{I}| = \frac{1}{2}$ rule, there are still some points which need further clarification. We mention in particular the t dependence of the form factors $f_+(t)$ and $f_-(t)$ and the different values of Reξ, obtained from the rate and polarization studies. Better data are obviously required before one can distinguish between the various theoretical models proposed to explain $f_+(t)$ and $f_-(t)$. In this paper we study the radiative K_{l3}^+ decay modes with the aim of obtaining information about $f_\pm(t)$ and the structure-dependent matrix elements in the $K\pi\gamma$ interaction. Radiative K_{l3}^0 decays will be the subject of a later paper. Before presenting our calculation, we would like to discuss the general problem of weak leptonic radiative pseudoscalar meson decays and summarize the work done by other authors.

Let us first consider the radiative leptonic weak decays $\pi \to l\nu\gamma$,[3] and $K \to l\nu\gamma$.[4] The two-body weak-matrix elements are proportional to the lepton mass, so the decay rates are much smaller for the electron mode than the muon mode. When a photon is emitted by the charged-particle lines (bremsstrahlung), the matrix element, which is still proportional to the lepton mass, can be calculated by standard quantum electrodynamics, and the decay rate has an infrared divergence associated with the zero photon mass. Photon emission from the interaction region has a normal spectrum, and the matrix element can be separated into a vector part and an axial-vector part due to parity violation. In general, the vector amplitude[5] can be related to the two-photon decay matrix element of the corresponding neutral meson, by the conserved-vector-current hypothesis of Feynman and Gell-Mann.[6] Of course, this is not trivial for the strangeness-changing vector current, because this current is not conserved. Nevertheless, some information can be obtained from sum rules and this point will be discussed in more detail later. The size of the axial-vector contribution can be roughly estimated from current-algebra techniques.[7] As far as experimental data are concerned, the decay $\pi \to \mu\nu\gamma$ is entirely dominated by inner bremsstrahlung radiation and yields no new information. Even though there is sufficient phase space in $K \to \mu\nu\gamma$ [a branching ratio $\Gamma(K \to \mu\nu\gamma)/\Gamma(K \to \text{all}) = 1.0 \times 10^{-4}$ is quoted in Ref. 4], there are no experimental results on this decay. The experiment[8] on $\pi \to e\nu\gamma$ yielded a rate consistent with the conserved-vector-current prediction and showed the presence of an axial-vector term. Note that there is

---

* Work supported under Contract No. AT(30-1)-3668B of the U. S. Atomic Energy Commission.

† Address as of September, 1969: The Niels Bohr Institute, Copenhagen Ø, Denmark.

[1] D. R. Botterill, R. M. Brown, A. B. Clegg, I. F. Corbett, G. Culligan, J. McL. Emmerson, R. C. Field, J. Garvey, P. B. Jones, N. Middlemas, D. Newton, T. W. Quirk, G. L. Salmon, P. H. Steinberg, and W. S. C. Williams, Phys. Rev. Letters **21**, 766 (1968); D. Cutts, R. Steining, C. Wiegand, M. Deutsch, Phys. Rev. Letters **20**, 955 (1968); L. B. Auerbach, A. K. Mann, W. K. McFarlane, and F. J. Sciulli, *ibid.* **19**, 464 (1967). Further references can be found in these papers and in the *Proceedings of the International Conference on High-Energy Physics, Vienna, 1968* (CERN, Geneva, 1968).

[2] H. T. Nieh, Phys. Rev. Letters **21**, 116 (1968); L. N. Chang and Y. C. Leung, *ibid.*, **21**, 122 (1968); B. W. Lee, *ibid.* **20**, 617 (1968); A. K. Mann and H. Primakoff, *ibid.* **20**, 32 (1968); D. P. Majumdar, *ibid.* **20**, 971 (1968). Further references can be found in these papers.

[3] S. G. Brown and S. A. Bludman, Phys. Rev. **136B**, 1160 (1964); V. G. Vaks and B. L. Ioffe, Nuovo Cimento **10**, 342 (1958).

[4] D. Neville, Phys. Rev. **124**, 2037 (1961).

[5] W. Kummer and W. Majerotto, Nuovo Cimento **55**, 558 (1968); V. F. Müller, Z. Physik **172**, 224 (1963).

[6] R. P. Feynman and M. Gell-Mann, Phys. Rev. **109**, 193 (1958). See also S. S. Gershtein and Ya. B. Zeldovich, Zh. Eksperim. i Teor. Fiz., **29**, 698 (1955) [English transl.: Soviet Phys.—JETP **2**, 576 (1957)].

[7] T. Das, V. S. Mathur, and S. Okubo, Phys. Rev. Letters **19**, 859 (1967); A. Q. Sarker, Phys. Rev. **173**, 1749 (1968); J. S. Vaishya, *ibid.* **173**, 1757 (1968); Riazuddin and Fayyazuddin, *ibid.* **171**, 1428 (1968); A. I. Vainshtein, Zh. Eksperim. i Teor. Fiz. Pis'ma v Redaktsiyu **6**, 341 (1967) [English transl.: Soviet Phys.—JETP Letters **6**, 815 (1967)]; R. Rockmore, Phys. Rev. **177**, 2573 (1968).

[8] P. Depommier, J. Heintze, C. Rubbia, and V. Soergel, Phys. Letters **7**, 285 (1963).

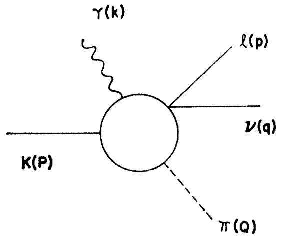

FIG. 1. The Feynman diagram for the decay $K\to\pi l\nu\gamma$.

no interference between the terms of different parity. Considering the small rate for $K\to e\nu$,[9] it is not surprising that the radiative mode has never been observed. [Typically $\Gamma(K\to e\nu\gamma)/\Gamma(K\to \text{all})\cong 10^{-8}$.]

The situation regarding four-body radiative semileptonic meson decays is entirely different. First of all, the matrix elements are not proportional to the lepton mass so there is no suppression of the decay rate into electrons. Indeed, this rate is larger than the muon decay rate because of increased phase space. Also, because of the presence of four particles in the final state, all interference terms contribute to the decay rate. We propose to study $K^+\to\pi^0e^+\nu\gamma$ and $K^+\to\pi^0\mu^+\nu\gamma$ but not $\pi\to\pi e\nu\gamma$. Experiment[10] gives a branching ratio

$$\Gamma(K^+\to\pi^0e^+\nu\gamma,\ E_\gamma>30\text{ MeV})/\Gamma(K^+\to\pi^0e^+\nu)=(1.2\pm0.8)\times10^{-2}$$

and more data, especially on the photon spectrum, would be necessary to determine the presence of structure-dependent radiation.[11] Note that the present paper deals only with hard-photon decays, where a photon is observed by the detection apparatus. The problem of radiative corrections to the K_{l3} spectra and rates, where the photon is not detected and its spectrum is combined with self-energy and vertex corrections, has been treated in detail by Ginsberg.[12] While the rates for the K_{e3} and $K_{e3\gamma}$ decays are only sensitive to the form factor $f_+(t)$, the $K_{\mu3}$ and $K_{\mu3\gamma}$ decays depend upon both $f_+(t)$ and $\xi(t)\equiv f_-(t)/f_+(t)$. In principle the rate for the radiative muon decay could give another estimate of $\xi(t)$.

Before concluding this general discussion, it would be appropriate to discuss the effects of time-reversal-violating interactions in these modes. MacDowell[13] and Gervais *et al.*[14] have already considered the mode $K\to\mu\nu\gamma$ with possible complex structure-dependent amplitudes. However, no experimental data are available. K_{l3} decay has been fitted with a complex value of ξ but so far all evidence is consistent with $\text{Im}\xi=0$. However, if there is a C-violating electromagnetic interaction with hadrons,[15] it would give rise to an asymmetry of the type $\mathbf{p}_\pi\cdot\mathbf{p}_e\times\mathbf{p}_\gamma$ in $K_{l3\gamma}$ decay. Such terms are probably very small and there is a problem of separating these events from $K\to\pi\pi e\nu\gamma$. The new value of the upper limit for the neutron dipole moment ($|d|<4\times10^{-23}$ cm) reported by Dress *et al.*[16] indicates no appreciable C violation in electromagnetic interactions with hadrons. Chu *et al.*[17] recently considered time-reversal invariance in $K\to\mu\nu e^+e^-$ decay. In the present paper we do not consider any effects of time-reversal violation.

The outline of this paper is as follows. Section II contains the derivation of the matrix element based on Low's theorem. This derivation is given in considerable detail so that we do not need to repeat this disscusion in a forthcoming paper on K^0 decay. The hypothesis of partially conserved axial-vector current (PCAC) is used in Sec. III to relate the structure-dependent form factors in $K^+\to\pi^0e^+\nu\gamma$ to those in $K^+\to e^+\nu\gamma$, which are known from vector-meson dominance and dispersion theoretic sum rules. A discussion of some technical details regarding the evaluation of the spectra and rates is given in Sec. IV, and our conclusions are given in Sec. V. Appendix A contains the results of the spin and polarization sums, and a tabulation of phase-space integrals is given in Appendix B.

II. MATRIX ELEMENT FOR $K^+\to\pi^0l^+\nu\gamma$

The matrix element for the process $K^+(P)\to\pi^0(Q)+l^+(p)+\nu(q)+\gamma(k)$ shown in Fig. 1 is given by

$$\mathfrak{M}={}_{\text{out}}\langle\pi^0l\nu\gamma|K^+\rangle_{\text{in}}=-i(2\pi)^4\delta^4(P-Q-p-q-k)\times\left(\frac{mm_\nu}{8P_0Q_0p_0q_0k_0V^5}\right)^{1/2}\frac{eG\sin\theta}{\sqrt{2}}T,\quad(2.1)$$

where T is defined by

$$T=-\bar{u}(p)i\gamma\cdot\epsilon\frac{i\gamma\cdot(p+k)-m}{2p\cdot k}[f_1(t)i\gamma\cdot P+f_2(t)i\gamma\cdot Q]\times(1+\gamma_5)v(q)+\langle\pi^0\gamma|V_\nu^{4-i5}(0)+A_\nu^{4-i5}(0)|K^+\rangle\times\bar{u}(p)\gamma_\nu(1+\gamma_5)v(q),\quad(2.2)$$

$$t=-(P-Q)^2=-\Delta^2.$$

[9] D. R. Botterill, R. M. Brown, I. F. Corbett, G. Culligan, J. McL. Emmerson, R. C. Field, J. Garvey, P. B. Jones, N. Middlemas, D. Newton, T. W. Quirk, G. L. Salmon, P. Steinberg, and W. S. C. Williams, Phys. Rev. **171**, 1402 (1968). Recently N. J. Carron and R. L. Schult [University of Illinois report 1969 (unpublished)] have argued that *structure-dependent* contributions to this decay may raise the branching ratio to $\sim10^{-5}$.

[10] E. Bellotti and A. Pullia, in *Proceedings of the Heidelberg International Conference on Elementary Particles* (North-Holland Publishing Co., Amsterdam, 1968), p. 278.

[11] C. S. Lai, Indiana University Report, 1967 (unpublished); J. S. Vaishya and K. C. Gupta, Phys. Rev. **165**, 1696 (1968).

[12] E. S. Ginsberg, Phys. Rev. **142**, 1035 (1966); **162**, 1570 (1967); N. P. Chang, *ibid.* **129**, 399 (1963); **131**, 1272 (1963).

[13] S. MacDowell, Phys. Rev. Letters **17**, 1116 (1966); **18E**, 227 (1967).

[14] J-L. Gervais, J. Iliopoulos, and J. M. Kaplan, Phys. Letters **20**, 432 (1966).

[15] J. Bernstein, G. Feinberg, and T. D. Lee, Phys. Rev. **139B**, 1650 (1965); S. Barshay, Phys. Letters **17**, 78 (1965).

[16] W. B. Dress, J. K. Baird, P. D. Miller, and N. F. Ramsey, Bull. Am. Phys. Soc. **13**, 1380 (1968).

[17] W. T. Chu, T. Ebata, and D. M. Scott, Phys. Rev. **166**, 1577 (1968). See also W. Flagg, *ibid.* **178**, 2387 (1969); E. S. Ginsberg, *ibid.* **135B**, 792 (1965).

In Eqs. (2.1) and (2.2) e ($e>0$) is the electric charge, G is the Fermi constant ($GM_P{}^2=1.02\times10^{-5}$), and θ is the Cabibbo angle ($\sin\theta\cong0.21$).[18] Throughout this paper, M, μ, m, and m_ν denote, respectively, the masses of the kaon, pion, lepton (e or μ), and neutrino. [The factor m_ν appearing in Eq. (2.1) cancels against the factor m_ν in Eqs. (A8)–(A12) when computing $|\mathfrak{M}|^2$, so that the limit $m_\nu\to0$ is well defined.] The two terms in Eq. (2.2) correspond, respectively, to bremsstrahlung from the external lepton [Fig. 2(a)] and to radiation from the hadrons. [The minus sign in the first term of Eq. (2.2) arises from our convention for the electromagnetic vertex: For emission of a photon by a particle of charge e the vertex factor is $-e\gamma_\mu$.] $f_1(t)$ and $f_2(t)$ are the usual K_{l3} form factors defined by

$$\langle\pi^0(Q)|V_\nu{}^{4-i5}(0)|K^+(P)\rangle =(4P_0Q_0V^2)^{-1/2}[f_1(t)P_\nu+f_2(t)Q_\nu]. \quad (2.3)$$

In the limit of exact $SU(3)$, $f_+(0)=\frac{1}{2}[f_1(0)+f_2(0)]=1/\sqrt{2}$, $f_-(0)=\frac{1}{2}[f_1(0)-f_2(0)]=0$. The $\Delta S=1$ semileptonic weak Hamiltonian density is assumed to be given by the conventional Cabibbo model

$$\mathcal{H}(x)=\frac{G}{\sqrt{2}}\sin\theta[V_\nu{}^{4-i5}(x)+A_\nu{}^{4-i5}(x)]^\star l_\nu(x)+\text{H.c.}$$
$$\mathcal{J}_\nu{}^\star(x)=[V_\nu(x)+A_\nu(x)]^\star=(-\mathcal{J}_i{}^\dagger(x),\mathcal{J}_4{}^\dagger(x)),\quad i=1,2,3 \quad (2.4)$$

where l_ν is the lepton current and

$$V_\nu{}^{4-i5}(x)=\mathfrak{F}_\nu{}^4(x)-i\mathfrak{F}_\nu{}^5(x),$$
$$A_\nu{}^{4-i5}(x)=\mathfrak{F}_{5\nu}{}^4(x)-i\mathfrak{F}_{5\nu}{}^5(x). \quad (2.5)$$

In Eqs. (2.5) the F-spin currents will be assumed to obey the usual $SU(3)\otimes SU(3)$ commutation relations[19]

$$[\mathfrak{F}_0{}^a(\mathbf{x},0),\mathfrak{F}_0{}^b(\mathbf{x}',0)]=if_{abc}\mathfrak{F}_0{}^c(\mathbf{x},0)\delta^3(\mathbf{x}-\mathbf{x}')$$
$$[\mathfrak{F}_0{}^a(\mathbf{x},0),\mathfrak{F}_{50}{}^b(\mathbf{x}',0)]=if_{abc}\mathfrak{F}_{50}{}^c(\mathbf{x},0)\delta^3(\mathbf{x}-\mathbf{x}') \quad (2.6)$$
$$[\mathfrak{F}_{50}{}^a(\mathbf{x},0),\mathfrak{F}_{50}{}^b(\mathbf{x}',0)]=if_{abc}\mathfrak{F}_0{}^c(\mathbf{x},0)\delta^3(\mathbf{x}-\mathbf{x}').$$

For later purposes we note that the two terms in Eq. (2.2) are not separately gauge-invariant. The decomposition of Eq. (2.2) is, however, useful since it permits us to extract from T a contribution which can be directly evaluated, namely, the lepton bremsstrahlung. The significance of this will become evident later. To proceed further we must analyze the radiation from the hadrons. We begin by performing a Lehmann-Symanzik-Zimmermann (LSZ) reduction on the photon giving

$$\langle\pi^0\gamma|V_\nu{}^{4-i5}(0)+A_\nu{}^{4-i5}(0)|K^+\rangle=i\epsilon_\mu\int d^4x\,e^{-ik\cdot x}(-\Box_x)$$
$$\times\langle\pi^0|T(\mathfrak{A}_\mu(x)[V_\nu{}^{4-i5}(0)+A_\nu{}^{4-i5}(0)])|K^+\rangle$$
$$\equiv i\epsilon_\mu(M_{\mu\nu}{}^V+M_{\mu\nu}{}^A), \quad (2.7)$$

where ϵ_μ is the polarization vector for the emitted photon, and $\mathfrak{A}_\mu(x)$ is the operator which annihilates the photon field. [Note that the normalization factor $(2k_0V)^{-1/2}$ in the LSZ reduction has been absorbed in the definition of T in Eq. (2.1).] $M_{\mu\nu}{}^V$ and $M_{\mu\nu}{}^A$ may be covariantly decomposed as follows:

$$M_{\mu\nu}{}^V=A\delta_{\mu\nu}+Bk_\mu k_\nu+CQ_\mu Q_\nu+DP_\mu P_\nu+Ek_\mu P_\nu+FP_\mu k_\nu +GP_\mu Q_\nu+HQ_\mu P_\nu+IQ_\mu k_\nu+Jk_\mu Q_\nu, \quad (2.8)$$

$$M_{\mu\nu}{}^A=\epsilon_{\mu\nu\alpha\beta}(bP_\alpha k_\beta+cQ_\alpha k_\beta+dP_\alpha Q_\beta). \quad (2.9)$$

In Eqs. (2.8) and (2.9), the coefficients $A,\cdots,J$, b, c, and d are, in general, functions of the variables $\nu=P\cdot k$, $t=-(P-Q)^2$, and $u=-(k-Q)^2$, and we have retained only those structures which are at most bilinear in the particle momenta in Eq. (2.9). In order to establish that it is $M_{\mu\nu}{}^A$ (rather than $M_{\mu\nu}{}^V$) which is proportional to $\epsilon_{\mu\nu\alpha\beta}$, we can examine the transformation properties of the effective Lagrangian density

$$\mathcal{L}(K\to\pi l\nu\gamma)=(g_1F_{\mu\nu}+g_2\tilde{F}_{\mu\nu})\pi\overleftrightarrow{\partial}_\mu Kl_\nu, \quad (2.10)$$

where

$$F_{\mu\nu}=\partial_\mu\mathfrak{A}_\nu-\partial_\nu\mathfrak{A}_\mu$$
$$\tilde{F}_{\mu\nu}=\tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta}F_{\alpha\beta}. \quad (2.11)$$

Evidently the term $\tilde{F}_{\mu\nu}\pi\overleftrightarrow{\partial}_\mu K$ has the transformation properties of an axial vector (since $\epsilon_{\mu\nu\alpha\beta}$ is a tensor *density*) which gives Eq. (2.9). We set aside for the moment the problem of calculating b, c, and d in Eq. (2.9) and turn to the form factors $A,\cdots,J$ in Eq. (2.8). The procedure we will adopt in obtaining information about these form factors is the following: We will relate $M_{\mu\nu}{}^V$ to the matrix element for K_{l3} decay via a Ward identity, and to the matrix element for $K\to l\nu\gamma$ by PCAC. It will be shown that in the soft-pion limit this is sufficient to determine the structure of $M_{\mu\nu}{}^V$. To derive the Ward identity, we begin by letting $-\Box_x$ in Eq. (2.7) act to the right, giving[20]

$$M_{\mu\nu}{}^V=i\int d^4x\,e^{-ik\cdot x}\{\langle\pi^0|T(j_\mu{}^\gamma(x)V_\nu{}^{4-i5}(0))|K^+\rangle$$
$$-\delta(x_0)\langle\pi^0|[\partial_0\mathfrak{A}_\mu(x),V_\nu{}^{4-i5}(0)]|K^+\rangle$$
$$-\partial_0(\delta(x_0)\langle\pi^0|[\mathfrak{A}_\mu(x),V_\nu{}^{4-i5}(0)]|K^+\rangle)\}, \quad (2.12)$$

where $j_\mu{}^\gamma(x)=\mathfrak{F}_\mu{}^3(x)+(1/\sqrt{3})\mathfrak{F}_\mu{}^8(x)$ is the electromagnetic current. It is claimed[20] that the second and third terms in Eq. (2.12) will cancel against Schwinger terms which arise from the first term in the process of deriving the Ward identity. Let us assume this for the present and retain only the first term in Eq. (2.12). Differentiating Eq. (2.12) with respect to x, we obtain

$$ik_\mu M_{\mu\nu}{}^V=i\int d^4x\,e^{-ik\cdot x}$$
$$\times\langle\pi^0|\delta(x_0)[j_0{}^\gamma(x),V_\nu{}^{4-i5}(0)]|K^+\rangle. \quad (2.13)$$

[18] N. Brene, M. Roos, and A. Sirlin, Nucl. Phys. **B6**, 255 (1968).
[19] M. Gell-Mann, Physics **1**, 63 (1964).
[20] See for example, S. L. Adler and R. F. Dashen, *Current Algebras* (W. A. Benjamin, Inc., New York, 1968), p. 218.

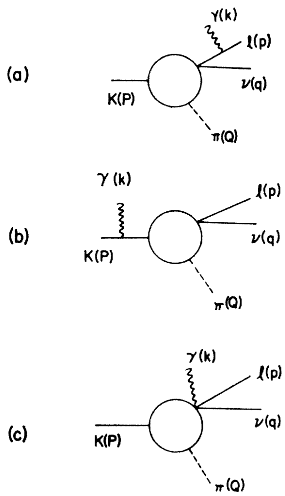

FIG. 2. (a) Inner bremsstrahlung from the lepton line. (b) Inner bremsstrahlung from the kaon line. (c) The seagull diagram.

If we neglect possible Schwinger terms arising from the commutator of j_0^γ with V_i $(i=1, 2, 3)$, we have from Eq. (2.6)

$$\delta(x_0)[j_0^\gamma(x), V_\nu^{4-i5}(0)] = -\delta^4(x)V_\nu^{4-i5}(x) \qquad (2.14)$$

whence

$$k_\mu M_{\mu\nu}{}^V = -\langle\pi^0|V_\nu{}^{4-i5}(0)|K^+\rangle = -[f_1(t)P_\nu + f_2(t)Q_\nu], \qquad (2.15)$$

which is the desired Ward identity. [As before, all normalization factors in Eq. (2.15) have been absorbed in the definition of T.] Next we return to the alleged cancellation of the Schwinger terms. To proceed,[21] we note that since T must be linear in the photon polarization vector, we can write

$$T = \epsilon_\mu M_\mu, \qquad (2.16)$$

with M_μ defined implicitly by Eq. (2.2). Gauge invariance (i.e., masslessness of the physical photon) then requires that T be invariant under the transformation $\epsilon_\mu \to \epsilon_\mu + \lambda k_\mu$ where λ is some scalar, which in turn requires that M_μ satisfy the condition

$$k_\mu M_\mu = 0 \quad \text{at} \quad k^2 = 0. \qquad (2.17)$$

From Eqs. (2.2) and (2.7), we can write $k_\mu M_\mu$ explicitly:

$$\begin{aligned} 0 = k_\mu M_\mu &= \bar{u}(p)\gamma\cdot k(\gamma\cdot(p+k)+im)/2p\cdot k \\ &\quad \times [f_1(t)i\gamma\cdot P + f_2(t)i\gamma\cdot Q](1+\gamma_5)v(q) \\ &\quad + k_\mu M_{\mu\nu}{}^V l_\nu + k_\mu M_{\mu\nu}{}^A l_\nu \\ &\equiv k_\mu M_\mu{}^l + k_\mu M_{\mu\nu}{}^V l_\nu + k_\mu M_{\mu\nu}{}^A l_\nu, \end{aligned} \qquad (2.18)$$

where $l_\nu = i\bar{u}(p)\gamma_\nu(1+\gamma_5)v(q)$. From Eq. (2.9) we note that $M_{\mu\nu}{}^A l_\nu$ transforms differently under parity than $M_\mu{}^l$ and $M_{\mu\nu}{}^V l_\nu$. Hence by use of the Dirac equation we deduce that $M_{\mu\nu}{}^V$ must satisfy

$$\begin{aligned} k_\mu M_{\mu\nu}{}^V l_\nu = -k_\mu M_\mu{}^l &= -\bar{u}(p)[f_1(t)i\gamma\cdot P + f_2(t)i\gamma\cdot Q] \\ &\quad \times (1+\gamma_5)v(q) \\ &= -[f_1(t)P_\nu + f_2(t)Q_\nu]l_\nu, \end{aligned} \qquad (2.19)$$

which just gives Eq. (2.15).[22] We conclude from Eqs. (2.12)–(2.19) that the Ward identity of Eq. (2.15) must be exact in the sense that the Schwinger terms which were dropped from Eq. (2.14) must exactly cancel the *divergence* of the second and third terms in Eq. (2.12) which were also dropped. Note that in order to exhibit this cancellation, it is unnecessary to assume anything about the detailed structure of the terms which were dropped from Eq. (2.12). [In fact, as the preceding argument indicates, $k_\mu M_{\mu\nu}{}^V$ could have been evaluated directly from Eqs. (2.2) and (2.7) by use of gauge invariance alone.] Proceeding in an analogous manner, we deduce from Eq. (2.18) the gauge condition for $M_{\mu\nu}{}^A$, namely,

$$k_\mu M_{\mu\nu}{}^A l_\nu = 0, \qquad (2.20)$$

and hence that $d=0$ in Eq. (2.9). Equation (2.20) is the result we would expect had we derived a Ward identity for $M_{\mu\nu}{}^A$. Proceeding as in Eqs. (2.12)–(2.15), we would have found

$$k_\mu M_{\mu\nu}{}^A = -\langle\pi^0|A_\nu{}^{4-i5}(0)|K^+\rangle = 0 \qquad (2.21)$$

by parity arguments, which then gives Eq. (2.20). Returning to the problem of evaluating the form factors $A, \cdots, J$ we have from Eqs. (2.8) and (2.15)

$$\begin{aligned} k_\mu M_{\mu\nu}{}^V &= (A + Bk^2 + FP\cdot k + IQ\cdot k)k_\nu \\ &\quad + (DP\cdot k + Ek^2 + HQ\cdot k)P_\nu + (CQ\cdot k + GP\cdot k + Jk^2)Q_\nu \\ &= -f_1(t)P_\nu - f_2(t)Q_\nu. \end{aligned} \qquad (2.22)$$

Equating coefficients of the independent vectors k, P, and Q, we find

$$\begin{aligned} A + Bk^2 + FP\cdot k + IQ\cdot k &= 0, \\ DP\cdot k + Ek^2 + HQ\cdot k &= -f_1(t), \\ CQ\cdot k + GP\cdot k + Jk^2 &= -f_2(t), \end{aligned} \qquad (2.23)$$

and solving for D, F, and G we can write

$$\begin{aligned} M_{\mu\nu}{}^V &= A(\delta_{\mu\nu} - P_\mu k_\nu/P\cdot k) + Bk_\mu k_\nu \\ &\quad + C(Q_\mu Q_\nu - Q\cdot kP_\mu Q_\nu/P\cdot k) \\ &\quad + Ek_\mu P_\nu + H(Q_\mu P_\nu - Q\cdot kP_\mu P_\nu/P\cdot k) \\ &\quad + I(Q_\mu k_\nu - Q\cdot kP_\mu k_\nu/P\cdot k) \\ &\quad + Jk_\mu Q_\nu - [f_1(t)P_\nu + f_2(t)Q_\nu]P_\mu/P\cdot k. \end{aligned} \qquad (2.24)$$

From Eqs. (2.2), (2.7), (2.9), and (2.24) we can then

---

[21] Our argument closely parallels that of Ref. 20. See also L. S. Brown, Phys. Rev. **150**, 1338 (1966); D. G. Boulware and L. S. Brown, *ibid.* **156**, 1724 (1967); R. P. Feynman, in *Proceedings of the 1967 International Conference on Particles and Fields* (Interscience, New York, 1967), p. 111.

[22] Evidently had we considered the decay of K^- instead of K^+, the sign of the commutator in Eq. (2.14) would have changed as would the sign of the first term in Eq. (2.2), so that the Ward identity argument would go through as before.

write

$$T=\bar{u}(p)\left(\frac{\epsilon\cdot p}{p\cdot k}+\frac{\gamma\cdot\epsilon\gamma\cdot k}{2p\cdot k}\right)[f_1(t)i\gamma\cdot P+f_2(t)i\gamma\cdot Q]$$
$$\times(1+\gamma_5)v(q)+A(\epsilon\cdot l-\epsilon\cdot Pk\cdot l/P\cdot k)$$
$$+C(\epsilon\cdot QQ\cdot l-Q\cdot k\epsilon\cdot PQ\cdot l/P\cdot k)$$
$$+H(\epsilon\cdot QP\cdot l-Q\cdot k\epsilon\cdot PP\cdot l/P\cdot k)$$
$$+I(\epsilon\cdot Qk\cdot l-Q\cdot k\epsilon\cdot Pk\cdot l/P\cdot k)-[f_1(t)P\cdot l+f_2(t)Q\cdot l]$$
$$\times\epsilon\cdot P/P\cdot k+\epsilon_{\mu\nu\alpha\beta}\epsilon_\mu l_\nu(bP_\alpha k_\beta+cQ_\alpha k_\beta). \quad (2.25)$$

In choosing to solve for D, F, and G from Eq. (2.23) we are motivated by the desire to exhibit explicitly the kinematic singularities arising from the kaon pole diagram Fig. 2(b) which we know contributes to $M_{\mu\nu}{}^V$. Since this diagram can be calculated explicitly, it is advantageous to do so at this point before discussing the consequences of PCAC and the $Q=0$ limit.

Before exhibiting the contribution of the kaon pole, it would be useful to state explicitly what we are driving after. By virtue of Low's theorem,[23,24] the sum of the known contributions from the kaon pole diagram [Fig. 2(b)] and the previously considered electron bremsstrahlung diagram [Fig. 2(a)] determine the structure of T up to (but not including) terms linear in the photon momentum k. Consequently, the unknown form factors A, C, H, $\cdots$ can contribute only to terms which are at least linear in k. What we are attempting to do is to rewrite Eq. (2.25) in a form consistent with Low's theorem.

Let T_L denote the contribution to T from the terms up to (but not including) those linear in k. It can be shown that T_L may be computed from the corresponding nonradiative amplitude by the following simple recipe[24]: (1) Write down T_{ex}, the sum of contributions in which the photon is radiated from an external charged line. (2) Expand T_{ex} in a Taylor series about $k=0$. (3) Drop all terms from T_{ex} which are explicitly independent of k or which are of order k or higher. Denote the result of this step by T_{ex}'. (4) Add to T_{ex}' a contribution ΔT *independent of* k so as to make $T_{\text{ex}}'+\Delta T$ gauge invariant. Then

$$T_L=T_{\text{ex}}'+\Delta T. \quad (2.26)$$

For the present problem we have

$$T_{\text{ex}}=\bar{u}(p)\left(\frac{\epsilon\cdot p}{p\cdot k}+\frac{\gamma\cdot\epsilon\gamma\cdot k}{2p\cdot k}\right)[f_1(t,0,0)i\gamma\cdot P+f_2(t,0,0)i\gamma\cdot Q](1+\gamma_5)v(q)-F_K((P-k)^2+M^2)\frac{\epsilon\cdot P}{P\cdot k}$$
$$\times\bar{u}(p)\{f_1(-(\Delta-k)^2,(P-k)^2+M^2,0)i\gamma(P-k)+f_2(-(\Delta-k)^2,(P-k)^2+M^2,0)i\gamma\cdot Q\}(1+\gamma_5)v(q), \quad (2.27)$$

where F_K is the electromagnetic form factor of the kaon $[F_K(0)\equiv1]$, and $f_{1,2}=f_{1,2}$ ($t=-\Delta^2$, $\Delta_1=P^2+M^2$, $\Delta_2=Q^2+\mu^2$) are the K_{l3} form factors when the kaon and the pion are on the mass shell. Expanding Eq. (2.27) in a Taylor series about $k=0$, we find

$$T_{\text{ex}}=\bar{u}(p)\left(\frac{\epsilon\cdot p}{p\cdot k}+\frac{\gamma\cdot\epsilon\gamma\cdot k}{2p\cdot k}\right)[f_1(t)i\gamma\cdot P+f_2(t)i\gamma\cdot Q](1+\gamma_5)v(q)-\bar{u}(p)\frac{\epsilon\cdot P}{P\cdot k}[f_1(t)i\gamma\cdot(P-k)+f_2(t)i\gamma\cdot Q](1+\gamma_5)v(q)$$
$$-\frac{2\epsilon\cdot P}{P\cdot k}k\cdot(P-Q)\frac{\partial}{\partial t}\bar{u}(p)[f_1(t)i\gamma\cdot(P-k)+f_2(t)i\gamma\cdot Q](1+\gamma_5)v(q)+2\epsilon\cdot P\frac{\partial}{\partial\Delta_1}F_K(\Delta_1)\bigg|_{\Delta_1=0}$$
$$\times\bar{u}(p)[f_1(t)i\gamma\cdot(P-k)+f_2(t)i\gamma\cdot Q](1+\gamma_5)v(q)+2\epsilon\cdot P\bar{u}(p)\left[\frac{\partial}{\partial\Delta_1}f_1(t,\Delta_1,0)i\gamma\cdot(P-k)\right.$$
$$\left.+\frac{\partial}{\partial\Delta_1}f_2(t,\Delta_1,0)i\gamma\cdot Q\right](1+\gamma_5)v(q), \quad (2.28)$$

where $f(t)\equiv f(t,0,0)$. Next we drop all terms which are either independent of k or which are of order k or higher yielding

$$T_{\text{ex}}'=\bar{u}(p)\left(\frac{\epsilon\cdot p}{p\cdot k}+\frac{\gamma\cdot\epsilon\gamma\cdot k}{2p\cdot k}\right)[f_1(t)i\gamma\cdot P+f_2(t)i\gamma\cdot Q](1+\gamma_5)v(q)-\bar{u}(p)\frac{\epsilon\cdot P}{P\cdot k}[f_1(t)i\gamma\cdot(P-k)+f_2(t)i\gamma\cdot Q](1+\gamma_5)v(q)$$
$$+2\frac{\epsilon\cdot PQ\cdot k}{P\cdot k}\frac{\partial}{\partial t}\bar{u}(p)[f_1(t)i\gamma\cdot P+f_2(t)i\gamma\cdot Q](1+\gamma_5)v(q). \quad (2.29)$$

[23] F. E. Low, Phys. Rev. **110**, 974 (1958); T. H. Burnett and N. M. Kroll, Phys. Rev. Letters **20**, 86 (1968).
[24] S. L. Adler and Y. Dothan, Phys. Rev. **151**, 1267 (1966); J. Pestieau, *ibid.* **160**, 1555 (1967).

Finally, checking Eq. (2.29) for gauge invariance, we find that we must add to Eq. (2.29) a ΔT given by

$$\Delta T=-f_1(t)\bar{u}(p)i\gamma\cdot\epsilon(1+\gamma_5)v(q)-2Q\cdot\epsilon\frac{\partial}{\partial t}[f_1(t)i\gamma\cdot P+f_2(t)i\gamma\cdot Q](1+\gamma_5)v(q). \quad (2.30)$$

ΔT corresponds to the "seagull" diagram Fig. 2(c). Hence

$$T_L=T_{\mathrm{ex}}'+\Delta T=\bar{u}(p)\left(\frac{\epsilon\cdot p}{p\cdot k}-\frac{\epsilon\cdot P}{P\cdot k}+\frac{\gamma\cdot\epsilon\gamma\cdot k}{2p\cdot k}\right)[f_1(t)i\gamma\cdot P+f_2(t)i\gamma\cdot Q](1+\gamma_5)v(q)$$
$$-f_1(t)\bar{u}(p)\left[i\gamma\cdot\epsilon-i\gamma\cdot k\frac{\epsilon\cdot P}{P\cdot k}\right](1+\gamma_5)v(q)-2\bar{u}(p)\left[\epsilon\cdot Q-Q\cdot k\frac{\epsilon\cdot P}{P\cdot k}\right]\frac{\partial}{\partial t}[f_1(t)i\gamma\cdot P+f_2(t)i\gamma\cdot Q](1+\gamma_5)v(q). \quad (2.31)$$

We can extract the contribution to T from T_L by defining new functions A', C', H', and I' via

$$\begin{aligned}A(\nu,t,u)&=-f_1(t)+(P\cdot k/M^2)A'(\nu,t,u)\\C(\nu,t,u)&=-2\partial f_2(t)/\partial t+(P\cdot k/M^2)C'(\nu,t,u)\\H(\nu,t,u)&=-2\partial f_1(t)/\partial t+(P\cdot k/M^2)H'(\nu,t,u)\\I(\nu,t,u)&=(P\cdot k/M^2)I'(\nu,t,u),\\ \nu&=P\cdot k.\end{aligned} \quad (2.32)$$

If we define the structure-dependent contribution S via

$$T=T_L+S \quad (2.33)$$

then S is given by

$$S=(A'/M^2)(\epsilon\cdot lP\cdot k-\epsilon\cdot Pk\cdot l)+(\epsilon\cdot QP\cdot k-\epsilon\cdot PQ\cdot k)$$
$$[(C'/M^2)Q\cdot l+(H'/M^2)P\cdot l+(I'/M^2)k\cdot l]$$
$$+\epsilon_{\mu\nu\alpha\beta}\epsilon_\mu l_\nu(bP_\alpha k_\beta+cQ_\alpha k_\beta). \quad (2.34)$$

It is understood that the unknown functions A', C', H', I', b, and c are finite as $k\rightarrow 0$, so that S is at least linear in k as is required by Low's theorem. Combining Eqs. (2.31) and (2.34) we have, after some Dirac algebra,

$$T=\bar{u}(p)\left(\frac{\epsilon\cdot p}{p\cdot k}-\frac{\epsilon\cdot P}{P\cdot k}+\frac{\gamma\cdot\epsilon\gamma\cdot k}{2p\cdot k}\right)[2f_+(t)i\gamma\cdot Q-mf_1(t)](1+\gamma_5)v(q)-2\left(Q\cdot\epsilon-Q\cdot k\frac{\epsilon\cdot P}{P\cdot k}\right)\frac{\partial}{\partial t}\ \bar{u}(p)[2f_+(t)i\gamma\cdot Q-mf_1(t)]$$
$$\times(1+\gamma_5)v(q)+\frac{A'}{M^2}(\epsilon\cdot lP\cdot k-\epsilon\cdot Pk\cdot l)+(\epsilon\cdot QP\cdot k-\epsilon\cdot PQ\cdot k)\left(\frac{C'}{M^2}Q\cdot l+\frac{H'}{M^2}P\cdot l+\frac{I'}{M^2}k\cdot l\right)$$
$$+\epsilon_{\mu\nu\alpha\beta}\epsilon_\mu l_\nu(bP_\alpha k_\beta+cQ_\alpha k_\beta),$$
$$2f_\pm(t)=f_1(t)\pm f_2(t). \quad (2.35)$$

It is worth observing that in going from Eqs. (2.8) and (2.9) to Eq. (2.35) we have reduced the number of unknown form factors from 13 to 6. We have written the matrix element in terms of $f_+(t)$ and $f_1(t)$ rather than $f_+(t)$ and $f_-(t)$ because the former choice of form factors gives rise to the smallest number of terms when taking the trace. To evaluate $\partial f_+(t)/\partial t$ and $\partial f_1(t)/\partial t$, we adopt the following parametrization for the momentum dependence of the K_{l3} form factors:

$$f(t)=f(0)\left(1+\frac{\Lambda t}{M^2}\right)=f(0)\left[1+\Lambda\frac{(m^2-2p\cdot q-2p\cdot k-2q\cdot k)}{M^2}\right], \quad (2.36)$$

where the connection with the usual notation is $\Lambda=\lambda M^2/\mu^2$, and $f(t)=f_+(t)$ or $f_1(t)$. We also define

$$\xi=f_-(0)/f_+(0),$$
$$\eta=f_1(0)/f_+(0)=1+\xi, \quad (2.37)$$

so that the relation between first-order quantities is

$$\eta\Lambda_1=\Lambda_++\xi\Lambda_-. \quad (2.38)$$

In principle, Eq. (2.35) is exact. However, in order to evaluate the unknown form factors A', C', H', I', b, and c, it is necessary to invoke PCAC and consider T in the $Q=0$ limit. More specifically, we will evaluate the unknown form factors in Sec. III by relating $K^+\rightarrow\pi^0l^+\nu\gamma$ to $K^+\rightarrow l^+\nu\gamma$ via PCAC and show that in the soft-pion (i.e., $Q=0$) limit the expression for T in Eq. (2.35) depends on only two unknown form factors.

III. EVALUATION OF THE STRUCTURE DEPENDENT FORM FACTORS

We assume PCAC in the form

$$\partial_\lambda\mathfrak{F}_{5\lambda}{}^j(x)=\frac{\mu^3a_\pi}{\sqrt{2}}\phi^j(x),\quad j=1,2,3 \quad (3.1)$$

where μ is the pion mass, $a_\pi=0.94$, and $\phi^j(x)$ is the field operator which annihilates a pion with isospin j. By an LSZ reduction on the pion field, we can express the left-hand side of Eq. (2.7) as

$$(2Q_0V)^{1/2}\langle\pi^0\gamma|\mathcal{J}_\nu{}^{4-i5}(0)|K^+\rangle=i\int d^4y\, e^{-iQ\cdot y}(-\Box_y+\mu^2)\langle\gamma|T(\phi^3(y)\mathcal{J}_\nu{}^{4-i5}(0))|K^+\rangle$$

$$=-\frac{\sqrt{2}(Q^2+\mu^2)}{\mu^3 a_\pi}\int d^4y\, e^{-iQ\cdot y}Q_\lambda\langle\gamma|T(\mathfrak{F}_{5\lambda}{}^3(y)\mathcal{J}_\nu{}^{4-i5}(0))|K^+\rangle$$

$$-\frac{\sqrt{2}i(Q^2+\mu^2)}{\mu^3 a_\pi}\int d^4y\, e^{-iQ\cdot y}\delta(y_0)\langle\gamma|[\mathfrak{F}_{50}{}^3(y),\mathcal{J}_\nu{}^{4-i5}(0)]|K^+\rangle. \quad (3.2)$$

In the last step in Eq. (3.2), we have made use of Eq. (3.1) and we have dropped surface terms arising from partial integration. Taking the limit $Q_\lambda\to 0$ in Eq. (3.2), we find

$$(2Q_0V)^{1/2}\langle\pi^0\gamma|V_\nu{}^{4-i5}(0)|K^+\rangle=(-i/\sqrt{2}\mu a_\pi)\langle\gamma|A_\nu{}^{4-i5}(0)|K^+\rangle,$$

$$(2Q_0V)^{1/2}\langle\pi^0\gamma|A_\nu{}^{4-i5}(0)|K^+\rangle=(i/\sqrt{2}\mu a_\pi)\langle\gamma|V_\nu{}^{4-i5}(0)|K^+\rangle. \quad (3.3)$$

In taking the limit $Q_\lambda\to 0$ in Eq. (3.2) the term proportional to Q_λ makes no contribution since covariance under parity rules out any intermediate state which could give rise to a pole in the $Q=0$ limit.[25] Evidently, the right-hand side of Eqs. (3.3) is just the matrix element for radiation from K^+ in the process $K^+\to l^+\nu\gamma$. From Eqs. (2.35) and (3.3) we see that the terms proportional to the form factors C', H', I', and c are zero in the soft-pion limit with the result that T now depends on only two unknown form factors A' and b. It should be emphasized that the $Q=0$ limit is taken only in S as is evident from the preceding discussion. For future reference, we exhibit the expression for T in the approximation to which we are now working,

$$T=\bar{u}(p)\left(\frac{\epsilon\cdot p}{p\cdot k}-\frac{\epsilon\cdot P}{P\cdot k}+\frac{\gamma\cdot\epsilon\gamma\cdot k}{2p\cdot k}\right)[2f_+(t)i\gamma\cdot Q-mf_1(t)](1+\gamma_5)v(q)+2\left(\frac{Q\cdot k\epsilon\cdot P}{P\cdot k}-Q\cdot\epsilon\right)\frac{\partial}{\partial t}\bar{u}(p)[2f_+(t)i\gamma\cdot Q-mf_1(t)]$$
$$\times(1+\gamma_5)v(q)+(A/M^2)(\epsilon\cdot lP\cdot k-\epsilon\cdot Pk\cdot l)+(B/M^2)\epsilon_{\mu\nu\alpha\beta}\epsilon_\mu l_\nu P_\alpha k_\beta, \quad (3.4)$$

where we have redefined $A'=A$ and $B=b$. To evaluate A and B we proceed to discuss the matrix element for $K^+\to l^+\nu\gamma$.

Following the same procedure used in the previous section, the matrix element for the process $K^+(P)\to l^+(p)+\nu(q)+\gamma(k)$ is given by

$$\mathfrak{M}={}_{\text{out}}\langle l\nu\gamma|K\rangle_{\text{in}}=-i(2\pi)^4\delta^4(P-p-q-k)\times\left(\frac{mm_\nu}{4P_0p_0q_0k_0V^4}\right)^{1/2}\frac{eG\sin\theta}{\sqrt{2}}T, \quad (3.5)$$

where T consists of the sum of the inner bremsstrahlung graph and the structure-dependent axial-vector and vector parts,

$$T=-imMf_K\bar{u}(p)\left(\frac{\epsilon\cdot p}{p\cdot k}-\frac{\epsilon\cdot P}{P\cdot k}+\frac{\gamma\cdot\epsilon\gamma\cdot k}{2p\cdot k}\right)\times(1+\gamma_5)v(q)+(i\tilde{a}/M)(\epsilon\cdot lP\cdot k-\epsilon\cdot Pk\cdot l)+(i\tilde{b}/M)\epsilon_{\mu\nu\alpha\beta}\epsilon_\mu l_\nu P_\alpha k_\beta, \quad (3.6)$$

where $\tilde{a}$, $\tilde{b}$ are dimensionless functions of the variable $P\cdot k$. An alternative derivation of Eq. (3.6) follows directly from Eq. (2.35) by setting $Q\equiv 0$, $f_1(t)=f_K$, and multiplying by a factor of i from the LSZ reduction and a factor of M to make the new coupling constants dimensionless.

An evaluation of the form factors $\tilde{a}=\tilde{a}(\nu=0)$ and $\tilde{b}=\tilde{b}(\nu=0)$ has been attempted by several authors.[5,7,11] The difficulty of the problem is indicated in part by the disparity in the numerical values obtained for $\tilde{a}$ and $\tilde{b}$ by the different groups. In an effort to unravel some of the difficulties, we will summarize in the following paragraphs the various approaches that have been used and state the results which have been obtained.

We begin with a discussion of $\tilde{b}(\nu)$ which has been treated by Gervais, Iliopoulos and Kaplan,[14] Rockmore,[7] and Sarker[7] using $K^*(890)$ dominance and by Kummer and Majerotto[5] using a Fubini-type sum rule. In the $K^*(890)$ dominance model of Fig. 3(a), $\tilde{b}(\nu)$ is given by

$$\left|\frac{\tilde{b}(\nu)}{M}\right|=\left|\frac{f_{K^*}G_{K^*K\gamma}}{-2\nu+M_{K^*}{}^2-M^2}\right|, \quad (3.7)$$

where f_{K^*} is defined by

$$(2Q_0V)^{1/2}\langle 0|V_\mu{}^{4-i5}(0)|K^{*+}(Q)\rangle=\epsilon_\mu f_{K^*} \quad (3.8)$$

[25] C. G. Callan and S. B. Treiman, Phys. Rev. Letters **16**, 153 (1966).

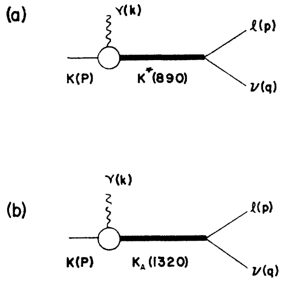

FIG. 3. (a) K^*-pole contribution to the structure-dependent vector matrix element in $K^+ \to l^+\nu\gamma$. (b) K_A-pole contribution to the structure-dependent axial-vector matrix element in $K^+ \to l^+\nu\gamma$.

in analogy to

$$(2Q_0V)^{1/2}\langle 0|\,V_\mu{}^{1-i2}(0)\,|\rho^+(Q)\rangle = \epsilon_\mu f_\rho\,. \qquad (3.9)$$

$G_{K^*K\gamma}$ is defined through the coupling

$$\mathcal{L}(K^* \to K\gamma) = eG_{K^*K\gamma}\tilde{F}_{\mu\nu}K_\mu{}^*\partial_\nu K + \text{H.c.} \qquad (3.10)$$

In the limit of exact $SU(3)$, we have

$$f_{K^*} = f_\rho = \sqrt{2}m_\rho{}^2/f_{\rho\pi\pi} \cong 0.26 m_\rho{}^2\,, \qquad (3.11)$$

with $f_{\rho\pi\pi}{}^2/4\pi \cong 2.4$. By use of spectral function sum rules obtained from asymptotic $SU(3)\otimes SU(3)$, the effects of $SU(3)$ symmetry breaking may be estimated[26] and give

$$f_{K^*} = f_\rho(m_{K^*}/m_\rho) \cong 0.30 m_\rho{}^2\,. \qquad (3.12)$$

Up to this point, there is a general level of agreement among the various authors, a level unfortunately not achieved in evaluating $G_{K^*K\gamma}$.

Rockmore and Sarker calculate $G_{K^*K\gamma}$ by assuming a Gell-Mann–Sharp–Wagner-type model,[27] in analogy to the conventional treatment of $G_{\omega\pi\gamma}$. Difficulties arise, however, from the fact that both isoscalar and isovector photons contribute to $K^* \to K\gamma$ while only isovector photons contribute to $\omega \to \pi\gamma$. As a consequence of this, ρ, ω, and ϕ intermediate states can contribute to $G_{K^*K\gamma}$ with the result that $G_{K^*K\gamma}$ depends somewhat sensitively on models of ω-ϕ mixing and $SU(3)$ symmetry breaking. Using the nonet model of Okubo[28] for ω-ϕ mixing, Rockmore finds

$$G_{K^*K\gamma} = G_{\rho\omega\pi}[f_\rho/m_\rho{}^2 - (\sqrt{\tfrac{2}{3}})f_\phi/m_\phi{}^2 - (\sqrt{\tfrac{1}{3}})f_\omega/m_\omega{}^2]\,, \qquad (3.13)$$

where f_ϕ and f_ω are defined as in Eqs. (3.8) and (3.9), and $G_{\rho\omega\pi}{}^2/4\pi \cong 0.40/\mu^2$ as determined from $\Gamma(\omega \to 3\pi)$. In the exact $SU(3)$ limit, $f_\phi = f_\rho$ and $f_\omega = 0$ (ϕ is a pure octet and ω is a pure singlet). The effects of $SU(3)$ symmetry breaking on f_ϕ and f_ω may be estimated again using spectral function sum rules and give[26] $f_\phi{}^2 \cong 1.03 f_\rho{}^2$, $f_\omega{}^2 \cong 0.43 f_\rho{}^2$. Hence

$$G_{K^*K\gamma} \cong 0.37/M\,. \qquad (3.14)$$

[Gervais *et al.*, using $SU(6)$ to relate $G_{K^*K\gamma}$ to $G_{\rho\pi\gamma}$, find $G_{K^*K\gamma} \cong 3.5/M$ using $\Gamma(\rho \to \pi\gamma) \cong 0.5$ MeV, a width which is probably too large.] From Eqs. (3.7), (3.12), and (3.14) we find

$$|\tilde{b}(0)| \cong 0.12\,. \qquad (3.15)$$

Note that since $\tilde{b}(\nu)$ multiplies an expression which is already linear in k, it is sufficient to evaluate $\tilde{b}(\nu)$ at $\nu \cong 0$ and treat $\tilde{b}(\nu)$ as a constant to the approximation in which we are working.

Kummer and Majerotto[5] evaluate $\tilde{b}(0)$ using the dispersion theoretic approach of Fubini[29] and find

$$\tilde{b}(0)/M = -(\sqrt{8})f_{\pi\gamma\gamma} + (3f_\rho G_{\rho\pi\gamma}/m_\rho{}^2)\,, \qquad (3.16)$$

where $f_{\pi\gamma\gamma}$ and $G_{\rho\pi\gamma}$ are defined by

$$(4k_0Q_0V^2)^{1/2}\langle\gamma(k)|\,V_\mu{}^3(0)\,|\pi^0(Q)\rangle = ief_{\pi\gamma\gamma}\epsilon_{\mu\nu\alpha\beta}\epsilon_\nu k_\alpha Q_\beta\,, \qquad (3.17)$$

$$(4k_0Q_0V^2)^{1/2}\langle\rho^+(k)|\,V_\mu{}^8(0)\,|\pi^+(Q)\rangle = \sqrt{3}G_{\rho\pi\gamma}\epsilon_{\mu\nu\alpha\beta}\epsilon_\nu Q_\alpha k_\beta\,. \qquad (3.18)$$

Note that $f_{\pi\gamma\gamma}$ is defined in analogy to $\tilde{b}(\nu)$

$$(4k_0P_0V^2)^{1/2}\langle\gamma(k)|\,V_\mu{}^{4-i5}(0)\,|K^+(P)\rangle = ie(\tilde{b}(\nu)/M)\epsilon_{\mu\nu\alpha\beta}\epsilon_\mu P_\alpha k_\beta\,. \qquad (3.19)$$

Since only the magnitudes (and not the signs) of the coupling constants in Eq. (3.16) can be determined theoretically, Eq. (3.16) gives rise to two solutions[5] for $|\tilde{b}(0)|$

$$|\tilde{b}(0)| = |0.43 \pm 0.37| = \begin{Bmatrix} 0.80 \\ 0.06 \end{Bmatrix}\,. \qquad (3.20)$$

We see that the various estimates of $|\tilde{b}(0)|$ differ quite appreciably from one another. From Eqs. (3.3) and (3.4) we can then write

$$|B(0)| \cong (M/\sqrt{2}\mu a_\pi)|\tilde{b}(0)| \cong 2.8|\tilde{b}(0)|\,. \qquad (3.21)$$

From the previous discussion we conclude that $|B(0)|$ probably lies in the range

$$0.2 \lesssim |B(0)| \lesssim 2 \qquad (3.22)$$

and consequently we have chosen the nominal value $|B(0)| = 1.0$ in plotting the structure-dependent contribution to the photon spectrum in Fig. (4).

We turn next to a discussion of $\tilde{a}(\nu)$ which has been treated by Rockmore,[7] Sarker,[7] and Vaishya and Gupta.[11] We assume that $\tilde{a}(\nu)$ is dominated by contributions from intermediate $I(J^P) = \frac{1}{2}(1^+)$ mesons such as the $K_A(1320)$ shown in Fig. 3(b).[30] A straightforward

[26] T. Das, V. S. Mathur, and S. Okubo, Phys. Rev. Letters **19**, 470 (1967).

[27] M. Gell-Mann, D. Sharp, and W. G. Wagner, Phys. Rev. Letters **8**, 261 (1962).

[28] S. Okubo, Phys. Letters **5**, 165 (1963).

[29] S. Fubini, Nuovo Cimento **43**, 475 (1966).

[30] Actually, there appear to be several resonances having the same quantum numbers as $K_A(1320)$. See, for example, N. Barash-Schmidt, A. Barbaro-Galtieri, L. R. Price, A. H. Rosenfeld, P. Söding, and C. G. Wohl, University of California Research Laboratory Report No. 8030, 1968 (unpublished); and Particle Data Group, Rev. Mod. Phys. **41**, 109 (1969).

evaluation of Fig. 3(b) yields

$$\frac{|\tilde{a}(\nu)|}{M}=\left|\frac{f_{K_A}h_{K_AK\gamma}(\nu)}{-2\nu+M_{K_A}^2-M^2}\right|, \quad (3.23)$$

where f_{K_A} and $h(\nu)$ are defined by

$$(2Q_0V)^{1/2}\langle 0|A_\mu^{4-i5}(0)|K_A^+(Q)\rangle=\eta_\mu f_{K_A} \quad (3.24)$$

$$-i(4P_0Q_0V^2)^{1/2}\langle K_A^+(Q)|j_\mu^\gamma(0)|K^+(P)\rangle =[2P\cdot k\eta_\mu-(P+Q)_\mu\eta\cdot k]h_{K_AK\gamma}(\nu) +k_\mu\eta\cdot kg_{K_AK\gamma}(\nu). \quad (3.25)$$

In Eqs. (3.24) and (3.25) η_μ is the polarization vector for K_A, and $k=P-Q$. By gauge invariance, the term proportional to $g_{K_AK\gamma}(\nu)$ makes no contribution to the radiative matrix element. In accordance with Low's theorem we know that $h_{K_AK\gamma}(\nu)$ is finite as $\nu\to 0$ and hence

$$|\tilde{a}(0)|/M=|f_{K_A}h_{K_AK\gamma}(0)|/(M_{K_A}^2-M^2). \quad (3.26)$$

In view of the previously mentioned uncertainties in the $I(J^P)=\frac{1}{2}(1^+)$ spectrum[30] and in particular of the K_A couplings, we have not attempted to evaluate $\tilde{a}=\tilde{a}(0)$ explicitly and consequently we have chosen the nominal value $A=A(0)=1.0$ in plotting the structure-dependent contribution in Fig. 6. [Recall that $A(0)$ is related to $\tilde{a}(0)$ through Eq. (3.3).]

In the approach of Vaishya and Gupta,[11] and of Sarker,[7] a sum rule is derived which permits the right-hand side of Eq. (3.26) to be expressed directly in terms of the kaon decay constant f_K thus avoiding a direct confrontation with the unknown coupling constants f_{K_A} and $h_{K_AK\gamma}$. The validity of this sum rule depends on (1) assumptions about subtracted or unsubtracted dispersion relations and (2) on carefully extracting the Born amplitude (including possible contributions from "seagull" diagrams) from the amplitude $\langle\gamma|A_\nu^{4-i5}(0)|K^+\rangle$. We quote without further comment the values $0.1\lesssim(|\tilde{a}(0)|/|\tilde{b}(0)|=|A(0)|/|B(0)|)\lesssim 0.9$ obtained by the various authors.

IV. SOME TECHNICAL DETAILS

Given the final form of the matrix element, the evaluation of the sum over spins and polarizations is tedious but straightforward. As note previously the infrared divergent terms of order k^{-1} in T arise exclusively from diagrams in which the photon is emitted from an external charged line. As a consequence of this, the terms in $|T|^2$ which are of order k^{-2} factorize in the following way:

$$|T(K^+\to\pi^0l^+\nu\gamma)|^2 =O(k^{-2})|T(K^+\to\pi^0l^+\nu)|^2+O(k^{-1})+\cdots. \quad (4.1)$$

The remaining terms in $|T|$ (the real "bremsstrahlung") are well behaved in the limit $k\to 0$ and require no special attention. From Eqs. 3.4) and (A5) the coefficient of order k^{-2} in Eq. (4.1) may be read off by inspection and is given by

$$|T(K^+\to\pi^0l^+\nu\gamma)|^2=\frac{-1}{4mm_\nu}\left|\frac{\epsilon\cdot p}{p\cdot k}-\frac{\epsilon\cdot P}{P\cdot k}\right|^2 \times\mathrm{Tr}[\Gamma(i\gamma\cdot q)\gamma_4\Gamma^\dagger\gamma_4(m-i\gamma\cdot p)]+\cdots, \quad (4.2)$$

where

$$\Gamma=2f_+(t)i\gamma\cdot Q-mf_1(t).$$

The trace in Eq. (4.2) is given by the quantity in curly brackets in Eq. (A6) and the full sum over spins and polarizations of the complete matrix element is given in Eqs. (A8)-(A12).

Note that we will use the photon energy as one of the integration variables in the expression for the rate and therefore evaluate the complete trace and polarization sum with zero photon mass. The final integral over the photon energy is cut off at a photon energy equal to the minimum photon energy to which the detecting apparatus is sensitive.

Note also that all terms in Eq. (A8) which contain the factor $(p\cdot k)^{-2}$ are multiplied by m^2. As we show in Appendix B, the phase space integral over $(p\cdot k)^{-2}$ behaves like m^{-2} so terms $m^2/(p\cdot k)^2$ are finite as $m^2\to 0$. This cancellation cannot be seen from the matrix element. If we examine the terms with $(p\cdot k)^{-1}$ in Eq. (3.4) they will have a singularity when $p\cdot k=0$ or $|\mathbf{k}|(|\mathbf{p}|\cos\theta-E)=0$, i.e., they will have an end point singularity in the $\cos\theta$ integration when $m=0$. At this point $\mathbf{p}\|\mathbf{k}$ so $p_\mu\propto k_\mu$. If we make this substitution in the numerator of Eq. (3.4), the $(p\cdot k)^{-1}$ terms have coefficients $k\cdot\epsilon$, which are zero by current conservation, and $\gamma\cdot p\gamma\cdot\epsilon$, which is proportional to the lepton mass by use of the Dirac equation. Therefore, the terms which can cause trouble have numerators at least of order m. At first sight the denominators seem to approach zero as m^2 but after explicit calculation of the integrals, we see that the $(p\cdot k)^{-2}$ terms in the trace approach zero as m^{-2}, and not as m^{-4}. Hence there is no linear mass divergence in the lepton mass. This is an example of mass singularities examined by Kinoshita[31] and the particular case discussed here uses an argument due to R. W. Brown (private communication). We are therefore not allowed to drop terms proportional to the lepton mass in calculating the decay rate into electrons.

Next we turn to the final-state integrations which yield the photon spectrum and decay rate. To illustrate the method of doing the integrations over the four-momenta of the neutrino, electron, and photon, we consider the basic integral

$$J=\int d^4k\,\delta(k^2)\theta(k_0)\int d^4p\,\delta(p^2+m^2)\theta(p_0)\int d^4q\,\delta(q^2)\theta(q_0) \times\int d^4Q\,\delta(Q^2+\mu^2)\theta(Q_0)\delta^4(P-Q-p-q-k). \quad (4.3)$$

[31] T. Kinoshita, J. Math. Phys. 3, 650 (1962).

Integration over the pion four-momentum is trivial using the final δ function. We then use the usual relation given by Källén[32] in the form

$$\int d^4q\,\delta(q^2+\epsilon)\theta(q_0)\delta((A-q)^2+\mu^2)\theta((A-q)_0)=\frac{\pi}{2a}\lambda^{1/2}(a,\epsilon,\mu^2)\theta(a)\theta(a-(\mu+\epsilon)^2)\,, \quad (4.4)$$

where

$$A^2+a=0\,,\quad \lambda(x,y,z)=x^2+y^2+z^2-2xy-2xz-2yz\,.$$

Hence

$$J=\tfrac{1}{2}\pi\int da\,\delta(A^2+a)\frac{\lambda^{1/2}(a,0,\mu^2)}{a}\theta(a-\mu^2)\theta(a)\int d^4k\,\delta(k^2)\theta(k_0)\int d^4p\,\delta(p^2+m^2)\theta(p_0)\,. \quad (4.5)$$

If we let $A=B-p$, where $B^2+b=0$, then

$$J=\tfrac{1}{2}\pi\int db\int da\frac{\lambda^{1/2}(a,0,\mu^2)}{a}\theta(a-\mu^2)\int d^4k\,\delta(k^2)\theta(k_0)\delta((P-k)^2+b)\theta((P-k)_0)$$

$$\times\int d^4p\,\delta(p^2+m^2)\theta(p_0)\delta((B-p)^2+a)\theta((B-p)_0)$$

$$=(\tfrac{1}{2}\pi)^3\int db\frac{\lambda^{1/2}(M^2,b,0)}{M^2}\theta(b)\theta(b-(\sqrt{a}+m)^2)\int da\frac{\lambda^{1/2}(b,m^2,a)}{b}\theta(a)\frac{\lambda^{1/2}(a,0,\mu^2)}{a}\theta(a-\mu^2)\theta(M^2-b)$$

$$=(\tfrac{1}{2}\pi)^3\int_{(m+\mu)^2}^{M^2}\frac{db}{b}\int_{\mu^2}^{(\sqrt{b}-m)^2}\frac{da}{a}\frac{1}{M^2}\lambda^{1/2}(M^2,b,0)\lambda^{1/2}(b,m^2,a)\lambda^{1/2}(a,\mu^2,0)\,. \quad (4.6)$$

The variable $b=-(P-k)^2=M^2-2P\cdot k=M^2-2MW$ in the kaon rest frame, where W denotes the photon energy. Hence the spectrum in b is essentially the negative of the photon energy spectrum. The cutoff on the low-photon energies now becomes a cutoff on high values of b. Specifically if we take $W_{\min}=30$ MeV as in the experiment of Ref. 10, $b_{\max}$ becomes $b_{\max}=(M-\Lambda)^2$ with $\Lambda=30$ MeV.

When we include a matrix element, dependent upon scalar products of four vectors, the pion integration is still trivial but the neutrino four-momentum appears in the numerator in powers up to two, and we require the following integrals:

$$\frac{2}{\pi}\int d^4q\,\delta(q^2)\theta(q_0)\delta((A-q)^2+\mu^2)\theta((A-q)_0)q_\mu$$

$$=\tfrac{1}{2}A_\mu\left(1-\frac{\mu^2}{a}\right)^2\theta(a)\theta(a-\mu^2)\,, \quad (4.7)$$

$$\frac{2}{\pi}\int d^4q\,\delta(q^2)\theta(q_0)\delta((A-q)^2+\mu^2)\theta((A-q)_0)q_\mu q_\nu$$

$$=\tfrac{1}{3}(A_\mu A_\nu+\tfrac{1}{4}a\delta_{\mu\nu})\left(1-\frac{\mu^2}{a}\right)^3\theta(a)\theta(a-\mu^2)\,. \quad (4.8)$$

Each integral involving q then reduces to a number of terms involving p. When we have tabulated all possible combinations of integrals over the lepton four-momentum (Appendix B), we can finally compute the spectrum and rate by integration over a and a,b, respectively. Luckily, only a few of the p integrations have to be actually integrated analytically. The others are related by recurrence relations as shown in Appendix B. Our final answer is enormous and cannot possibly be reproduced here. However, if anyone wants the rates and spectra for different vaules of the couplings from those used in the last section, we will be happy to rerun our program.

Finally, we note that, since the structure-dependent terms are at least linear in k, their contribution to the $K^+\to\pi^0l^+\nu\gamma$ decay will be small compared to the "structureless" bremsstrahlung contributions from T_L. This implies that contributions from S can be extracted only with great difficulty given a knowledge of the $K^+\to\pi^0l^+\nu\gamma$ rate and spectrum. Unfortunately, this is a general feature of radiative decays and implies that structure-dependent effects can only be seen with ease in radiative decays where the bremsstrahlung decay is forbidden by a selection rule, e.g., $\eta\to\pi^+\pi^-\gamma$ because the decay $\eta\to\pi^+\pi^-$ violates parity.

V. CONCLUSIONS

We present next the results of the numerical computations. Let us first discuss the rates. Our procedure is to evaluate the direct square of the infrared divergent part of the matrix element up to terms linear in Λ. This gives seven terms, i.e., $f_+^2(0)$, $f_+(0)f_1(0)$, $f_1^2(0)$, $f_+^2(0)\Lambda_+$, $f_1^2(0)\Lambda_1$, $f_+(0)f_1(0)\Lambda_+$, and $f_+(0)f_1(0)\Lambda_1$. The square of the term which is of order k^0 (as well as the square of the structure-dependent terms) was found to be so small that it could be safely neglected. The remaining terms come from the interference between the

[32] G. Källén, *Elementary Particle Physics* (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1964).

inner-bremsstrahlung term and the other three terms. Here we are justified in keeping only $f_+(0)$ and $f_-(0)$ in the inner bremsstrahlung, and we thus generate the terms $f_+^2(0)\Lambda_+$, $f_+^2(0)\Lambda_1$, $f_1^2(0)\Lambda_+$, $f_1^2(0)\Lambda_1$, $f_+(0)A$, $f_+(0)B$, $f_1(0)A$, and $f_1(0)B$. The rate in both cases is the sum of the fifteen terms. In practice the decay rate for the electron mode has only five terms because all the terms proportional to the lepton mass are so small that they can be discarded. The results are as follows:

$$\Gamma(K^+\to\pi^0e^+\nu\gamma,\,E_\gamma>30\text{ MeV})=(G^2\sin^2\theta M^5/64\pi^3)\times10^{-3}[1.0082f_+^2(0)+0.1704f_+^2(0)\Lambda_+ +0.1242f_+^2(0)\Lambda_+-0.0022f_+(0)A-0.0013f_+(0)B]. \quad (5.1)$$

As noted above, the two terms proportional to $f_+^2(0)\Lambda_+$ in Eq. (5.1) have different origins in Eq. (3.4) and have not been combined in order to exhibit their relative magnitudes. We follow the same procedure in Eqs. (5.2) and (5.3).

$$\Gamma(K^+\to\pi^0\mu^+\nu\gamma,\,E_\gamma>30\text{ MeV})=(G^2\sin^2\theta M^5/64\pi^3)\times10^{-5}[1.8736f_+^2(0)+0.0930f_1^2(0)+0.1955f_+(0)f_1(0) +0.6111f_+^2(0)\Lambda_++0.0446f_1^2(0)\Lambda_1+0.0413f_+(0)f_1(0)(\Lambda_++\Lambda_1)+0.9106f_+^2(0)\Lambda_+-0.0028f_1^2(0)\Lambda_1 -0.0026f_1^2(0)\Lambda_+-0.0111f_+^2(0)\Lambda_1-0.0370f_+(0)A-0.0242f_+(0)B+0.0073f_1(0)A-0.0060f_1(0)B]. \quad (5.2)$$

Taking now the rate for $K^+\to\pi^0e^+\nu$ from Eq. (A7), we find

$$R_1=\frac{\Gamma(K^+\to\pi^0e^+\nu\gamma,\,E_\gamma>30\text{ MeV})}{\Gamma(K^+\to\pi^0e^+\nu)}=\frac{[1.0082+0.1704\Lambda_++0.1242\Lambda_+-0.0022A/f_+(0)-0.0013B/f_+(0)]\times10^{-1}}{4(1.2067+0.3335\Lambda_+)}. \quad (5.3)$$

The usual model of $K^*(890)$ dominance of the vector form factor yields $\Lambda_+=M^2/M_{K^*}^2=0.31$. If we assume $\sin\theta=0.21$, then $f_+(0)=0.76$ from the K_{l3} rate $[\Gamma(K_{l3})=4.07\times10^6\text{ sec}^{-1}]$. Hence the branching ratio is

$$R_1=(2.100-0.006A-0.003B)\times10^{-2}. \quad (5.4)$$

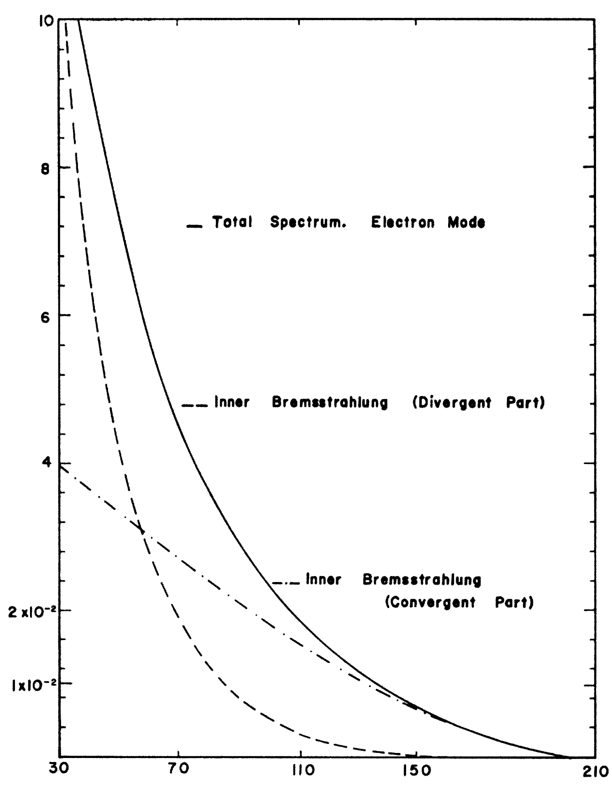

FIG. 4. Photon spectrum in $K^+\to\pi^0e^+\nu\gamma$, with $f_+(0)=0.76$, $\Lambda_+=0.31$, $A=B=0$. The ordinate has been normalized by dividing by $K^+\to\pi^0e^+\nu$ rate.

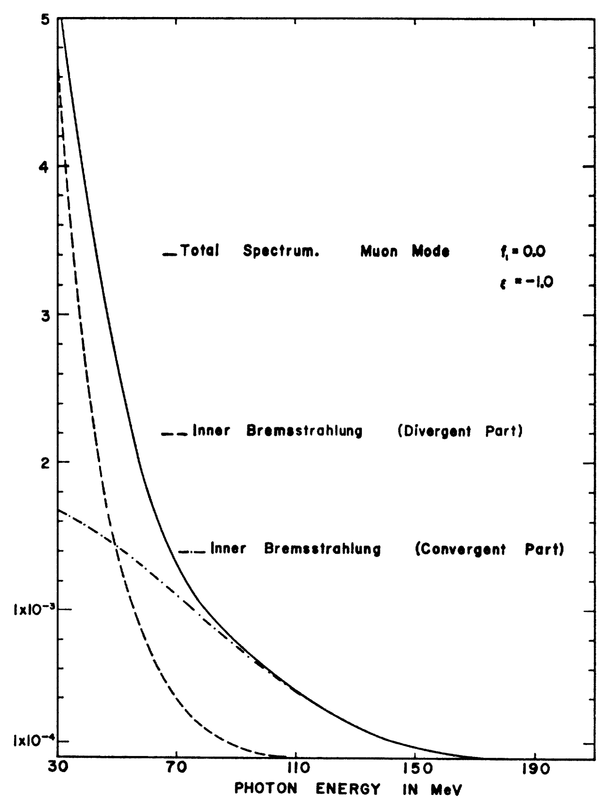

FIG. 5. Photon spectrum in $K^+\to\pi^0\mu^+\nu\gamma$ with $f_+(0)=0.76$, $f_1(0)=0.0$, $(\xi=-1.0)$, $\Lambda_+=0.31$, $\Lambda_1=0.31$, $A=B=0$. The ordinate has been normalized by dividing by the $K^+\to\pi^0\mu^+\nu$ rate.

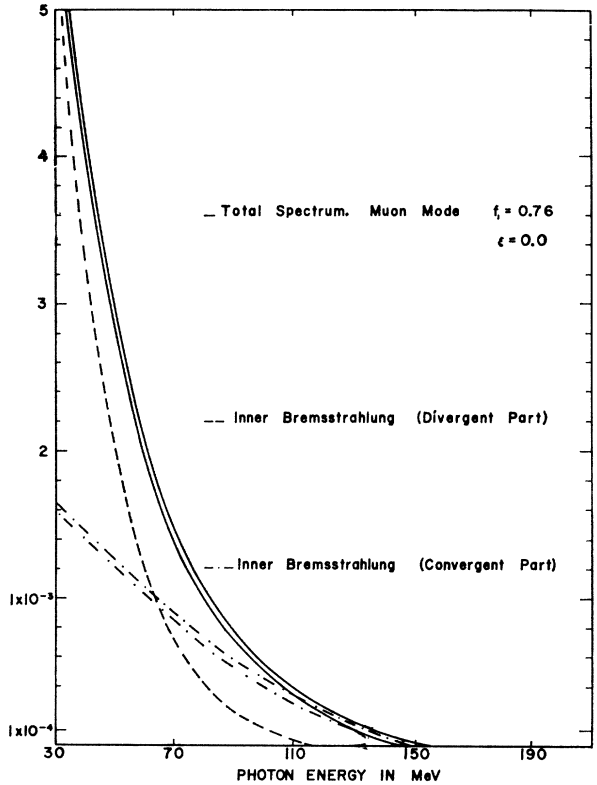

FIG. 6. Photon spectrum in $K^+\to\pi^0\mu^+\nu\gamma$, with $f_+(0)=0.76$, $f_1(0)=0.76$, $(\xi=0.0)$, $\Lambda_+=0.31$, $\Lambda_1=0.31$ and $A=B=0$. The lower curves include interference with $A=B=1.0$. The ordinate has been normalized by dividing by the $K^+\to\pi^0\mu^+\nu$ rate.

The experimental number $R_1=(1.2\pm0.8)\times10^{-2}$ can only be fitted with A and B large and positive. If A and B are smaller, then R_1 can be fitted with a smaller value of Λ_+. We cannot place too much reliance on the results of one experiment, and clearly better data would be very useful. The large positive values of A and B are incompatible with the estimates of $\tilde{a}$ and $\tilde{b}$ obtained in Sec. III. Our result for the electron mode clearly shows that measurements of $\tilde{a}$ and $\tilde{b}$ from the mode $K^+\to\mu^+\nu\gamma$ can, in principle, be combined with measurements of $\tilde{a}$ and $\tilde{b}$ from the decay $K^+\to\pi^0e^+\nu\gamma$ to give these parameters.

Now let us turn to the branching ratio ,

$$R_2=\frac{\Gamma(K^+\to\pi^0\mu^+\nu\gamma,\,E_\gamma>30\text{ MeV})}{\Gamma(K^+\to\pi^0\mu^+\nu)} \tag{5.5}$$

which is a rather complicated function of many parameters. Assuming, for example, that $\eta=1.0$, $(\xi=0.0)$, $f_+(0)=f_1(0)=0.76$, $\Lambda_+=\Lambda_1=0.31$, and $A=B=2.5$, we find $R_2=0.70\times10^{-3}$. The branching ratio R_2 can be used to solve for Λ_1 and $f_1(0)$, but as no experimental data are available we only quote a single result.

We complete our analysis by plotting the photon spectra for different values of the couplings. Figure 4 shows the photon spectrum in the decay $K^+\to\pi^0e^+\nu\gamma$ with $A=B=0$. The corrections due to small finite values of A and B are almost unobservable and have not been included. Figure 5 shows the photon spectrum in the decay $K^+\to\pi^0\mu^+\nu\gamma$ with $\xi=-1.0$ (result from polarization data[1]), $\Lambda_+=\Lambda_1=0.31$, and $A=B=0$. Figure 6 shows the spectrum with $\xi=0$ (rough average of polarization and rate data[1]) and both $A=B=0$ and $A=B=1.0$. We have not drawn the diagrams with different values of ξ because in all cases the variation is small and the effect could be misleading until we know the other couplings.

Our results may be summarized as follows. Accepting the estimates of A and B from Sec. III, the rates and spectra for both decays are primarily determined by the K_{l3} form factors. It is unlikely that measurements will be made in the near future with enough accuracy to detect the A and B terms. Noting that the ratio R_1 is independent of the Cabibbo angle and $f_+(0)$, we would like to stress the possibility of directly obtaining Λ_+ from this branching ratio. R_2 is a rather complicated function of the K_{l3} form factors and has a smaller value so it is probably academic to discuss this decay at present. However, we feel that it will eventually be measured and may provide a useful supplement to $K_{\mu3}$ decay as a means of checking the values of η and λ_1 (ξ and λ_+).

ACKNOWLEDGMENTS

The authors are indebted to Robert W. Brown of Brookhaven National Laboratory for valuable suggestions and for making the C.D.C. SCHOONSCHIP program available to us. We also wish to thank E. S. Ginsberg, A. K. Mann, and H. Primakoff for helpful discussions.

APPENDIX A. THE TRACE CALCULATION AND RELATED CONVENTIONS

In this appendix, we summarize some conventions relating to the Dirac equation and present several identities which were found useful in performing the trace calculations. The necessary traces were calculated by hand and checked against the results of SCHOONSCHIP, a CDC 6600 program for symbolic evaluation of algebraic expressions, written by M. Veltman. The results for $\Gamma(K^+\to\pi^0l^+\nu)$ and $\Gamma(K^+\to\pi^0l^+\nu\gamma)$ are given in Eqs. (A6) and (A13).

We have worked in the Pauli metric of Jackson,[33] Källén,[32] Mandl,[34] and Bernstein.[35] The Dirac equation for the positive-energy spinor $u(p,s)$ describing a spin-$\frac{1}{2}$ fermion of momentum p and spin s, and for the negative-

[33] J. D. Jackson, in *Elementary Particle Physics and Field Theory, 1962 Brandeis Lectures* (W. A. Benjamin, Inc., New York, 1963), Vol. 1, p. 263.

[34] F. Mandl, *Introduction to Quantum Field Theory* (Interscience Publishers, Inc., New York, 1959).

[35] J. Bernstein, *Elementary Particles and Their Currents* (W. H. Freeman and Co., San Francisco, 1968).

energy spinor $v(p,s)$ is

$$(i\gamma\cdot p+m)u(p,s)=\bar{u}(p,s)(i\gamma\cdot p+m)=0,$$
$$(-i\gamma\cdot p+m)v(p,s)=\bar{v}(p,s)(-i\gamma\cdot p+m)=0,$$
$$\gamma\cdot p=\boldsymbol{\gamma}\cdot\mathbf{p}+i\gamma_4 p_0, \quad \text{(A1)}$$
$$p_\mu=(\mathbf{p},\, p_4=ip_0),$$
$$p\cdot k=p_\mu k_\mu=\mathbf{p}\cdot\mathbf{k}-p_0k_0.$$

The Dirac matrices γ_μ satisfy

$$\{\gamma_\mu,\gamma_\nu\}=2\delta_{\mu\nu},\ \{\gamma_5,\gamma_\mu\}=0, \quad \text{(A2)}$$

with

$$\gamma_5=\gamma_1\gamma_2\gamma_3\gamma_4=(1/4!)\epsilon_{\mu\nu\lambda\rho}\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\rho,$$

where $\epsilon_{\mu\nu\lambda\rho}$ is the completely antisymmetric permutation symbol:

$$\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_5=(\delta_{\mu\nu}\gamma_\lambda-\delta_{\mu\lambda}\gamma_\nu+\delta_{\nu\lambda}\gamma_\mu)\gamma_5-\epsilon_{\mu\nu\lambda\rho}\gamma_\rho,$$
$$\gamma_\mu\gamma_\nu\gamma_5=\delta_{\mu\nu}\gamma_5-\tfrac{1}{2}\epsilon_{\mu\nu\lambda\rho}\gamma_\lambda\gamma_\rho, \quad \text{(A3)}$$
$$\gamma_\mu\gamma_5=(1/3!)\epsilon_{\mu\nu\lambda\rho}\gamma_\nu\gamma_\lambda\gamma_\rho.$$

The trace of γ_5 with three or fewer γ_μ vanishes and

$$\mathrm{Tr}(\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\rho\gamma_5)=4\epsilon_{\mu\nu\lambda\rho},$$
$$\mathrm{Tr}(\gamma_i\gamma_j\gamma_k\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_5)=4(\delta_{ij}\epsilon_{k\mu\nu\lambda}-\delta_{ik}\epsilon_{j\mu\nu\lambda}+\delta_{jk}\epsilon_{i\mu\nu\lambda}+\delta_{\mu\nu}\epsilon_{ijk\lambda}-\delta_{\mu\lambda}\epsilon_{ijk\nu}+\delta_{\nu\lambda}\epsilon_{ijk\mu}), \quad \text{(A4)}$$
$$i,\ j,\ k,\ \mu,\ \nu,\ \lambda=1,\ 2,\ 3,\ 4.$$

For other trace formulas the reader is referred to Källén.[32] The K_{l3} matrix element T is given by

$$T(K^+\to\pi^0 l^+\nu)=(4P_0Q_0V^2)^{1/2}\langle\pi^0(Q)|V_\mu^{4-i5}(0)|K^+(P)\rangle l_\mu$$
$$=\bar{u}(p)[2f_+(t)i\gamma\cdot Q-mf_1(t)](1+\gamma_5)v(q) \quad \text{(A5)}$$

so the rate is

$$\Gamma(K^+\to\pi^0l^+\nu)=\frac{G^2\sin^2\theta}{4M}\frac{1}{(2\pi)^5}\int d^4p\,\delta(p^2+m^2)\theta(p_0)\int d^4q\,\delta(q^2)\theta(q_0)\delta((P-p-q)^2+\mu^2)$$
$$\times\{-8[4f_+^2(t)(m^2p\cdot q-M^2p\cdot q-2P\cdot pP\cdot q-2m^2P\cdot q)+4f_+(t)f_1(t)m^2(P\cdot q-p\cdot q)+m^2f_1^2(t)p\cdot q]\}, \quad \text{(A6)}$$

where

$$f_+(t)=f_+(0)[1+\Lambda_+(m^2-2p\cdot q)/M^2],$$

with a corresponding formula for $f_1(t)$. Using the invariant-mass technique described in Sec. IV, we reduce the rate down to a single integral, which is computed numerically. We give the results as functions of $\eta=f_1(0)/f_+(0)$, Λ_+, Λ_1, and $f_+(0)$:

$$\Gamma(K^+\to\pi^0e^+\nu)=\frac{G^2\sin^2\theta M^5}{16\pi^3}f_+^2(0)\times10^{-2}[1.2067+0.3335\Lambda_+],$$

$$\Gamma(K^+\to\pi^0\mu^+\nu)=\frac{G^2\sin^2\theta M^5}{16\pi^3}f_+^2(0)\times10^{-2}$$
$$\times[0.6497+0.1063\eta+0.0232\eta^2+0.2720\Lambda_++0.0277(\Lambda_+\eta+\Lambda_1\eta)+0.0152\Lambda_1\eta^2]. \quad \text{(A7)}$$

The sum over spins and polarizations for the $K_{l3\gamma}$ matrix element gives

$$4mm_\nu\sum|M_{IB}|^2$$
$$=8\left[\frac{m^2}{(p\cdot k)^2}+\frac{M^2}{(P\cdot k)^2}+\frac{2P\cdot p}{P\cdot kp\cdot k}\right][4f_+^2(t)(m^2p\cdot q-M^2p\cdot q-2P\cdot pP\cdot q-2m^2P\cdot q)$$
$$+4f_+(t)f_1(t)m^2(P\cdot q-p\cdot q)+m^2f_1^2(t)p\cdot q]+8\left\{4f_+^2(t)\left[4P\cdot q-\frac{2P\cdot kP\cdot q}{p\cdot k}-\frac{4P\cdot pP\cdot q}{p\cdot k}-\frac{M^2P\cdot q}{P\cdot k}+\frac{2M^2q\cdot k}{P\cdot k}-\frac{M^2q\cdot k}{p\cdot k}\right.\right.$$
$$-\frac{M^2p\cdot q}{p\cdot k}-\frac{2P\cdot pp\cdot q}{p\cdot k}+\frac{2m^2P\cdot q}{p\cdot k}-\frac{2m^2P\cdot kP\cdot q}{(p\cdot k)^2}+\frac{2M^2P\cdot pq\cdot k}{(P\cdot k)^2}+\frac{2M^2P\cdot qp\cdot k}{(P\cdot k)^2}-\frac{2M^2p\cdot kq\cdot k}{(P\cdot k)^2}+\frac{2m^2M^2q\cdot k}{(P\cdot k)^2}-\frac{m^2M^2q\cdot k}{(p\cdot k)^2}$$
$$-\frac{M^2P\cdot pq\cdot k}{P\cdot kp\cdot k}+\frac{6P\cdot pP\cdot q}{P\cdot k}-\frac{2P\cdot pq\cdot k}{P\cdot k}+\frac{2P\cdot qp\cdot k}{P\cdot k}+\frac{m^2P\cdot q}{P\cdot k}+\frac{m^4q\cdot k}{(p\cdot k)^2}-\frac{m^2q\cdot k}{p\cdot k}-\frac{m^2p\cdot q}{p\cdot k}+\frac{2(P\cdot p)^2q\cdot k}{P\cdot kp\cdot k}$$
$$\left.+\frac{3m^2P\cdot pq\cdot k}{P\cdot kp\cdot k}\right]+m^2f_1^2(t)\left[\frac{q\cdot k}{p\cdot k}+\frac{p\cdot q}{p\cdot k}-\frac{P\cdot q}{P\cdot k}+\frac{m^2q\cdot k}{(p\cdot k)^2}+\frac{P\cdot pq\cdot k}{P\cdot kp\cdot k}\right]$$

$$-4m^2 f_+(t) f_1(t)\left[\frac{M^2 q\cdot k}{(P\cdot k)^2}+m^2\frac{q\cdot k}{(p\cdot k)^2}+\frac{2P\cdot pq\cdot k}{P\cdot kp\cdot k}\right]\Big\}+16\left[\frac{2P\cdot p}{P\cdot k}+\frac{P\cdot pq\cdot k}{P\cdot kp\cdot k}+\frac{m^2}{p\cdot k}-\frac{p\cdot q}{p\cdot k}+\frac{M^2p\cdot k}{(P\cdot k)^2}+\frac{M^2q\cdot k}{(P\cdot k)^2}+\frac{P\cdot q}{P\cdot k}\right]$$

$$\times\left[4\frac{\partial}{\partial t}f_+^2(t)(m^2p\cdot q-M^2p\cdot q-2P\cdot pP\cdot q-2m^2P\cdot q)+2\frac{\partial}{\partial t}[f_+(t)f_1(t)]m^2(P\cdot q-p\cdot q)+m^2\frac{\partial}{\partial t}f_1^2(t)p\cdot q\right]$$

$$+8\Big\{4\frac{\partial}{\partial t}f_+^2(t)\Big[-2(P\cdot q)^2-4P\cdot pP\cdot q+2P\cdot pq\cdot k+2P\cdot qp\cdot k+4P\cdot qq\cdot k-4M^2p\cdot q-2P\cdot kp\cdot q-6P\cdot pp\cdot q+4m^2P\cdot q$$

$$-6P\cdot qp\cdot q+2p\cdot kp\cdot q-2m^2q\cdot k+2p\cdot qq\cdot k-m^2p\cdot q+\frac{2M^2(q\cdot k)^2}{P\cdot k}+\frac{2P\cdot k(p\cdot q)^2}{p\cdot k}-\frac{2m^2P\cdot kP\cdot q}{p\cdot k}+\frac{2P\cdot kP\cdot qp\cdot q}{p\cdot k}$$

$$-\frac{2P\cdot pP\cdot qq\cdot k}{p\cdot k}+\frac{4M^2P\cdot p(q\cdot k)^2}{(P\cdot k)^2}+\frac{4M^2P\cdot q(p\cdot k)^2}{(P\cdot k)^2}-\frac{M^2P\cdot pq\cdot k}{P\cdot k}-\frac{M^2P\cdot qp\cdot k}{P\cdot k}-\frac{M^2P\cdot qq\cdot k}{P\cdot k}-\frac{2M^2p\cdot k(q\cdot k)^2}{(P\cdot k)^2}$$

$$-\frac{2M^2(p\cdot k)^2q\cdot k}{(P\cdot k)^2}+\frac{2M^2p\cdot kq\cdot k}{P\cdot k}+\frac{4m^2M^2(q\cdot k)^2}{(P\cdot k)^2}-\frac{2M^2p\cdot kp\cdot q}{P\cdot k}-\frac{2M^2p\cdot qq\cdot k}{P\cdot k}-\frac{m^2M^2q\cdot k}{p\cdot k}+\frac{M^2p\cdot qq\cdot k}{p\cdot k}+\frac{2m^2P\cdot pq\cdot k}{p\cdot k}$$

$$-\frac{4P\cdot pp\cdot qq\cdot k}{p\cdot k}-\frac{2m^2P\cdot kp\cdot q}{p\cdot k}+\frac{4M^2P\cdot pp\cdot kq\cdot k}{(P\cdot k)^2}+\frac{4M^2P\cdot qp\cdot kq\cdot k}{(P\cdot k)^2}-\frac{M^2P\cdot p(q\cdot k)^2}{P\cdot kp\cdot k}+\frac{4m^2M^2p\cdot kq\cdot k}{(P\cdot k)^2}-\frac{2P\cdot p(q\cdot k)^2}{P\cdot k}$$

$$+\frac{6(P\cdot p)^2q\cdot k}{P\cdot k}-\frac{4P\cdot q(p\cdot k)^2}{P\cdot k}+\frac{4(P\cdot q)^2p\cdot k}{P\cdot k}+\frac{3m^4q\cdot k}{p\cdot k}+\frac{10P\cdot pP\cdot qp\cdot k}{P\cdot k}+\frac{10P\cdot pP\cdot qq\cdot k}{P\cdot k}-\frac{6P\cdot pp\cdot kq\cdot k}{P\cdot k}-\frac{6P\cdot qp\cdot kq\cdot k}{P\cdot k}$$

$$+\frac{7m^2P\cdot pq\cdot k}{P\cdot k}+\frac{m^2P\cdot qp\cdot k}{P\cdot k}+\frac{5m^2P\cdot qq\cdot k}{P\cdot k}+\frac{2(P\cdot p)^2(q\cdot k)^2}{P\cdot kp\cdot k}+\frac{3m^2P\cdot p(q\cdot k)^2}{P\cdot kp\cdot k}-\frac{3m^2p\cdot qq\cdot k}{p\cdot k}\Big]+m^2\frac{\partial}{\partial t}f_1^2(t)$$

$$\times\Big[p\cdot q+\frac{P\cdot pq\cdot k}{P\cdot k}-\frac{P\cdot qp\cdot k}{P\cdot k}-\frac{P\cdot qq\cdot k}{P\cdot k}+m^2\frac{q\cdot k}{p\cdot k}-\frac{p\cdot qq\cdot k}{p\cdot k}+\frac{P\cdot p(q\cdot k)^2}{P\cdot kp\cdot k}\Big]+4m^2f_+(t)\frac{\partial}{\partial t}f_1(t)\Big[P\cdot q+p\cdot q-\frac{2M^2(q\cdot k)^2}{(P\cdot k)^2}$$

$$+\frac{M^2q\cdot k}{P\cdot k}+\frac{P\cdot pq\cdot k}{p\cdot k}-\frac{P\cdot kp\cdot q}{p\cdot k}+\frac{2P\cdot qq\cdot k}{p\cdot k}-\frac{2M^2p\cdot kq\cdot k}{(P\cdot k)^2}+\frac{M^2(q\cdot k)^2}{P\cdot kp\cdot k}-\frac{3P\cdot pq\cdot k}{P\cdot k}-\frac{P\cdot qp\cdot k}{P\cdot k}-\frac{3P\cdot qq\cdot k}{P\cdot k}-m^2\frac{q\cdot k}{p\cdot k}$$

$$+\frac{p\cdot qq\cdot k}{p\cdot k}-\frac{P\cdot p(q\cdot k)^2}{P\cdot kp\cdot k}\Big]+4m^2f_1(t)\frac{\partial}{\partial t}f_+(t)\Big[-P\cdot q-p\cdot q-\frac{2M^2(q\cdot k)^2}{(P\cdot k)^2}-\frac{M^2q\cdot k}{P\cdot k}-\frac{P\cdot pq\cdot k}{p\cdot k}+\frac{P\cdot kp\cdot q}{p\cdot k}-\frac{2P\cdot qq\cdot k}{p\cdot k}$$

$$-\frac{2M^2p\cdot qq\cdot k}{(P\cdot k)^2}-\frac{M^2(q\cdot k)^2}{P\cdot kp\cdot k}-\frac{5P\cdot pq\cdot k}{P\cdot k}+\frac{P\cdot qp\cdot k}{P\cdot k}-\frac{P\cdot qq\cdot k}{P\cdot k}-3m^2\frac{q\cdot k}{p\cdot k}+\frac{3p\cdot qq\cdot k}{p\cdot k}-\frac{3P\cdot p(q\cdot k)^2}{P\cdot kp\cdot k}\Big]\Big\}, \tag{A8}$$

$$\frac{4mm_\nu}{B^2}\sum|M^A|^2=\frac{4mm_\nu}{A^2}\sum|M^V|^2=\frac{-16}{M^4}[P\cdot kP\cdot qp\cdot k+P\cdot kP\cdot pq\cdot k+M^2p\cdot kq\cdot k], \tag{A9}$$

$$4mm_\nu \sum (M_{IB}M^{V*}+M^VM_{IB}^*)$$
$$=\frac{16A}{M^2}\bigg\{2f_+(t)\bigg[-2P\cdot kP\cdot q-3P\cdot pP\cdot q-M^2q\cdot k+2P\cdot pq\cdot k+2P\cdot qp\cdot k-m^2P\cdot q-\frac{(P\cdot p)^2q\cdot k}{p\cdot k}+\frac{P\cdot kP\cdot pp\cdot q}{p\cdot k}$$
$$-\frac{m^2P\cdot kP\cdot q}{p\cdot k}-\frac{M^2P\cdot pq\cdot k}{P\cdot k}-\frac{M^2P\cdot qp\cdot k}{P\cdot k}+\frac{2M^2p\cdot kq\cdot k}{P\cdot k}-\frac{m^2M^2q\cdot k}{P\cdot k}+\frac{m^2P\cdot kq\cdot k}{p\cdot k}-\frac{m^2P\cdot pq\cdot k}{p\cdot k}+\frac{m^2P\cdot kp\cdot q}{p\cdot k}\bigg]$$
$$+f_1(t)\bigg[m^2P\cdot q+m^2M^2\frac{q\cdot k}{p\cdot k}-\frac{m^2P\cdot kq\cdot k}{p\cdot k}+m^2\frac{P\cdot pq\cdot k}{p\cdot k}-m^2\frac{P\cdot kp\cdot q}{p\cdot k}\bigg]\bigg\}, \quad \text{(A10)}$$

$$4mm_\nu \sum (M_{IB}M^{A*}+M^AM_{IB}^*)$$
$$=\frac{-16B}{M^2}\bigg\{2f_+(t)\bigg[2P\cdot kP\cdot q+P\cdot pP\cdot q+M^2q\cdot k+M^2p\cdot q-\frac{(P\cdot p)^2q\cdot k}{p\cdot k}+\frac{P\cdot kP\cdot pp\cdot q}{p\cdot k}+\frac{m^2P\cdot kP\cdot q}{p\cdot k}+m^2M^2\frac{q\cdot k}{p\cdot k}$$
$$+\frac{m^2P\cdot kq\cdot k}{p\cdot k}+2P\cdot pq\cdot k-2P\cdot qp\cdot k\bigg]-f_1(t)m^2\frac{P\cdot kq\cdot k}{p\cdot k}\bigg\}, \quad \text{(A11)}$$

$$4mm_\nu \sum (M^AM^{V*}+M^VM^{A*})=-\frac{32AB}{M^4}(P\cdot kP\cdot qp\cdot k-P\cdot kP\cdot pq\cdot k). \quad \text{(A12)}$$

As explained in the text, we drop all terms proportional to Λ in Eqs. (A10) and (A11). The final form for the rate is

$$\Gamma(K^+\to\pi^0l^+\nu\gamma)=\frac{e^2G^2\sin^2\theta}{4M}\frac{1}{(2\pi)^8}\int d^4k\int d^4p\int d^4q\int d^4Q\,\delta^4(P-Q-p-q-k)4mm_\nu\sum|M_{IB}+M^V+M^A|^2, \quad \text{(A13)}$$

where we drop the δ and θ functions for simplicity, i.e.,

$$\Gamma(K^+\to\pi^0l^+\nu\gamma)=\frac{G^2\sin^2\theta M^5}{256\pi^3}\frac{\alpha}{\pi}\int_{(m+\mu)^2/M^2}^{(1-\Lambda/M)^2}\frac{db}{M^2}\int_{\mu^2/M^2}^{(\sqrt{(b/M^2)}-m/M)^2}\frac{da}{M^2}f\bigg(\frac{a}{M^2},\frac{b}{M^2},\frac{m^2}{M^2},\frac{\mu^2}{M^2}\bigg), \quad \text{(A14)}$$

where f is dependent upon all the masses and couplings and is too complicated to be reproduced here.

APPENDIX B: PHASE-SPACE INTEGRALS

Define the basic integral, dropping the θ functions

$$I_{(m,n)}=\bigg(\frac{2}{\pi}\bigg)^2\int d^4k\,\delta(k^2)\delta((P-k)^2+b)$$
$$\times\int d^4p\,\delta(p^2+m^2)\delta((B-p)^2+a)\frac{(P\cdot p)^n}{(p\cdot k)^m}, \quad \text{(B1)}$$

where $B=P-k$. We have already shown that

$$I_{(0,0)}=\frac{\lambda^{1/2}(M^2,b,0)}{M^2}\frac{\lambda^{1/2}(b,a,m^2)}{b}. \quad \text{(B2)}$$

By similar methods, involving the evaluation of some angular integrations,

$$I_{(1,0)}=\frac{2}{M^2}\ln\bigg|\frac{\omega-\lambda^{1/2}(b,m^2,a)}{\omega+\lambda^{1/2}(b,m^2,a)}\bigg|, \quad \text{(B3)}$$
$$\omega=b+m^2-a,$$

$$I_{(2,0)}=\frac{4}{m^2M^2}\frac{\lambda^{1/2}(b,m^2,a)}{M^2-b},$$

$$I_{(0,1)}=-\frac{(b+M^2)}{4b}\omega I_{(0,0)},$$

$$I_{(-1,0)}=\frac{(b-M^2)}{4b}\omega I_{(0,0)},$$

$$I_{(0,2)}=\bigg[-\frac{M^2}{12b}\lambda(b,m^2,a)+\frac{1}{12b}(b+M^2)^2\bigg(\frac{\omega^2}{b}-m^2\bigg)\bigg]I_{(0,0)},$$

$$I_{(-2,0)}=\frac{1}{12b}(b-M^2)^2\bigg(\frac{\omega^2}{b}-m^2\bigg)I_{(0,0)},$$

$$I_{(0,3)}=\left[-\frac{(b+M^2)^3\omega^3}{32b^3}+\frac{M^2\omega^3(b+M^2)}{16b^2}+\frac{m^2\omega(b+M^2)^3}{16b^2}-\frac{m^2M^2\omega(b+M^2)}{4b}\right]I_{(0,0)},$$

$$I_{(-3,0)}=\left[\frac{(b-M^2)^3\omega^3}{32b^3}-\frac{m^2\omega(b-M^2)^3}{16b^2}\right]I_{(0,0)}, \qquad \text{(B4)}$$

$$I_{(1,1)}=-\tfrac{1}{2}\omega I_{(1,0)}+I_{(0,0)},$$
$$I_{(1,2)}=-\omega I_{(0,0)}+\tfrac{1}{4}\omega^2 I_{(1,0)}+I_{(-1,0)},$$
$$I_{(1,3)}=-\tfrac{1}{8}\omega^3 I_{(1,0)}+\tfrac{3}{4}\omega^2 I_{(0,0)}-\tfrac{3}{2}\omega I_{(-1,0)}+I_{(-2,0)},$$
$$I_{(2,1)}=-\tfrac{1}{2}\omega I_{(2,0)}+I_{(1,0)},$$
$$I_{(2,2)}=-\omega I_{(1,0)}+\tfrac{1}{4}\omega^2 I_{(2,0)}+I_{(0,0)},$$
$$I_{(-1,1)}=-\tfrac{1}{2}\omega I_{(-1,0)}+I_{(-2,0)},$$
$$I_{(-2,1)}=-\tfrac{1}{2}\omega I_{(-2,0)}+I_{(-3,0)},$$
$$I_{(-1,2)}=\tfrac{1}{4}\omega^2 I_{(-1,0)}-\omega I_{(-2,0)}+I_{(-3,0)}. \qquad \text{(B5)}$$

The recurrence relations may be checked by noting that $\delta((B-p)^2+a)$ fixes $P\cdot p$, i.e., using $p\cdot B=-\frac{1}{2}\omega$,

$$P\cdot p=-\tfrac{1}{2}\omega+p\cdot k. \qquad \text{(B6)}$$

We can then express $I_{(1,1)}$ as

$$I_{(1,1)}=\left(\frac{2}{\pi}\right)^2\int d^4k\,\delta(k^2)\delta((P-k)^2+b)\times\int d^4p\,\delta(p^2+m^2)\delta((B-p)^2+a)\left(\frac{-\frac{1}{2}\omega+p\cdot k}{p\cdot k}\right)=-\tfrac{1}{2}\omega I_{(1,0)}+I_{(0,0)}. \qquad \text{(B7)}$$

The other recurrence relations may be derived analogously and in general permit the integrals $I_{(m,n)}$ to be expressed in terms of $I_{(m,0)}$ and $I_{(0,n)}$.

PHYSICAL REVIEW VOLUME 184, NUMBER 5 25 AUGUST 1969

Electric Dipole Moment of the Neutron: Expected Order of Magnitude

G. BARTON AND E. D. WHITE
University of Sussex, Brighton, England
(Received 3 April 1969; revised manuscript received 8 May 1969)

Sideways dispersion relations yield a theoretically well-defensible "lower limit" on the expected order of magnitude of the neutron electric dipole moment. If T violation is due to the weak interactions, then 10^{-24} e cm appears as a reasonable expectation, and 10^{-23} e cm the most one can hope for. If T violation is due to the electromagnetic interactions of hadrons, we still cannot safely expect more than 10^{-23} e cm, although optimistic guesses can easily yield 6×10^{-22} e cm or more.

THE electric dipole moment $e\beta$ of the neutron would vanish under reflection (P) invariance, or under time-reversal (T) invariance[1]; since neither is exact,[2] one should estimate the theoretically expected order of magnitude of β. The experimental upper limit, $\beta<5\times10^{-23}$ cm,[3] already falls below some predictions.[4] Some other models of T violation predict that β should effectively vanish; we call these null-β theories.[5] We discuss first the case where T violation is due to the weak interactions, and second, the case where it is due to the electromagnetic (EM) interactions of the hadrons[6] (which remain, however, P-invariant).

In comparison with our predecessors, we claim only (i) that our input assumptions bridge only those gaps in the experimental situation that cannot at present be sidestepped, (ii) that we are conservative rather than optimistic about the dynamics, and (iii) that we have isolated a less model-dependent and better calibrated expression for certain almost unavoidable contributions, which should set a theoretically well defensible order-of-magnitude "lower limit" on β, unless there are accidental cancellations, or conspiracies which effectively reduce the theory to the null-β type.

We begin with weak T violation. To motivate a fairly careful treatment, recall that a quasidimensional estimate would read thus: $\beta=$ (strength of T violation) $\times$ (strength of P violation) $\times$ (typical hadronic length). The first factor is generally agreed at around 10^{-3}; but

[1] L. D. Landau, Nucl. Phys. 3, 127 (1957).

[2] For references to T violation, see R. C. Casella, Phys. Rev. Letters **22**, 554 (1969).

[3] J. K. Baird *et al.*, Phys. Rev. **179**, 1285 (1969).

[4] Some recent predictions are P. Babu and M. Suzuki, Phys. Rev. **162**, 1359 (1967): $\beta>2.2\times10^{-22}$ cm; K. Nishijima, Progr. Theoret. Phys. (Kyoto) **41**, 739 (1969): 2×10^{-22} cm; P. McNamee and J. C. Pati, Phys. Rev. **178**, 2273 (1968): (0.9 to 1.5)$\times10^{-22}$ or (5 to 8)$\times10^{-24}$ cm, in two alternative models.

[5] L. Wolfenstein, Phys. Rev. Letters **13**, 562 (1964); R. J. Oakes, *ibid.* **20**, 1539 (1968). For a difficulty in Oakes' theory: B. H. J. McKellar, *ibid.* **21**, 1822 (1968).

[6] J. Bernstein, G. Feinberg, and T. D. Lee, Phys. Rev. **139**, B1650 (1960).