Analytic Solution of the Linear-Trajectory Bootstrap*

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The quark-model bootstrap previously proposed is extended to include the lower trajectories. The extension is obtained by showing that the factorization properties of the multiparticle Veneziano amplitude can be generalized to the resonances on the lower trajectories. All trajectories other than the leading trajectory are degenerate, and the degeneracy increases rapidly as we go downward. Apart from the fundamental narrow-resonance approximation, the solution of the bootstrap model is exact. If we make a hypothesis regarding the simplicity of the spectrum of the lower trajectories, then the solution appears to be unique to within a finite number of constants. The inconsistency associated with negative widths remains, and any improvement in this respect will probably require going beyond the narrow-resonance approximation.

1. INTRODUCTION

THE purpose of the present paper is to show that the relativistic quark model, which the present authors have proposed in conjunction with Halpern,[1,2] can be extended to include the lower trajectories. The model therefore provides a formally exact solution of the linear-trajectory bootstrap, in which we make no approximation besides the narrow-resonance approximation. It may be questioned whether such an exact solution is of much value, since the narrow-resonance approximation is not expected to be valid for the lower trajectories, besides which the model possesses inconsistencies associated with negative widths. In order to remove the inconsistencies, it will almost certainly be necessary to go beyond the narrow-resonance approximation. We feel, however, that a formally exact solution may well be of value in providing a starting point for an improved treatment. It is also of interest to observe that one can probably answer questions on the uniqueness of the Veneziano formula within the framework of such a model. If we ask for that solution which contains the minimum number of lower trajectories, we obtain a result which, as far as we can tell, is unique to within a finite number of parameters.

The fundamental question which must be resolved concerns the factorization of the multiparticle Veneziano formula. If we examine the N-point amplitude (Fig. 1) and isolate the pole which occurs when the trajectory $1j$ passes through a certain integer, the residue of the pole must be expressible as a finite sum of products of two factors, one associated with each half of Fig. 1. It is already known that the residue associated with the leading trajectory can be expressed as a single product of two factors if all external particles are spinless. We shall show in the present paper that the residues associated with the subsidiary trajectories can also be expressed as a sum of such products, but the sum will now consist of more than one term. In other words, the subsidiary trajectories will consist of several degenerate numbers.

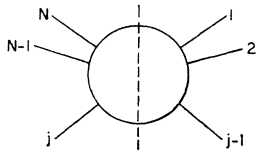

FIG. 1. Decomposition of the N-point Veneziano amplitude into two factors.

Once we know that the Veneziano amplitude is factorable, we can construct a solution of the linear-trajectory bootstrap as we have explained in Refs. 1 and 2. From a multiparticle Veneziano amplitude we isolate the pole term corresponding to the four-point amplitude. Any of the resonances which correspond to the individual factors associated with the leading or nonleading trajectories can play the role of external particles in such an amplitude. The same resonances will appear as internal particles in the four-point amplitude, and the vertices will possess the required symmetry properties. Thus all the requirements of the bootstrap are fulfilled. Production amplitudes can similarly be constructed.

If we replace the single-term Veneziano formula by a sum of leading and nonleading terms with arbitrary coefficients, the amplitude will no longer be factorizable. Such a formula does not, therefore, provide a solution to the bootstrap equations. The question arises whether one can restore the factorization requirement by suitably adjusting the coefficients of the nonleading terms. One way of doing so would be to take a system with a number of quarks of different masses. The masses can be adjusted so that trajectories in new positions are not introduced. However, the number of degenerate trajectories would be increased by such a process, and the degeneracy increases rapidly as we go downward. We therefore make the hypothesis that the scattering amplitudes which occur in nature do not have an unnecessarily complicated spectrum of lower trajectories. It might be hoped that the hypothesis could be proved from the unitarity condition, but at the moment it appears that such a proof would require a considerable improvement in our understanding of the whole prob-

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[1] K. Bardakci and M. B. Halpern, (to be published).
[2] S. Mandelstam, (to be published).

lem. We feel that a hypothesis of this nature provides a reasonable starting-point for constructing a dynamical scheme. It can at least be partly justified by the remarkable agreement of the predictions of the simple quark model with experiment.

One may still ask the question whether there is any other way of adding nonleading terms without destroying the factorization condition. In particular, one would like to know whether it is possible to do so without increasing the number of lower trajectories. We have attempted to construct amplitudes which fulfill these conditions, but we have not succeeded. On the other hand, we have no proof that such amplitudes do not exist. The position at the moment is therefore that we have an analytic solution of the rising-trajectory bootstrap. The solution possesses no obvious nonuniqueness once we make our hypothesis regarding the simplicity of the spectrum of lower trajectories, but we cannot prove that no other solution exists.

With the approach outlined above, we can discard the assumptions made in Ref. 2 regarding the nonleading Veneziano terms. The results quoted in Refs. 1 and 2 will depend on the hypothesis mentioned above, and they will be fundamentally dependent on the narrow-resonance approximation, but no other approximation is necessary.

We shall carry out our proofs in terms of a model with neutral-scalar quarks, since the extension to actual quarks is straightforward. The spectrum of the quark model will be obtained by superimposing the $SU(6)$ spectrum of particles, or the $SU(6,6)$ spectrum of trajectories, on the spectrum derived in the present paper. In Sec. 2 we shall show that any particular multiparticle Veneziano residue can be expressed as a finite sum of factorizable terms. The details of the spectrum of lower trajectories are not necessary for an understanding of the model, but we shall outline a prescription for calculating the number of satellites in Secs. 3 and 4. The second trajectory will turn out to be twofold degenerate; the third will be sevenfold degenerate. Thus, even when we choose the solution with the minimum number of trajectories, this number still increases rapidly as we go towards the lower trajectories.

2. FACTORIZATION OF THE GENERAL RESIDUE OF THE N-POINT VENEZIANO AMPLITUDE

We now examine the nth pole in a particular channel of the N-point Veneziano amplitude. We divide the particles into two groups $1,\ldots,j$ and $j+1,\ldots,N$, and we examine the residue when the corresponding trajectory function α_{1j} is equal to n (Fig. 1). If we integrate only over the variable u_{1j} relative to the channel in question, it is not difficult to show that the residue in the pole of the remaining integrand is given by the following formula:

$$R=-\frac{1}{n!}I_1I_2\frac{\partial^n}{\partial w^n}\prod_{i=1}^{j}\prod_{k=j+1}^{n}(1-wy_iy_k)^{2p_ip_k+\epsilon_{ik}}\Big|_{w=0}, \quad (2.1)$$

where

$$y_i=u_{1i}\cdots u_{1,j-1}, \quad i\leqq 2\leqq j-1 \quad (2.2a)$$

$$y_j=1, \quad y_1=0, \quad (2.2b)$$

$$y_k=u_{N,j+2}\cdots u_{Nk}, \quad j+2\leqq k\leqq N-1 \quad (2.2c)$$

$$y_{j+1}=1, \quad y_N=0, \quad (2.2d)$$

$$\epsilon_{ik}=0 \text{ unless } i=j \text{ and } k=j+1, \quad (2.2e)$$

$$\epsilon_{j,j+1}=b-1, \quad (2.2f)$$

and b is the intercept of the Regge trajectory. (We are using the mass scale for which the slope of the trajectories is equal to unity.) The functions I_1 and I_2 are the integrands of the multiparticle Veneziano formulas associated with the two halves of Fig. 1. They will of course be functions of all the appropriate u's and α's.

In order to evaluate the expression on the right of (2.1), it is convenient to take the logarithm of the product, so as to turn it into a sum. Thus

$$\begin{aligned}R&=-\frac{1}{n!}I_1I_2\left(\frac{\partial^n}{\partial w^n}\right)\prod_{i=1}^{j}\prod_{k=j+1}^{N}(1-wy_iy_k)^{2p_ip_k+\epsilon_{ik}}\Big|_{w=0}\\&=-\frac{1}{n!}I_1I_2\left(\frac{\partial^n}{\partial w^n}\right)\exp\Big[\sum_{i=1}^{j}\sum_{k=j+1}^{N}(2p_ip_k+\epsilon_{ik})\\&\qquad\times\ln(1-wy_iy_k)\Big]\Big|_{w=0}\\&=-\frac{1}{n!}I_1I_2\left(\frac{\partial^n}{\partial w^n}\right)\exp\Big[-\sum_{i=1}^{j}\sum_{k=j+1}^{N}(2p_ip_k+\epsilon_{ik})\\&\qquad\times\sum_{r=1}^{\infty}\frac{w^r}{r}y_i{}^ry_k{}^r\Big]\Big|_{w=0}. \quad (2.3)\end{aligned}$$

We now observe that the coefficient of w^r, excluding the term $\epsilon_{j,j+1}$, can be written as a product of two factors, one involving only p_i and y_i and the other only p_j and y_j. Thus

$$R=-\frac{1}{n!}I_1I_2\frac{\partial^n}{\partial w^n}\exp\left\{-2\sum_{r=1}^{\infty}\frac{w^r}{r}\Big[\Big(\sum_{i=1}^{j}p_iy_i{}^r\Big)\Big(\sum_{k=j+1}^{N}p_ky_k{}^r\Big)+b-1\Big]\right\}\Big|_{w=0}. \quad (2.4)$$

In writing down (2.4) we have used Eq. (2.2f) for ϵ_{ik}.

To find the nth derivative in (2.4), we expand the exponential in powers of r and isolate the coefficient of

w^r. The result will consist of a sum of terms of the form

$$cI_1I_2\prod_{k_r}[(\sum_{i=1}^{j}p_iy_i^r)(\sum_{k=j+1}^{N}p_ky_k^r)+b-1]^{k_r}, \quad (2.5a)$$

with

$$\sum r\kappa_r=n. \quad (2.5b)$$

The product is over all possible sets of integers κ_r which satisfy (2.5b) with $0\leqq r\leqq n$. Since we are interested in making an angular-momentum analysis of the residue, we shall work in the c.m. system of the channel $1j$, and shall separate the scalar product into its spatial and temporal parts. Thus F will consist of terms of the form

$$cI_1I_2\prod_{\kappa r}[(\sum p_{0i}y_i^r)\cdot(\sum p_{0k}y_k^r) - (\sum \mathbf{p}_i y_i^r)\cdot(\sum \mathbf{p}_k y_k^r)+b-1]^{\kappa r}, \quad (2.6)$$

with the indices κ_r still being limited by (2.5b). All summations over the indices i and k run from 1 to j and from $j+1$ to N, respectively.

We may next expand the expression (2.6) so that the amplitude will be represented by a sum of terms of the form

$$cI_1I_2\prod_{r,s}(\sum \mathbf{p}_i y_i^r)^{\lambda_r}\cdot(\sum \mathbf{p}_k y_k^r)^{\lambda_r} \times(\sum p_{0i'}y_{i'}^s)^{\mu_s}(\sum p_{0k'}y_{k'}^s)^{\mu_s}, \quad (2.7a)$$

with

$$\sum r\lambda_r+\sum s\mu_s\leqq n. \quad (2.7b)$$

By making an angular-momentum analysis of each term of (2.7a), we can resolve it into a finite number of terms, each consisting of two factors which depend only on the initial-state and final-state variables, respectively. However, it is possible to find certain identities with the aid of which we can reduce the number of terms, and we shall now explain how such identities may be obtained.

3. RESTRICTIONS DUE TO CHARGE CONJUGATION

In choosing the variables u corresponding to Fig. 1 which we regard as independent, we could have labeled the particles by starting from the bottom of the diagram and proceeding anticlockwise. The particles numbered 1 to j would then be numbered j to 1, while those numbered $j+1$ to N would be numbered N to $j+1$. Under such a transformation it is not difficult to show that the y's transform as follows (with $u_{1j}=0$):

$$y_i\rightarrow 1-y_{j+1-i}, \quad y_k\rightarrow 1-y_{N+j+1-k}. \quad (3.1)$$

The factors I_1 and I_2 in our amplitude will change by a Jacobian factor under the transformation in question, since a change of the variables of integration is involved, but the extra factors (2.7a) should be unchanged. The expressions as written do not have this property.

To investigate the origin of this apparent discrepancy, we notice that the left-hand factors of the terms of (2.7) were obtained by n-fold differentiation with respect to the variable u_{1j}, with $u_{ii'}$ kept constant ($1\leqq i\leqq j-2$, $2\leqq i'\leqq j-1$). If we had adopted the other choice of variables, we would again differentiate with respect to u_{1j}, but the variables kept constant would now be $u_{ii'}$ ($2\leqq i\leqq j-1$, $3\leqq i'\leqq j$). Under the change of variables, the integrand I simply changes by the Jacobian factor, but the derivative of I with respect to u_{1j} acquires in addition a term

$$\frac{\partial}{\partial u_{1,j-1}}\left(\frac{\partial u_{1,j-1}}{\partial u_{1j}}\bigg|_{uij}I\right)\bigg|_{ui,j-1}, \quad 2\leqq i\leqq j-2. \quad (3.2)$$

The expression (3.2) is not zero, but it does give a zero contribution to our final result, which involves an integration over $u_{1,j-1}$. We therefore have to supplement (2.7) with the information that terms of the form (3.2) give no contribution to the integral. When we do so, the symmetry under the change of variables must appear automatically.

We now quote the results:

$$\frac{\partial u_{1,j-1}}{\partial u_{1j}}\bigg|_{uij}=u_{1,j-1}(1-u_{1,j-1}), \quad (u_{1j}=0) \quad (3.3)$$

$$\frac{\partial}{\partial u_{1,j-1}}[I'u_{1,j-1}(1-u_{1,j-1})]\bigg|_{ui,j-1} =2I'P\cdot(\sum p_iy_i-\tfrac{1}{2}P) \equiv 2I'P_0[\sum p_{0i}(y_i-\tfrac{1}{2})], \quad (3.4a)$$

where

$$P=\sum_{i=1}^{j}p_i. \quad (3.4b)$$

I' is the Veneziano integrand with independent variables $u_{1,j-1}$. We could also examine the effect of applying the operator

$$\frac{\partial}{\partial u_{1,j-1}}\frac{\partial u_{1,j-1}}{\partial u_{1j}}\bigg|_{uij}$$

more than once. It will be more convenient, however, to use instead the followlng formula for the derivative of y_i with respect to $u_{1,j-1}$:

$$\frac{\partial y_i}{\partial u_{1,j-1}}\bigg|_{ui,j-1}=-\left[1/u_{1,j-1}(1-u_{1,j-1})\right](y_i^2-y_i). \quad (3.5)$$

From the form of (3.4a), one can conclude immediately that the two equations (3.4) and (3.5) contain all the information that could be obtained by repeated application of operator

$$\frac{\partial}{\partial u_{1,j-1}}\frac{\partial u_{1,j-1}}{\partial u_{1,j}}\bigg|_{uij}.$$

Equations (3.4) and (3.5) can be used to differentiate expressions of the form $u_{1,j-1}(1-u_{1,j-1})I'(\sum \mathbf{p}_i y_i^r)^{\lambda_r}$

$\times(\sum p_{0i}y_i^s)^{\mu_s}$ with respect to the variable $u_{1,j-1}$. The resulting expressions will be sums of terms of the form $I'(\sum \mathbf{p}_i y_i^r)^{\lambda_r}(\sum p_{0i}y_i^s)^{\mu_s}$. By making use of the fact that such sums will give zero when integrated over $u_{1,j-1}$, we can reduce the degeneracy of the lower trajectories.

The change of the ordering of the particles in the initial and final states is equivalent to applying the charge-conjugation operator. It follows that the effect of the initial- or final-state factor under the transformation (3.1) gives us the charge conjugation of the corresponding state.

4. COUNTING OF THE SATELLITES

We now return to examine the expressions of the form (2.7). The initial-state factor will be

$$I_1\prod_{r,s}(\sum \mathbf{p}_i y_i^r)^{\lambda_r}(\sum p_{0i'}y_{i'}^s)^{\mu_s}, \tag{4.1}$$

where each factor $(\sum \mathbf{p}_i y_i^r)^{\lambda_r}$ must be interpreted as a λ_r-index symmetrical tensor. The tensor can be decomposed into representations of the rotation group with angular momentum λ_r, λ_r-2, $\lambda_r-4,\ldots$. The direct product of the representations corresponding to factors of (4.1) with different r can be resolved into irreducible representations of the rotation group in the usual way.

It is convenient to regard each of these representations, for each combination of the indices λ_r and μ_r, as forming a state in a vector space. The condition (2.7b) shows that the dimensionality of the space is finite, and the expression (2.7a) shows that the scattering amplitude is a diagonal matrix in this space.

We can next redefine our basis in such a way that the states are eigenfunctions of C. We simply replace the expressions (4.1) by the following expressions:

$$I_1\prod_{r,s}[\sum \mathbf{p}_i(y_i-\tfrac{1}{2})^r]^{\lambda_r}[\sum p_{0i'}(y_{i'}-\tfrac{1}{2})^s]^{\mu_s}. \tag{4.2}$$

The charge conjugation of the state (4.2) will be

$$C=(-1)^\nu, \tag{4.3a}$$

where

$$\nu=\sum_r r\lambda_r+\sum_s s\mu_s. \tag{4.3b}$$

One can apply an angular-momentum analysis to the states (4.2) in the same way as to the states (4.1). The scattering amplitude will not necessarily be diagonal in the new basis, but it will contain no elements between states of different C. We can therefore rediagonalize the amplitude by defining a third basis in each of the subspaces with definite C.

With the aid of the relations (3.4) and (3.5), we can now reduce the number of states. Let us denote the integer ν, defined by (4.3b), as the index of the state. The state vector (4.2), corresponding to a state of index $\nu-1$, will have the form

$$I_1\prod_{r,s}[\sum \mathbf{p}_i(y_i-\tfrac{1}{2})^r]^{\lambda_r}[\sum p_{0i'}(y_{i'}-\tfrac{1}{2})]^{\mu_s},\quad \sum_r r\lambda_r+\sum_s s\mu_s=\nu-1. \tag{4.4}$$

We may now take the expression

$$u_{1,j-1}(1-u_{1,j-1})I_1\times\prod_{r,s}[\sum \mathbf{p}_i(y_i-\tfrac{1}{2})^r]^{\lambda_r}[\sum p_{0i'}(y_{i'}-\tfrac{1}{2})]^{\mu_s},$$

and differentiate it with respect to $u_{1,j-1}$, using (3.4) and (3.5). It is easy to see that the result is a linear combination of a state with index ν and states with lower index. Such a state does not contribute to our scattering amplitude, since the state vector gives zero when integrated over $u_{1,j-1}$. Furthermore, the angular momentum of the state is unchanged by the operation of differentiation. The number of states with index ν and angular momentum J is thus decreased by the number of states of index $\nu-1$ and angular momentum J.

We shall now indicate how the number of states of a given angular momentum may be counted with the aid of a simple-harmonic-oscillator model. We construct a system of independent simple harmonic oscillators, each corresponding to one value of the index r of (4.2). The rth oscillator will have energy levels 0, r, $2r$, etc. Corresponding to the factor $[\sum p_i(y_i-\frac{1}{2})^r]^{\lambda_r}$, we suppose the rth oscillator to be in the λ_rth energy level, so that it has energy $r\lambda_r$. We have seen that the factor $[\sum p_i(y_i-\frac{1}{2})^r]^{\lambda_r}$ corresponds to a series of states with angular momentum λ_r, λ_r-2, etc. Such states will correspond to the states of the oscillator with radial quantum number 0, 1, etc. The angular momentum of all the oscillators can then be combined in the usual way.

We must modify our prescription somewhat to take into account the factors $[\sum p_{0i'}(y_{i'}-\frac{1}{2})]^{\mu_s}$. Instead of an ordinary simple harmonic oscillator, we consider a "Lorentz simple harmonic oscillator." For each level of the ordinary oscillator with energy E and angular momentum J, there will be a series of daughters of the Lorentz oscillator with energy E and angular momentum J, $J-1$, $J-2$, etc. The factor $[\sum \mathbf{p}_i(y_i-\frac{1}{2})^r]^{\lambda_r}\times[\sum p_{0i'}(y_{i'}-\frac{1}{2})^r]^{\mu_r}$ will correspond to the $(\lambda_r+\mu_r)$th level of the rth oscillator, and to the μ_rth daughter of the Lorentz sequence.

The product of factors (4.2) will thus correspond to a system of Lorentz oscillators. The definition (4.3b) indicates that the total energy of the system of oscillators is equal to the index of the state in question. The number of states of index ν is equal to the number of states of the system of oscillators with total energy equal to ν, and the angular momentum of any state will correspond to the angular momentum of the state of the system of oscillators.

The inequality (2.7b) indicates that the total number of trajectories at the nth pole is equal to the number of

states with $\nu \leqq n$. On the other hand, we have seen that the number of states with $\nu=n$ is decreased by the number of states with $\nu=n-1$, as a consequence of (3.4) and (3.5). Hence the total number of trajectories of angular momentum J passing through the nth pole is equal to the number of states of our system of oscillators with energy n and angular momentum J. The charge conjugation of the states with index less than n will be equal to the charge conjugation of the corresponding states at the $(n-1)$th pole, while the charge conjugation of the additional states will be $(-1)^n$. Thus, if the charge conjugation of the states at the $(n-1)$th pole with angular momentum J is known, there will be an equal number of states at the nth pole with the same value of J and C. All remaining states at the nth pole will have $C=(-1)^n$.

Following the above method, we may easily count the states corresponding to the first few trajectories. The leading trajectory will of course not be degenerate. The second trajectory will be a doublet, with one member of each charge conjugation. The trajectory with $C=-(-1)^j$ will be the first daughter of the leading trajectory, while the second trajectory will be an additional satellite which is not required by Lorentz symmetry. The third trajectory will be sevenfold degenerate. Six members will have $C=(-1)^j$, while the seventh will have $C=-(-1)^j$. Of the trajectories with normal C, one will have an abnormal parity of $-(-1)^j$.

5. CONCLUDING REMARKS

The obvious limitation of our work is the lack of any systematic analysis concerning the factorization properties of multiparticle Veneziano formulas with nonleading terms. In order for the factorization property to be relevant to the question of the uniqueness of the simple Veneziano formula, one has to make the conjecture that one cannot add nonleading terms in such a way that factorization is maintained and that the degeneracy of the lower trajectories is not increased. At present all we can say is that we have been unable to find a method of doing so.

A related question concerns the factorization properties of the generalized Veneziano formulas. To our knowledge, such generalized formulas have not been constructed for amplitudes with more than five external lines. It would be surprising if the formulas were really limited to four- and five-point amplitudes, and the factorization of the generalized formulas is obviously a subject for investigation.

When counting the lower trajectories, we have not proved that all our trajectories are really distinct. It could be that some of our initial and final states are linear combinations of one another, due to some property of the Veneziano formula which we have overlooked.

The hypothesis restricting the number of lower trajectories is not to be taken to imply that we ask for the solution with the absolute minimum number, but that the number of lower trajectories is not too large to provide any restriction at all. Thus, if we really insisted on the minimum number of trajectories, we could conclude that the quarks were neutral scalar particles. We obviously have to relax the hypothesis at least enough to allow a triplet of quarks with spin $\frac{1}{2}$. By relaxing the hypothesis further, it appears that we can introduce $SU(6)$ or $SU(3)$ symmetry breaking.

Another point worth making is that we are not necessarily assuming that the complete spectrum of lower trajectories which exists in nature is simple, but that a weak-coupling approximation with a minimum number of lower trajectories provides an appropriate starting point.

The problem of repulsive trajectories, or negative widths, which was described in Refs. 1 and 2, occurs for the lower trajectories. In fact, the problem is not now confined to the case where the external particles have spin $\frac{1}{2}$; it occurs for spin-zero external particles as well. For practical purposes, the problem is more serious with quarks of spin $\frac{1}{2}$, since it then exists for the leading trajectory, but as a matter of principle the problem exists in any factorized Veneziano model. Its solution will probably require going beyond the narrow-resonance approximation, and may involve the introduction of Goldstone bosons.

In spite of the somewhat *ad hoc* assumptions which we have had to make, we feel that the analytic solution of the linear-trajectory bootstrap scheme does indicate the great power inherent in the developments of the past year.

We understand that similar results regarding factorization of the lower trajectories have been obtained by Fubini and Veneziano.[3]

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[3] G. Veneziano, private communication.