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# Relativistic Quark Model Based on the Veneziano Representation. I. Meson Trajectories*†

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We base a bootstrap dynamical scheme on the Veneziano model, We assume that the internal trajectories in all channels are similar in their spin and isotopic-spin properties to the external particles, and that vertices are symmetric in their internal and external particles. It turns out that quarks form a very useful entity for expressing the solution, even though the system contains no physical quarks. The spin and multiplicity of the quarks are not determined by the solution, but must be assumed. The spectrum of particles is the same as that predicted by the nonrelativistic quark model, and the particles from multiplets of $SU(6)$. The trajectories of the system, on the other hand, form multiplets of $SU(12)$ or $SU(6,6)$, and they are doubled both in their parity and in their Toller $M$ value. Some of the trajectories have residues whose sign corresponds to repulsion rather than attraction, and an interpretation of such trajectories is suggested. If we assume $\rho$ universality in order to fix certain parameters, we can determine all further coupling constants to within a scale factor, and our vertices are those predicted by $SU(6)_W$. In much of the work the detailed form of the Veneziano model is not used, and a comparison is given with certain previous approaches.

## 1. INTRODUCTION

THE Veneziano formula[1] provides a promising alternative to the finite-energy sum rules for constructing a dynamical scheme based on rising Regge trajectories. Since the formula has Regge asymptotic behavior in the three channels, all finite-energy sum rules will automatically be satisfied at all values of $t$. Questions of how many moments to use in the finite-energy sum rules, or of which value of $t$ to take, become irrelevant. The whole scheme thus takes on a clearer form. It now becomes a practicable objective to include completely the lowest $SU(6)$ multiplets of mesons and baryons in a fairly simple calculation. General results related to the quark-model and $SU(6)$ can be derived.

As in any dynamical scheme so far suggested, one has to begin by making approximations whose accuracy can only be estimated by comparison with experiment, or possibly by extending the calculation to a higher approximation. In the scheme based on finite-energy sum rules, the approximation was to assume that only the resonances of lowest energy contribute. Such an approximation is similar in spirit to that of neglecting multiparticle intermediate states in the unitarity equation. Our assumption in the new bootstrap scheme is to take only a finite number of Veneziano terms, or, in the lowest approximation, only a single term. Higher terms emphasize the higher resonances, and are to be taken into account in subsequent stages of the approximation scheme. Actually, we shall be able to obtain many of our conclusions from a somewhat weaker assumption than the absence of non-leading terms. Our calculations for the variation of the Regge residues as a function of $t$ will require the full assumption, however.

It is not completely out of the question that one may be able to treat the lower terms analytically. If the remarkable factorization and vertex-symmetry properties of the $n$-point Veneziano amplitude[2,3] can be extended to nonleading trajectories, one may be able to say something about them.[4] With sufficient optimism one may even hope to show that they cannot change our conclusions about the nature of the spectrum. We shall not attempt to examine such points in the present paper, however.

The scattering as given by the Veneziano formula can take place through an infinite number of resonances. If one simply examines the amplitude for the scattering of four given particles one can obviously not insert all the physical requirements. It is therefore to be expected that any sufficiently general formula will have an infinite number of constants; in the Veneziano formula these constants enter as the coefficients of the nonleading terms.

Any four of the infinite number of internal resonances can, themselves, form the external particles of a scattering amplitude which is given by the Veneziano formula. We thus have a multichannel problem, and the residues are restricted by factorization. In the amplitude $A+B\rightarrow C+D$, the residue at the pole due to the internal resonance $E$ must factorize in the form $g_{ABE}g_{ECD}$. As is well known, such factorization decreases the number of independent parameters in the many-channel problem. Furthermore, if one examines an

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† A preliminary account of this work, based on finite-energy sum rules instead of the Veneziano formula, was presented to the International Conference on High-Energy Physics at Vienna, 1968. The treatment based on the Veneziano formula is much cleaner and, since it is equivalent to satisfying the finite-energy sum rules at all values of $t$, it allows more detailed predictions on the nature of the trajectories.

[1] G. Veneziano, Nuovo Cimento **57A**, 190 (1968).

[2] K. Bardakci and H. Ruegg, Phys. Letters **28B**, 342 (1968); M. A. Virasoro, Phys. Rev. Letters **22**, 37 (1969); H. M. Chan and S. T. Tsou, Phys. Letters **28B**, 485; C. J. Goebel and B. Sakita, Phys. Rev. Letters **22**, 257 (1969).

[3] K. Bardakci and M. B. Halpern (to be published).

[4] Whatever extension may be possible, it is certainly not true that factorization will take place with single subsidiary trajectories at integral distances below the main trajectory.

amplitude such as $E+B\rightarrow C+D$, where the internal resonance of the previous amplitude is now made an external particle, the residue at the pole due to the intermediate state $A$ will factorize in the form $g_{ABE}g_{ACD}$. The coupling constants $g_{ABE}$ in the two residues must be the same. We therefore obtain an infinite number of relations connecting the infinite number of constants in the Veneziano formula. The power of the factorization requirement was pointed out in Veneziano's original paper.

Factorization is itself a consequence of unitarity. Thus, while the unitarity equation is not used explicitly in a narrow-resonance approximation, it does play a major role through the factorization requirement.

One now has to adopt an approximation scheme in order to reduce the infinite set of equations to a finite set. In the first approximation, one would demand that the internal resonances on the leading trajectories should also play the role of external particles. Nonleading terms in the Veneziano formula, which emphasize the lower trajectories, would be left out. One would then obtain a finite number of equations for a finite number of unknowns. In higher approximations one would demand that internal particles on nonleading trajectories should play the role of external particles, and one would introduce nonleading terms. At the moment we do not know whether the system of equations is overdetermined, uniquely determined, or underdetermined; indeed, we have not formulated the problem precisely enough to ask that question.

As we have already remarked, we can obtain our most important results without making the assumption that all nonleading terms are zero. Certain types of nonleading terms are simply not relevant when one asks questions about the lowest particles on the leading trajectories, and no assumption need be made about them. On the other hand, one obviously has to make some assumption about the nonleading terms. Without such an assumption, the crossing relation would impose no restrictions. If one were to allow arbitrary nonleading terms, one could start from a Veneziano term such as

$$R_{l_1l_2,l_3l_4}\frac{\Gamma(-as-b)\Gamma(-at-b-n)}{\Gamma(-as-at-2b-n)},$$

where $R$ is a matrix which involves the spin and unitary-spin variables of the particles. The Veneziano term has a leading trajectory in the $s$ channel, but no leading trajectory in the $t$ channel. One could now add the term

$$R^c{}_{l_1l_2l_3,l_4}\frac{\Gamma(-as-b-n)\Gamma(-at-b)}{\Gamma(-as-at-2b-n)},$$

where $R^c$ is the matrix obtained from $R$ by crossing. The sum of the two terms satisfies the crossing relation, and the matrix $R$ is unrestricted. The properties of $R$ in the $t$ channel are irrelevant, since there is no leading trajectory in this channel.

We therefore introduce an assumption that the leading trajectory bootstrap itself, and we shall first explain our assumption in a system with unitary spin but no ordinary spin. The external particles will consist of certain multiplets, which may be singlets, octets, etc. We demand that the internal particles consist of the same multiplets as the external particles. The internal particles on the highest trajectory must be identical to the external particles, and the amplitude must satisfy the factorization requirement. We insist that no trajectories, leading or nonleading, occur in multiplets other than those corresponding to the external particles.

It may be that there are nonleading trajectories with more complicated multiplicities than the leading trajectories. Our bootstrap assumption is that such trajectories need only be considered at a subsequent stage in the approximation scheme, where the higher multiplets are introduced as external particles.

One makes a similar assumption regarding ordinary spin. We characterize a trajectory by means of its lowest particle, so that we regard the $\pi$ trajectory as pseudoscalar, the $\rho$ trajectory as vector, and so on. Again we make the assumptions that the internal particles on the leading trajectory are identical to the external particles, that the factorization requirement is satisfied, and that the only internal trajectories which occur are those with similar characteristics to the external particles.

A second assumption or, more accurately, a point of interpretation, concerns repulsive trajectories. In any model with linear trajectories, parity doublets must occur for all fermion trajectories and for all boson trajectories with $M\neq 0$. The sign of the residues associated with the parity doublets corresponds to repulsion rather than attraction, unless the residue has a zero at some point below the mass of the lowest particle. Repulsive trajectories cannot occur in a narrow-resonance approximation. We shall discuss the interpretation of the trajectories in the following section; we shall have to regard the narrow-resonance approximation as analogous to a Born approximation rather than as an accurate representation of the amplitude. We should emphasize that the problem of the parity doubling occurs generally in relativistic quark models, and is not peculiar to our present scheme.

A further subsidiary assumption has to be made in order to fix two free parameters $\lambda_V$ and $\lambda_\Pi$, which are associated with the vector and pseudoscalar trajectories, respectively. It turns out that one choice of this parameter gives us the $SU(6)_W$ results. If, therefore, we assume one $SU(6)_W$ result, such as $\rho$ universality, we can predict all other $SU(6)_W$ results. We shall refer to this assumption as "assumption 3." The parameters $\lambda_V$ and $\lambda_\Pi$ specify which combination of degenerate states correspond to the physical pseudoscalar and vector nonets, and their arbitrariness does not constitute a real arbitrariness within the framework of the model itself. By going somewhat beyond the limitations of the model, one may make arguments to indicate that the $\lambda$'s should have approximately their $SU(6)_W$ value. We shall discuss such arguments at the end of this section and in Sec. 6.

In a bootstrap calculation based on the principles just outlined, we shall be mainly concerned with the crossing matrix in spin and $SU(3)$ space. It is necessary to find eigenfunctions of the crossing matrix which are consistent with all factorization requirements. Our approach thus has features in common with previous approaches by Singh and Udgaonkar,[5] Capps and Cutkosky,[6] and especially Capps.[7] By considering dispersion relations for backward scattering (fixed $t$ dispersion relations for the $su$ crossing relations), and by making reasonable assumptions about the subtraction terms, Capps obtained and solved an albegraic problem which, in unitary space, is identical to ours. Harari[8] has obtained similar results from the assumption that there are no exotic resonances. In ordinary space, the results of Capps are also similar but not identical to ours. Since we do not restrict ourselves to backward scattering, we are able to obtain more detailed properties of the trajectories.

Our method of constructing the solution to the problem leads us naturally to regard the mesons as composed of quarks in both their spin and unitary-spin degrees of freedom. One can obtain a solution with quarks of any spin and multiplicity, and we shall assume that the quarks have spin $\frac{1}{2}$ and that the symmetry group is $SU(3)$.

Subject to a qualification which we shall make below, we shall find that the spectrum of *particles* in our model is the same as that predicted by the nonrelativistic quark model. The particles will thus occur in multiplets of $SU(6)$ and, in our model, all members of the multiplet will be degenerate. We have already mentioned that we can adjust the constants $\lambda_V$ and $\lambda_\Pi$ in such a way that the vertex functions are those predicted by $SU(6)_W$.

The spectrum of *trajectories* is more extensive than the nonrelativistic quark-model spectrum, and the trajectories are doubled in both the parity and the Toller quantum number. Thus, in addition to the $\rho$ nonet with $M=0$, we have an axial-vector nonet with $M=0$, $C=-(-1)^j$, as well as an $M=1$ conspiracy between a second vector nonet and an axial-vector nonet with $M=0$, $C=(-1)^j$. There are similarly two scalar and two pseudoscalar nonets, all of which have $M=0$. The scalar and axial-vector trajectories are repulsive and do not correspond to particles. Both of the vector and pseudoscalar trajectories are attractive, but it turns out that the residue of the second vector trajectory and of the second pseudoscalar trajectory at the mass of the $\rho$ or the $\pi$ is only $\frac{1}{9}$ as large as that of the first. We shall remark in the following section that it is plausible to interpret these trajectories in the same way as the repulsive trajectories. They would therefore have no particles associated with them, but our interpretation is not unambiguous.

All our trajectories are exchange degenerate, as in the ordinary quark model. Since we are unable to obtain a solution without exchange degeneracy, we must use the oridinal Veneziano formula, and not the generalizations proposed by Virasoro[9] and Mandelstam.[10]

Once we have made assumption 3, our model has two free parameters, the over-all coupling constant and the intercept of the trajectories (or the mass of the mesons). The over-all coupling constant can obviously not be determined in a narrow-resonance model. We might determine the mass of the mesons if we use the generalized Veneziano supplementary condition,[11] but there are certainly no compelling reasons for doing so. The choice $n=0$ would be inconsistent, since it would make $\mu^2=-1$, while the choice $n=1$ leads to $\mu^2=1$. The masses are of course measured in units of the reciprocal of the slopes of the Regge trajectories, which is 1 BeV. The predicted mass would thus be too large, but corrections due to finite widths would be expected to decrease the mass.

In Secs. 3–5 of the paper, the external particles will be restricted to the lowest members of the leading trajectory. The idea of Bardakci and Halpern[3] then enables us to obtain a solution with any members of the leading trajectory as external particles. Their proposal is to use the properties of the eight-point Veneziano amplitude to construct a bootstrap model where the external particles possess angular momentum. By combining the properties of their model with the spin and unitary-spin properties treated in the present paper, one is led naturally to consider the eight external particles as quarks. We shall combine our model with the Bardakci-Halpern model in Sec. 6. It is evident from such a construction that quarks having space, spin, and $SU(3)$ degrees of freedom may be a very useful concept, even though the quarks do not exist as real particles.

## 2. REPULSIVE RISING TRAJECTORIES

The problem of the parity doublets of rising trajectories was noticed before the Veneziano representation was found.[12] Various solutions have been suggested, other than that which we shall propose here, but none of them is very attractive. We shall treat the problem by

[5] V. Singh and B. M. Udgaonkar, Phys. Rev. **139B**, 1585 (1965).
[6] R. E. Cutkosky, Phys. Rev. **131**, 1888 (1963); R. H. Capps, Phys. Rev. Letters **14**, 842 (1965); Phys. Rev. **148**, 1332 (1966); *ibid.* **158**, 1433 (1967).
[7] R. H. Capps, Phys. Rev. **168**, 1731 (1968).
[8] H. Harari, Phys. Rev. Letters **22**, 562 (1969).
[9] M. A. Virasoro, Phys. Rev. **177**, 2309 (1969).
[10] S. Mandelstam, Phys. Rev. **183**, 1374 (1969).
[11] G. Veneziano, Ref. 1; S. Mandelstam, Phys. Rev. Letters **21**, 1724 (1968).
[12] This problem has been emphasized by P. G. O. Freund (private communication).

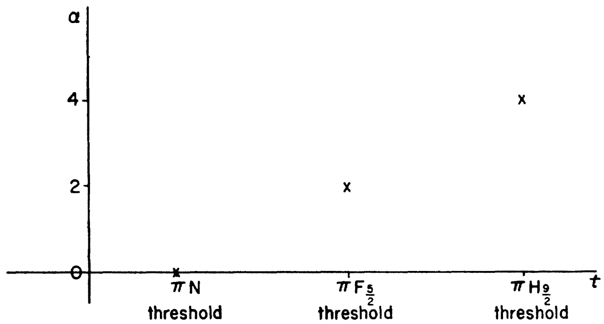


FIG. 1. Gribov-Pomeranchuk singularities for baryon trajectories.

referring to fermion channels, where the possibility $M=0$ does not exist and where more experimental information is available. Once one agrees that the problem exists for fermion channels, we do not consider it a drawback of a particular model that it exists in the boson channels as well.

Let us examine the parity doublet of the baryon Regge trajectories with zero strangeness. The first possibility might be that particles do exist where the trajectories pass through the half-integers. No such particles have been found at the point where the trajectories pass through $l=0$, where $l$ refers to the orbital angular momentum of the quarks. It could happen that the trajectories passed through the value $l=0$ at an energy greater than that for which the quantum numbers of the resonances have been identified, but this would require a marked departure from linearity. All the resonances in the $l=1$ region have been identified with the normal-parity resonances predicted by the quark model, so, unless we are prepared to destroy the beautiful agreement between theory and experiment, we conclude that there are no resonances when the parity-doublet trajectories pass through $l=1$. In this case it would require a smaller deviation from linearity to push the resonances above the region where we have detailed experimental information.

The second possibility is that no particles exist where the parity-doublet trajectories pass through the half-integers, either because the residues vanish or because the trajectories choose nonsense. We have never encountered a model where the trajectories fail to produce particles at all the half integers, and it is probably impossible to construct one. A more reasonable modification of this possibility would be to assume that the trajectories choose nonsense (or have zero residues) at $l=0$ and $l=1$ only. This would not be within the spirit of the quark model, where trajectories should only choose nonsense at negative values of $l$. One may also have an uneasy feeling about requiring that the trajectories choose nonsense at points where experimental information is available, and then choose sense.

If the Regge residue of the trajectories has no zero, the parity doublets will be repulsive. Hence, if there are particles on the parity doublet, the residue must change sign at some point between the lowest particles on the trajectory on the two sides of the MacDowell axis. A similar result is true for boson trajectories with $M\neq 0$; one of the two trajectories must have a zero between $t=0$ and the position of the lowest particle. Had there been no strong experimental evidence against the existence of particles on the parity-doublet trajectories, it would be natural to assume that the zeros were present. We shall attempt an interpretation under the hypothesis that the trajectories remain repulsive.

We wish to link this problem with another problem connected with rising Regge trajectories, namely the problem of the Gribov-Pomeranchuk threshold singularities.[13] Gribov and Pomeranchuk showed that an infinite number of trajectories converge on the point $l=-\frac{1}{2}$ at threshold, where $l$ is the orbital angular momentum. As the variable $t$ is moved through its threshold value, the trajectories approach $l=-\frac{1}{2}$ along the imaginary axis and recede again along the imaginary axis. In a Schrödinger model with an attractive potential, a finite number of trajectories will move to the right of $l=-\frac{1}{2}$ as $t$ is increased from $-\infty$ to threshold. These trajectories will not take part in the Gribov-Pomeranchuk phenomenon. If the potential is repulsive, all trajectories take part in the Gribov-Pomeranchuk phenomenon.

The lowest two-particle channel which communicates with the baryon trajectory is the pion-nucleon channel. At $j=0$, the orbital angular momentum can be equal to $-\frac{1}{2}$. A Gribov-Pomeranchuk singularity will therefore exist at the threshold for $\pi N$ scattering at $j=0$. Another channel which communicates with the baryon trajectory is the $\pi$-$F_{5/2}$ channel. One of its subchannels will have $l=-\frac{1}{2}$ at $j=2$. Another Gribov-Pomeranchuk singularity will thus be present at the threshold for $\pi F_{5/2}$ scattering at $j=2$. Similarly there will be a singularity at the threshold for $\pi H_{9/2}$ scattering at $j=4$, and so on. The Gribov-Pomeranchuk singularities have been shown in Fig. 1.

At each of the singularities indicated in Fig. 1, an infinite number of trajectories must approach the real axis from the imaginary direction as $t$ passes through the appropriate value. The trajectories must recede again as $t$ is increased further. The arguments of Gribov and Pomeranchuk cannot tell us how the trajectories behave away from the singularities, but it is reasonable to suppose that trajectories exist which jump from singularity to singularity, remaining in the complex $j$ plane except at the singularities. At any rate, rising trajectories must exist which approach the Gribov-Pomeranchuk singularities from the complex plane. Such trajectories will supplement the ordinary trajectories which rise roughly linearly.

As long as we work within the framework of the

[13] V. N. Gribov and I. Ya. Pomeranchuk, Phys. Rev. Letters 9, 238 (1962).

narrow-resonance approximation, we cannot accommodate trajectories which do not rise linearly. We shall therefore regard the Veneziano formula as analogous to the Born approximation in potential scattering, and we do not insist that it reproduce correctly all features of the scattering amplitude. We attempt to relate the trajectories in the actual scattering amplitude, some of which take part in the Gribov-Pomeranchuk phenomenon, to their counterparts in the narrow-resonance approximation where they all rise linearly. We again remind the reader of the corresponding relationship in potential theory. The leading attractive trajectories, which correspond to the particles or resonances, do not pass through the Gribov-Pomeranchuk singularities. The repulsive trajectories, and also the subsidiary attractive trajectories, do. In a model with rising Regge trajectories, we therefore require that the leading attractive trajectories do not pass through the Gribov-Pomeranchuk singularities. The respulsive trajectories do pass through the singularities and, if a trajectory is repulsive along its entire length, it may pass through an infinite number of Gribov-Pomeranchuk singularities. Such a behavior has no counterpart in potential theory, where there are only a finite number of singularities.

Since the parity-doublet trajectories are repulsive, it is proposed that they pass through the Gribov-Pomeranchuk singularities. They will not be near the real axis at the half-integers, and they do not produce particles or resonances. As long as the trajectories are sufficiently far from the real axis except near the thresholds, there is no obvious contradiction between the sign of the residue and the requirements of analyticity and unitarity.

Besides the trajectories just treated, our model predicts a second vector and pseudoscalar trajectory. They are attractive, but their residues when they pass through $s=\mu^2$ are much smaller than those of the first trajectory. We shall assume that these trajectories also take part in the Gribov-Pomeranchuk phenomenon and move into the complex plane away from the thresholds; nothing in our previous reasoning implies that *only* the repulsive trajectories can have this property. Such an interpretation is certainly not unique, nor is it necessary that the trajectory have the same behavior near all the integers.

Except for the $SU(6)$ singlet, all channels with repulsive trajectories also have nonleading attractive trajectories one unit below. This may be a significant point, since analyticity and unitarity do place restrictions on the position and residue of a repulsive trajectory in the complex plane. It is a reasonable conjecture that the restrictions will be weaker when attractive trajectories are present as well. If so, it may be that one can obtain a consistent solution if and only if the residues of the repulsive trajectories are not too large relative to those of the attractive trajectories. Making the assumption that the maximum repulsive residue is not too large relative to the attractive residues, we can resolve certain ambiguities in the predictions of the model. In particular, there exists another solution where the roles of the attractive and repulsive trajectories are reversed, so that the lowest-mass particles are scalars and axial vectors instead of pseudoscalars and vectors. In such a model some of the repulsive trajectories have residues three times as large as those of the attractive trajectories, whereas in our model they are about $\frac{2}{3}$ as large. We therefore prefer our original model.

Another application of the assumption regarding the magnitude of the repulsive residues is to fix the approximate value of the parameters $\lambda$ mentioned in the previous section. The optimum value appears to be around the value given by assumption 3.

The conclusions of this section are therefore that repulsive rising trajectories may be allowed, provided that we regard the narrow-resonance approximation as analogous to the Born approximation in potential scattering. It is very plausible that the inconsistency associated with the wrong sign of the residue will disappear in an improved treatment with finite widths.

## 3. BOOTSTRAP FOR SPINLESS MESONS

Before dealing with the solution of our problem which corresponds to nature and which possesses mesons of spin 0 and 1, we shall deal with a hypothetical problem where the mesons have $SU(3)$ symmetry but are spinless. The solution we expect will correspond to a model with a triplet of spinless quarks. Most of our results will agree with those predicted by other models, but it will be helpful to use this problem to explain our methods before we go on to the more difficult problem where the quarks have spin.

We may write the general Veneziano amplitude as follows:

$$A_{l_1l_2l_3l_4}(s,t)=\sum_r R_{l_1l_2l_3l_4}{}^{(r)}W^{(r)}(s,t). \qquad (3.1)$$

The indices $l_1$, $l_2$, $l_3$, and $l_4$ on the amplitude $A$ represent the components in unitary space, and they specify both the multiplet involved (e.g., singlets or octets) and the component within the multiplet. The summation on the right is over leading and nonleading Veneziano terms $W^{(r)}$, and $R$ is a matrix in unitary space which is independent of $S$ and $t$. One can resolve $R$ into multiplets and components for the incoming state 34 as a whole and, according to our bootstrap assumption, the only multiplets which appear in this resolution must be the multiplets corresponding to the external particles.

From the crossing relation, we may write

$$A_{l_1l_2l_3l_4}(s,t)=\sum_r R_{c,l_1l_2l_3l_4}{}^{(r)}W^{(r)}(t,s), \qquad (3.2)$$

where $R_c$ is obtained from $R$ by applying the crossing matrix. If we resolve $R_c$ into multiplets and components corresponding to the incoming state as a whole, it too must only contain multiplets corresponding to the external particles.

Adding (3.1) and (3.2), we find that

$$A_{l_1l_2l_3l_4}=\sum_r R_{+,l_1l_2l_3l_4}{}^{(r)}W_+{}^{(r)}(s,t)+\sum_r R_{-,l_1l_2l_3l_4}{}^{(r)}W_-{}^{(r)}(s,t)\,, \quad (3.3a)$$

where

$$R_\pm{}^{(r)}=R^{(r)}\pm R_c{}^{(r)}\,, \quad (3.3b)$$

$$W_\pm{}^{(r)}(s,t)=\tfrac{1}{2}\{W^{(r)}(s,t)+W^{(r)}(t,s)\}\,. \quad (3.3c)$$

The matrices $R_+{}^{(r)}$ and $R_-{}^{(r)}$ will be eigenvectors of the crossing matrix, with eigenvalues 1 and $-1$, respectively. We shall show later in this section that there exists no suitable eigenvector of the crossing matrix with eigenvalue $-1$. If we anticipate this result, we may drop the second term of (3.3a). On doing so and concentrating on the leading term, we obtain the final result

$$A_{l_1l_2l_3l_4}=R_{+l_1l_2l_3l_4}W(s,t)+\text{nonleading terms}\,, \quad (3.4)$$

where $R_+$ is an eigenvector of the crossing matrix with eigenvalue 1, and $W$ is the leading Veneziano term.

The matrix $R_+$ must satisfy a number of properties which restrict it fairly severely. It must be decomposable into a sum of factors as follows:

$$R_{+l_1l_2l_3l_4}=\sum_{l_5} V_{l_1l_2l_5}V_{l_5l_3l_4}\,, \quad (3.5)$$

where $l_5$ runs over the same multiplets and components of multiplets as the indices $l_1, l_2, l_3, l_4$ associated with the external particles. Furthermore, $V$ must be symmetric in the three indices $l_3$, $l_4$, $l_5$. When $l_3$, $l_4$, $l_5$ represent different components of the same multiplet, the symmetry follows from $SU(3)$ invariance and Bose statistics in all interesting cases, but the symmetry is an additional requirement when $l_3$, $l_4$, and $l_5$ represent different multiplets. Finally, $V$ must be an eigenvector of the crossing matrix with eigenvalue 1. We shall find that these requirements are too strong to allow any solution, but we shall be able to solve the problem by adding an exchange-degenerate trajectory.

Our algebraic problem is in fact identical to the problem encountered by Capps,[4] who investigated the dispersion relation at $u=0$ and made similar bootstrap assumptions to ours. Since the reasoning will be crucial in the following sections, we shall rederive his result. We shall first explain how a solution may be found, and shall then discuss its uniqueness.

The solution will be based on a mathematical quark model. To each meson we associate a quark index and an antiquark index. Thus, if the quarks form an $SU(3)$ triplet, the mesons form a nonet, which can be resolved into an octet and a singlet in the usual way. The indices are not intended to correspond to real, physical quarks. Each meson will be represented by a nine-component vector $f$, which possesses the quark and antiquark indices. We adopt the following convention:

Outgoing particles: The quark index is a superscript $\alpha_i$, the antiquark index a subscript $\beta_i$.

Incoming particles: The quark index is a subscript $\beta_i$, the antiquark index a superscript $\alpha_i$.

The $f$'s associated with the four particles will thus be written $f_{\beta_i}{}^{\alpha_i}$, $i=1, \cdots, 4$. We obtain $SU(3)$-invariant amplitudes by contracting upper and lower indices.

We now examine the following possible forms for the matrix $R$:

$$\delta_{\alpha_1}{}^{\beta_2}\delta_{\alpha_2}{}^{\beta_4}\delta_{\alpha_4}{}^{\beta_3}\delta_{\alpha_3}{}^{\beta_1}\,, \quad (3.6a)$$

$$\delta_{\alpha_1}{}^{\beta_3}\delta_{\alpha_3}{}^{\beta_4}\delta_{\alpha_4}{}^{\beta_2}\delta_{\alpha_2}{}^{\beta_1}\,. \quad (3.6b)$$

Both these forms will be $SU(3)$ invariant, since they are obtained by contracting upper and lower indices. They can both be represented in the form (3.5) in either the $s$ or the $t$ channel. In the $s$ channel of (3.6a), for instance, one quark index $(\alpha_2,\beta_4)$ and one antiquark index $(\alpha_3,\beta_1)$ are contracted between the incoming and outgoing states, so that the intermediate-state summation will again be over a nonet of mesons. Similar arguments apply to the $t$ channel of (3.6a) and to both channels of (3.6b). The factors $V$ do not yet satisfy the symmetry conditions. The crossing matrix interchanges the particles 2 and 3, and therefore interchanges (3.6a) and (3.6b). If we take the sum of (3.6a) and (3.6b), we obtain an eigenfunction of the crossing matrix with eigenvalue 1. Thus

$$R_{+\alpha_1\alpha_2\alpha_3\alpha_4}{}^{\beta_1\beta_2\beta_3\beta_4}=\delta_{\alpha_1}{}^{\beta_2}\delta_{\alpha_2}{}^{\beta_4}\delta_{\alpha_4}{}^{\beta_3}\delta_{\alpha_3}{}^{\beta_1}+\delta_{\alpha_1}{}^{\beta_3}\delta_{\alpha_3}{}^{\beta_4}\delta_{\alpha_4}{}^{\beta_2}\delta_{\alpha_2}{}^{\beta_4}\,. \quad (3.7)$$

Now let us investigate the factorization of (3.7) in the form (3.5). It is not difficult to see that

$$R_+=\sum_{\alpha_5,\beta_5}(V_{+\alpha_1\alpha_2\alpha_5}{}^{\beta_1\beta_2\beta_5}V_{+\beta_5\alpha_4\alpha_3}{}^{\alpha_5\beta_4\beta_3}+V_{-\alpha_1\alpha_2\alpha_5}{}^{\beta_1\beta_2\beta_5}V_{-\beta_5\alpha_4\alpha_3}{}^{\alpha_5\beta_4\beta_3})$$

$$\equiv R_{+,1}+R_{+,2}\,, \quad (3.8a)$$

where

$$V_{\pm\alpha_1\alpha_2\alpha_5}{}^{\beta_1\beta_2\beta_5}=\tfrac{1}{2}(\delta_{\alpha_1}{}^{\beta_2}\delta_{\alpha_2}{}^{\beta_5}\delta_{\alpha_5}{}^{\beta_1}\pm\delta_{\alpha_1}{}^{\beta_5}\delta_{\alpha_5}{}^{\beta_2}\delta_{\alpha_2}{}^{\beta_1})\,. \quad (3.8b)$$

The vertex function $V_+$ is symmetric in the particles 1, 2, and 3. It is, therefore, the appropriate vertex for three-meson coupling, and the intermediate states in the summation of the first term of (3.8a) are identical to the external particles.

The second term of (3.8a) can be interpreted as a summation over the intermediate-state particles on the exchange partner of the trajectory under consideration. It is immediately obvious that the first term of (3.8a) is symmetric under interchange of the final-state particles 1 and 2, so that it has $D$-type octet coupling and scalar coupling, whereas the second term is antisymmetric and has $F$-type octet coupling. It then follows from Bose statistics that the two terms couple to intermediate states with even and odd angular momentum respectively. Alternatively, we may write Eq. (3.4) with an $st$ and a $tu$ term as follows:

$$A_{\alpha_1\alpha_2\alpha_3\alpha_4}{}^{\beta_1\beta_2\beta_3\beta_4}(s,t)=R_{+\alpha_1\alpha_2\alpha_3\alpha_4}{}^{\beta_1\beta_2\beta_3\beta_4}W(s,t) +R_{+\alpha_1\alpha_2\alpha_4\alpha_3}{}^{\beta_1\beta_2\beta_4\beta_3}W(s,u)\,. \quad (3.9a)$$

If we decompose the $R$'s according to (3.8a) and make use of their symmetry properties under interchange of the final-state particles, the equation becomes

$$A(s,t)=R_{+,1}[W(s,t)+W(s,u)] +R_{+,2}[W(s,t)-W(s,u)]\,, \quad (3.9b)$$

and we observe that the matrices $R_1$ and $R_2$ are associated with positive- and negative-signature trajectories, respectively.

Our conclusion is thus that we can solve our problem provided we have a leading trajectory whose lowest member is identical to the external particle, together with the exchange partner of the trajectory. Exchange degeneracy is required in order to satisfy our bootstrap assumptions.

We now return to examine the uniqueness of the solution (3.7). If we assume that the external particles, and therefore the internal particles, have no higher multiplets than octets, we can easily see that the only possible $SU(3)$-invariant forms for $R$ are given by (3.6). All other possibilities fail to satisfy at least one of our requirements. It is of course possible to contract the indices in combinations different from (3.8a) or (3.8b), but most combinations involve a contraction of more than two indices between the initial and final states of either the $s$ channel or the $t$ channel. The intermediate particles in that channel would then consist of 27's as well as octets and singlets. The combinations in which two indices of the same external particle are contracted do not possess this defect but, if the matrix $R$ is then represented in the form (3.5), it is found that the vertex function $V$ is not symmetric in the three particles.

We might try and construct a solution where the external particles consist of higher multiplets than octets. One obvious way of constructing such a solution would be to start with a higher $SU(3)$ multiplet of quarks. However, if we started with an $n$-fold multiplet, we would obtain a solution invariant under $SU(n)$, whereas one of our assumptions was that the symmetry group is $SU(3)$. We conjecture that no other solutions exist, but we have no proof of this fact.

We could have solved our problem by working directly with the octet-octet crossing matrix given by de Swart.[14] The matrix has one eigenvector with eigenvalue 1 which involves only singlets and octets, namely,

$$5\,\mathbf{8}_{DD}+9\,\mathbf{8}_{FF}+16\,\mathbf{1}\,. \quad (3.10)$$

[14] J. J. de Swart, Nuovo Cimento 31, 420 (1964).

By adding external singlets, we could then satisfy all our factorization conditions. We have preferred the direct method, first because it illustrates the connection with the quark model, and second because it is applicable, with modifications, to the case where the quarks have spin. However, we shall use (3.10) in order to calculate the ratio between the vertex functions for various components of the octets and singlets, since the direct use of (3.8) would be rather cumbersome. The ratio between the vertices for the different components of the individual octets is given by the appropriate $SU(3)$ Clebsch-Gordan coefficient. Equation (3.10) then implies that one must associate over-all factors of $\sqrt{5}$, 3, and 4 with the vertices $(V_{888})_D$, $(V_{888})_F$, and $V_{881}$. All components of the vertex $V_{881}$ will be equal to $\sqrt{2}$, since all the Clebsch-Gordan coefficients are equal to $1/\sqrt{8}$. Finally, by considering the crossing relation between the processes $\mathbf{8}+\mathbf{8}\rightarrow\mathbf{1}+\mathbf{1}$ and $\mathbf{8}+\mathbf{1}\rightarrow\mathbf{8}+\mathbf{1}$, we conclude that the vertex $V_{111}$ is also equal to $\sqrt{2}$.

We shall now verify the assertion that there is no suitable eigenvector of the crossing matrix with eigenvalue $-1$. Since we have observed that the matrices which fulfill our requirements must be linear combinations of (3.6), the only possibility would be to replace the $+$ sign in (3.7) by a minus sign. If one did so, one would obtain a matrix which changes a state symmetric in the two particles to a state antisymmetric in the particles. Such a matrix is not allowed by Bose statistics and angular-momentum conservation. It corresponds to the following eigenvector of de Swart's matrix:

$$\mathbf{8}_{DF}+\mathbf{8}_{FD}\,. \quad (3.11)$$

The result which we have obtained is not particularly enlightening as regards the spectrum of particles. We have found that a system with octets necessarily possesses singlets as well. This is the prediction of the quark model, but it has also been predicted on the basis of many other models whose assumptions have some overlap in physical content with ours. It is of interest to note that we can obtain a solution to the Veneziano bootstrap model on this basis, provided we include the exchange partners of the main trajectories. When we apply our method to particles with spin we shall obtain more interesting results, and it is to this problem we now turn.

## 4. SOLUTION FOR MESONS WITH SPIN

### A. Kinematics and Elementary Solutions

In the present section we shall show that the reasoning which we used for the unitary-spin problem may be modified to treat the problem of ordinary spin. In order to simplify the writing, we shall begin by treating ordinary spin in the absence of unitary spin, though the combination of both spins poses no essentially new problems.

We shall work in terms of $M$-functions rather than scattering amplitudes, since the $M$-functions have a much simpler behavior under crossing.[15] We associate with each particle a quark index and an antiquark index, which we denote by $a_i$ and $b_i$; each index can take the values 1 and 2. We adopt the same convention with regard to upper and lower indices as in the previous section. The indices transform as the undotted indices of the conventional two-spinor notation.[16] Thus, if a lower index transforms according to a representation $(\frac{1}{2},0)$ of the Lorentz group, the upper index will transform according to an equivalent representation where all transformation matrix are replaced by their reciprocals. This implies that the two types of indices must be subjected to inverse boosts when converting the $M$-function to the scattering amplitude.

We now construct the possible forms of the matrix $R$. Lorentz-invariance allows two contractions analogous to the $\delta$-function contractions of the previous section, namely,

$$\delta_{a_i}{}^{b_j} \tag{4.1a}$$

and

$$p_{1,a_i\mathbf{c}}p_2{}^{\mathbf{c}b_j}, \tag{4.1b}$$

where

$$p_{a\mathbf{c}}=p_0\delta_{a\mathbf{c}}+(\boldsymbol{\sigma}\mathbf{p})_{a\mathbf{c}}, \tag{4.1c}$$

$$p^{\mathbf{c}b_j}=p_0\delta_{\mathbf{c}b_j}-(\boldsymbol{\sigma}\mathbf{p})_{\mathbf{c}b_j}, \tag{4.1d}$$

and $p_1$ and $p_2$ represent any linear combination of the momenta. For reasons of printing we have denoted a dotted index by a bold-face subscript or superscript. In the expression (4.1b) and subsequent similar expressions, we shall define the momenta with a minus sign if the subscript refers to an antiquark; the expressions are then unchanged under crossing. The requirement that the intermediate trajectories be no more complicated than the initial particles than restricts (4.1) to the factors

$$\delta_{a_i}{}^{b_j} \tag{4.2a}$$

and

$$p_{i,a_i\mathbf{c}}p_j{}^{\mathbf{c}b_j}, \tag{4.2b}$$

where the subscripts $i$ and $j$ on $p$ indicate that the momentum $p_{a_i\mathbf{c}}$, which carries a subscript relating to the $i$th particle, must be the momentum of the $i$th particle itself. If a matrix $R$ with any of the more general factors (4.1) is constructed, intermediate particles with spin greater than 1 exist in at least one of the two channels, because it always turns out that more than two spinor indices are contracted between the initial- and final-state variables $f_b{}^a$ and $p_{a\mathbf{c}}$.

We notice that repeated factors of the form of (4.1b) combine as follows:

$$p_{i,a_j\mathbf{c}}p_j{}^{\mathbf{c}b_j}p_{j_jb_j\mathbf{d}}p_k{}^{\mathbf{d}ek}=-p_j^2p_{i,a_i\mathbf{c}}p_k{}^{\mathbf{c}ek}. \tag{4.3}$$

Let us attempt to construct a matrix $R$ according to the models of Sec. 3. We might try the following expression, which is analogous to (3.6a):

$$\delta_{a_1}{}^{b_2}\delta_{a_2}{}^{b_4}\delta_{a_4}{}^{b_3}\delta_{a_3}{}^{b_1}+\delta_{a_1}{}^{b_3}\delta_{a_3}{}^{b_4}\delta_{a_4}{}^{b_2}\delta_{a_2}{}^{b_1}. \tag{4.4}$$

The amplitude would then factorize as before into terms with the expected intermediate trajectory and its exchange partner. Unfortunately the expression (4.4) is not invariant under reflection and is therefore unsatisfactory.

Another possibility is the following:

$$\mu^{-8}p_{1,a_1\mathbf{e}}p_2{}^{\mathbf{e}b_2}p_{2,a_2\mathbf{f}}p_4{}^{\mathbf{f}b_4}p_{4,a_4\mathbf{g}}p_3{}^{\mathbf{g}b_3}p_{3,a_3\mathbf{h}}p_1{}^{\mathbf{h}b_1} +12431\rightarrow 13421. \tag{4.5}$$

Using (4.3), we can again represent the amplitude in a form similar to (3.8), with the $\delta$ functions replaced by the appropriate factors (4.2b). Equation (4.5) is also not invariant under reflection. The sum of (4.4) and (4.5) is invariant under reflection, but it no longer factorized into contributions from the expected trajectory and its exchange partner. If an attempt is made to express the matrix in the form (3.8), it is found that the intermediate trajectories become parity doubled.

The conclusion just reached is not surprising. A solution to our problem gives an amplitude with the correct analytic properties at all values of $s$ and $t$, including $t=0$, and the trajectories therefore have an $M$ quantum number. Unless $M=0$, the trajectories must be doubled in a parity-conserving system. What we have found is that it is impossible to solve our problem with $M=0$ trajectories alone.

If we allow a doubling of trajectories (in addition to the exchange doubling), the sum of (4.4) and (4.5) does provide a solution to our problem. We shall point our very shortly that it is not an acceptable solution, however.

### B. Doubling of Quark States

To investigate further the solution just obtained, it is convenient to regard (4.4) and (4.5) as the matrices $R$ for the scattering of two different types of mesons. By taking linear combinations of the meson states with coefficient $1/\sqrt{2}$, we obtain a meson of definite parity, and the $R$ matrix of its scattering amplitude is obtained by adding (4.4) and (4.5). The mesons which have the coupling (4.4) are composed of two quarks which form a representation $(\frac{1}{2},0)$ of the Lorentz group. The product of two such representations gives us the representations $(1,0)$ and $(0,0)$ which correspond to (axial-) vector mesons with $M=1$ and (pseudo-) scalar mesons with $M=0$. We shall show below that the value of $M$ obtained in this way is the correct assignment for the Toller quantum number of the leading trajectories. In the coupling (4.5), the operators $p_{a\mathbf{b}}$ change the undotted to dotted indices, and the quarks form a representation $(0,\frac{1}{2})$. The (axial-) vector mesons with this

[15] I should like to thank Kuo-Hsiang Wang for pointing out to me the usefulness of $M$-functions in this problem.

[16] A brief discussion of two-spinor notation will be found in the paper by H. P. Stapp, Phys. Rev. **125**, 2139 (1962).

coupling thus have $M=-1$, and mesons with fixed parity can be obtained by taking the sum and difference of the states with $M=\pm 1$.

Since the $\rho$ trajectory is known to have $M=0$, the scheme we have outlined evidently does not correspond to nature. Also, it turns out that the scheme fails to satisfy our assumption 3. In order to obtain vector mesons with $M=0$, we must combine the representations $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ of the Lorentz group. We are thus led to consider a model in which the $(\frac{1}{2},0)$ quarks can combine with themselves, the $(0,\frac{1}{2})$ quarks can combine with themselves, and the $(\frac{1}{2},0)$ quarks can combine with the $(0,\frac{1}{2})$ quarks. We thereby obtain a further doubling of trajectories. It will be found, however, that the total contribution of the repulsive trajectories in the new scheme is not greater than the contribution of the single repulsive trajectories in the scheme which we have just discarded.

The matrix $R$ for the scattering of any four particles will again be obtained by contracting the indices in the order 12431, and in the order 13421. Two indices associated with a $(\frac{1}{2},0)$ quark are contracted by a matrix such as $\delta_{a_1}{}^{b_2}$, two indices associated with a $(0,\frac{1}{2})$ quark by a matrix such as $p_{1,a_1e}p_2{}^{eb_2}$, while an index associated with a $(\frac{1}{2},0)$ quark cannot be contracted with an index associated with a $(0,\frac{1}{2})$ quark.

It is convenient to extend our notation so as to treat both kinds of quarks symmetrically. The state vectors $f_{b_id_i}{}^{a_ic_i}$ in spin space will have two quark indices $a_i$ and $c_i$ (or $b_i$ and $d_i$) and two antiquark indices $b_i$ and $d_i$. The indices $a_i$ and $b_i$ specify the spin states as before, while the indices $b_i$ and $d_i$ take the value 1 for a $(\frac{1}{2},0)$ quark and $-1$ for a $(0,\frac{1}{2})$ quark. The coupling will then be given by the expression

$$(-1)^{-(d_2+c_4-2)/2}\delta_{a_1}{}^{b_2}\delta_{c_1}{}^{d_2}\delta_{a_2}{}^{b_4}\delta_{c_2}{}^{d_4}\delta_{a_4}{}^{b_3}\delta_{c_4}{}^{d_3}\delta_{a_3}{}^{b_1}\delta_{c_3}{}^{d_1} + (-1)^{(d_1+c_3-2)/2}\delta_{a_1}{}^{b_3}\delta_{c_1}{}^{d_3}\delta_{a_3}{}^{b_4}\delta_{c_3}{}^{d_4} \times\delta_{a_4}{}^{b_2}\delta_{c_4}{}^{d_2}\delta_{a_2}{}^{b_1}\delta_{c_2}{}^{d_1}. \quad (4.6)$$

As before, the boosts applied to the outgoing particles when converting the $M$-function to the scattering amplitude will be the inverse of the boosts applied to the incoming particles. Also, the boosts applied to the states with $c$ or $d$ equal to 1 will be the inverse of those applied to the states with $c$ or $d$ equal to $-1$. By reversing the boosts in this way, we can replace the factors $p_{1,a_1e}p_2{}^{eb_2}$ in (4.5) by $\delta$ functions, as we have done in (4.6). The phase factors in front of the two terms of (4.6) result from our definition of the sign of $p$ in expressions such as (4.1).

The $R$ matrix (4.6) can be written in a form similar to (3.8). The intermediate-state summation will be over all indices $a$, $b$, $c$, and $d$. Thus, in addition to summing over all spin states of the (axial-) vector and (pseudo-) scalar Reggeons and their exchange partners, we have to sum over the four Reggeons obtained by allowing the two indices $c$ and $d$ to take their two values. The vertex functions will be given by the equations

$$V_{\pm a_1c_1a_2c_2a_5c_5}{}^{b_1d_1b_2d_2b_5d_5} =\tfrac{1}{2}[(-1)^{(d_2-1)/2}\delta_{a_1}{}^{b_2}\delta_{c_1}{}^{d_2}\delta_{a_2}{}^{b_5}\delta_{c_2}{}^{d_5}\delta_{a_5}{}^{b_1}\delta_{c_5}{}^{d_1} \pm(-1)^{(d_1-1)/2}\delta_{a_1}{}^{b_5}\delta_{c_1}{}^{d_5}\delta_{a_5}{}^{b_2}\delta_{c_5}{}^{d_2}\delta_{a_2}{}^{b_1}\delta_{c_2}{}^{d_1}]. \quad (4.7)$$

We now convert the $M$-function vertex (4.7) into a scattering-amplitude vertex and, at the same time, we shall change our indices $a$ and $b$ into helicity subscripts. The indices $c$ and $d$ remain unchanged, but we shall write them all as subscripts. Each of the quarks associated with the incoming particles 1 and 2 will be subjected to a boost $e^{-c\lambda\xi}$, $e^{-c\mu\xi}$, $e^{-d\lambda\xi}$, or $e^{-d\mu\xi}$, where

$$\cosh\xi=s^{1/2}/2\mu. \quad (4.8)$$

Also, contraction of helicity indices between the two outgoing particles one and two gives rise to an extra factor $-i(-1)^{\lambda}$. We shall define the helicity of the intermediate Reggeon 5 to be its spin in the direction of motion of particle one, so that contractions between particles two and five will involve a Kronecker delta $\delta_{-\lambda,\mu}$. Thus[17]

$$\mathbf{V}_{\pm,\lambda_1\mu_1c_1d_1,\lambda_2\mu_2c_2d_2,\lambda_5\mu_5c_5d_5}=-\tfrac{1}{2}ie^{-(c_1\lambda_1+d_1\mu_1+c_2\lambda_2+d_2\mu_2)\xi} \times[(-1)^{d_2\mu_2}\delta_{\lambda_1\mu_2}\delta_{c_1d_2}\delta_{\lambda_2,-\mu_5}\delta_{c_2d_5}\delta_{\lambda_5\mu_1}\delta_{c_5d_1} \pm(-1)^{d_1\mu_1}\delta_{\lambda_1\mu_5}\delta_{c_1d_5}\delta_{-\lambda_5,\mu_2}\delta_{c_5d_2}\delta_{\lambda_2\mu_1}\delta_{c_2d_1}]. \quad (4.9)$$

### C. Characteristics of the Trajectories

Let us now discuss more fully the four sets of Reggeons obtained by letting the subscripts $c$ and $d$ take the values 1 and $-1$. As we remarked in the introduction, we shall characterize all trajectories by means of their lowest particle. We shall only concern ourselves with those trajectories which choose sense at the mass $\mu$, where the leading trajectory in the Veneziano function $W$ passes through $l=0$. The spin, parity, and charge conjugation of the trajectory will, therefore, be given completely by its spin degrees of freedom. The spin and orbital angular momentum of the vector trajectories will combine with one another to give two subsidiary trajectories in addition to the leading trajectory, but the subsidiary trajectories choose nonsense at $s=\mu^2$ and do not interest us as the moment.

We first notice that the two state vectors $\delta_{c1}\delta_{d1}+\delta_{c,-1}\delta_{d,-1}$ and $\delta_{c1}\delta_{d,-1}+\delta_{c,-1}\delta_{d1}$ have negative intrinsic parity, while the state vectors $\delta_{c1}\delta_{d1}-\delta_{c,-1}\delta_{d,-1}$ and $\delta_{c1}\delta_{d,-1}-\delta_{c,-1}\delta_{d1}$ have positive intrinsic parity. The positive-parity combinations can be further separated on the basis of charge conjugation, since the state vectors $\delta_{c1}\delta_{d1}-\delta_{c,-1}\delta_{d,-1}$ and $\delta_{c1}\delta_{d,-1}-\delta_{c,-1}\delta_{d1}$ have $C=(-1)^j$ and $-(-1)^j$, respectively.

We are more interested in the two negative-parity combinations, which both have $C=(-1)^j$ and cannot be separated from one another by selection rules. If the

[17] The vertex (4.9) is still regarded as a vertex in the spin degrees of freedom alone. We therefore do not include the factor $[(2l+1)/(2j+1)]^{1/2}$.

trajectories were really degenerate, we could define the individual Reggeons by taking any two orthogonal linear combinations of the vectors $\delta_{c1}\delta_{d1}+\delta_{c,-1}\delta_{d,-1}$ and $\delta_{c1}\delta_{d,-1}+\delta_{c,-1}\delta_{d1}$. We do not expect the trajectories to be exactly degenerate, however, and we therefore do have to decide which linear contributions correspond to the individual Reggeons. For this purpose we shall make use of the assumption 3. There are two vector Reggeon states $f_{\lambda\mu}{}^{v}(\delta_{c1}\delta_{d1}+\delta_{c,-1}\delta_{d,-1})$ and $f_{\lambda\mu}{}^{v}(\delta_{c1}\delta_{d,-1}+\delta_{c,-1}\delta_{d1})$, and two pseudoscalar Reggeon states $f_{\lambda\mu}{}^{s}(\delta_{c1}\delta_{d1}+\delta_{c,-1}\delta_{d,-1})$ and $f_{\lambda\mu}{}^{s}(\delta_{c1}\delta_{d,-1}+\delta_{c,-1}\delta_{d1})$, where $f^v$ and $f^s$ are the usual nonrelativistic state vectors associated with vector and scalar states. We wish to take orthogonal linear combinations corresponding to the trajectories $V$ and $\Pi$ on which the $\rho$ and $\pi$ nonets lie, and two further trajectories $V'$ and $\Pi'$.

We can obtain some restrictions on the basis of the $M$-values. The state vectors $f_{\lambda\mu}{}^{v(s)}(\delta_{c1}\delta_{d1}\pm\delta_{c,-1}\delta_{d,-1})$ are obtained by combining two $(\frac{1}{2},0)$ representations or two $(0,\frac{1}{2})$ representations. The resultant representations will be (1,0) or (0,1) and (0,0), which correspond to (axial) vectors with $|M|=1$ and (pseudo) scalars with $M=0$. The state vectors $f_{\lambda\mu}{}^{v(s)}(\delta_{c1}\delta_{d,-1}\pm\delta_{c,-1}\delta_{d1})$ are obtained by combining a $(\frac{1}{2},0)$ representation with a $(0,\frac{1}{2})$ representation, and will consist of a vector and axial vector with $M=0$ and their daughters, a scalar and pseudoscalar with $M=0$.

If we anticipate the result that our $M$-value corresponds to the Toller quantum numbers $M$, we conclude that it is a good quantum number when $s$, the energy of the Reggeon, is zero. It will not be a good quantum number at any other value of $s$. The linear combinations which we require will thus be the vectors $f_{\lambda\mu}{}^{v(s)}(\delta_{c1}\delta_{d1}+\delta_{c,-1}\delta_{d,-1})$ and $f_{\lambda\mu}{}^{v(s)}(\delta_{c1}\delta_{d,-1}+\delta_{c,-1}\delta_{d1})$ themselves at $s=0$. Furthermore, the $V$ trajectory should have $M=0$ at $s=0$, and the $\Pi$ should not be the daughter of another trajectory. Thus, at $s=0$, the state vectors of the $V$ and $\Pi$ will be $f_{\lambda\mu}{}^{v}(\delta_{c1}\delta_{d,-1}+\delta_{c,-1}\delta_{d1})$ and $f_{\lambda\mu}{}^{s}(\delta_{c1}\delta_{d1}+\delta_{c,-1}\delta_{d,-1})$. We may accordingly write

$$f^{V}{}_{\lambda\mu,cd}=2^{-1/2}[(s/\mu^2)+\lambda_V{}^2]^{-1/2}f_{\lambda\mu}{}^{v}\times[(s^{1/2}/\mu)(\delta_{c1}\delta_{d1}+\delta_{c,-1}\delta_{d,-1})+\lambda_V(\delta_{c1}\delta_{d,-1}+\delta_{c,-1}\delta_{d1})], \quad (4.10a)$$

$$f^{V'}{}_{\lambda\mu,cd}=2^{-1/2}[(s/\mu^2)+\lambda_V{}^2]^{-1/2}f_{\lambda\mu}{}^{v}\times[\lambda_V(\delta_{c1}\delta_{d1}+\delta_{c,-1}\delta_{d,-1})-(s^{1/2}/\mu)\times(\delta_{c1}\delta_{d,-1}+\delta_{c,-1}\delta_{d1})], \quad (4.10b)$$

$$f^{\Pi}{}_{\lambda\mu,cd}=2^{-1/2}[(s/\mu^2)+\lambda_\Pi{}^2]^{-1/2}f_{\lambda\mu}{}^{s}\times[(s^{1/2}/\mu)(\delta_{c1}\delta_{d,-1}+\delta_{c,-1}\delta_{d1})+\lambda_\Pi(\delta_{c1}\delta_{d1}+\delta_{c,-1}\delta_{d,-1})], \quad (4.10c)$$

$$f^{\Pi'}{}_{\lambda\mu,cd}=2^{-1/2}[(s/\mu^2)+\lambda_\Pi{}^2]^{-1/2}f_{\lambda\mu}{}^{s}\times[\lambda_\Pi(\delta_{c1}\delta_{d,-1}+\delta_{c,-1}\delta_{d1})-(s^{1/2}/\mu)\times(\delta_{c1}\delta_{d1}+\delta_{c,-1}\delta_{d,-1})], \quad (4.10d)$$

where $\lambda_V$ and $\lambda_\Pi$ are constants to be determined below.

We employ factors of $s^{1/2}$ rather than $s$ in (4.10), since it is known that the vertices associated with the Toller quantum numbers in question differ by factors of $s^{1/2}$. We shall see that the definitions (4.10) do lead to vertex functions with the correct analytic properties at $s=0$.

The singularity in the expressions on the right of (4.10) at $s=-\lambda^2\mu^2$ does not represent a singularity of the amplitude, since, in our model, it corresponds to an arbitrary definition of the individual vector Reggeons. When the degeneracy between the trajectories is broken, the complete amplitude will remain analytic at $s=-\lambda^2\mu^2$ provided that the trajectories intersect at this point. The individual vertex functions are then allowed to have singularities of the form $[(s/\mu^2)+\lambda^2]^{-1/2}$. We could have multiplied the two terms in the expressions on the right of (4.10) by polynomials in $s$. The trajectories would then have intersected in more than one point.

To find the vertex function $v$ between our external particles and internal trajectories, we must sandwich the expression (4.9) between the appropriate vectors $f_{\lambda_1\mu_1,c_1d_1}$, $f_{\lambda_2\mu_2,c_2d_2}$, and $f_{\lambda_5\mu_5,c_5d_5}$. The external particles 1 and 2 will be vectors and pseudoscalars, and the state vectors $f^V$ and $f^\Pi$ must be taken at the value $s=\mu^2$ corresponding to the un-Reggeized particle. For the internal particle 5 we are interested in any Reggeon at any value of $s$. By using this prescription it is not difficult to see that the choice $\lambda_V=\lambda_\Pi=1$ in (4.10) gives us the $SU(6)_W$ vertex.[18] For, if the internal vertex $f_5$ is left arbitrary, while $f_1$ and $f_2$ are expressed by (4.10c) with $s=\mu^2$, $\lambda_V=\lambda_\Pi=1$, we obtain the following vertex:

$$v^{V(\Pi),V(\Pi),X}=\tfrac{1}{4}if_{\lambda_1\mu_1}{}^{v(s)}f_{\lambda_2\mu_2}{}^{v(s)}f^{X}{}_{\lambda_5\mu_5,c_5d_5}\times[(-1)^{\mu_2}\sinh(2\mu_2\xi)e^{-(c_5\lambda_5-d_5\mu_5)\xi}\delta_{\lambda_1\mu_2}\delta_{\lambda_2,-\mu_5}\delta_{\lambda_5,\mu_1}+(-1)^{\mu_1}\sinh(2\mu_1\xi)e^{(c_5\lambda_5-d_5\mu_5)}\delta_{\lambda_2\mu_1}\delta_{\lambda_1\mu_5}\delta_{-\lambda_5,\mu_2}]\times(\delta_{d_51}+\delta_{d_5,-1})(\delta_{c_51}+\delta_{c_5,-1}). \quad (4.11)$$

The superscript $X$ refers to any of the Reggeons. We may compare (4.11) with the nonrelativistic vertex:

$$v^{v(s),v(s),v(s)}=-\tfrac{1}{2}if_{\lambda_1\mu_1}{}^{v(s)}f_{\lambda_2\mu_2}{}^{v(s)}f_{\lambda_5\mu_5}{}^{v(s)}\times[(-1)^{\mu_2}\delta_{\lambda_1\mu_2}\delta_{\lambda_2,-\mu_5}\delta_{\lambda_5\mu_1}+(-1)^{\mu_1}\delta_{\lambda_2\mu_1}\delta_{\lambda_1\mu_5}\delta_{-\lambda_5,\mu_2}]. \quad (4.12)$$

In its dependence on the helicity subscripts for the particles one and two, the expression on the right of (4.11) differs from that on the right of (4.12) only by the factor $\sinh(2\mu\xi)$, which may be written $-i(-1)^\mu\sinh\xi$. The extra factor $(-1)^\mu$ is just the extra factor required by $SU(6)_W$.

If one further uses (4.10a) and (4.10c) for $f_5$ in (4.11),

[18] Strictly speaking, we should speak of the $SU(2)_W$ vertex rather than the $SU(6)_W$ vertex, since we have no unitary spin in the problem at this point. However, we feel that it will be less confusing to the reader if we use the usual terminology.

and sets $\lambda_V=\lambda_\Pi=1$, $s=\mu^2$, one obtains the result

$$v^{V(\Pi),V(\Pi),V(\Pi)}(\mu)$$
$$=\tfrac{1}{2}f_{\lambda_1\mu_1}{}^{v(s)}f_{\lambda_2\mu_2}{}^{v(s)}f_{\lambda_5\mu_5}{}^{v(s)}\cosh(\lambda_5\xi)\cosh(\mu_5\xi)$$
$$\times[(-1)^{\mu_2}\sinh(2\mu_2\xi)\delta_{\lambda_1\mu_2}\delta_{\lambda_2,-\mu_5}\delta_{\lambda_5\mu_1}$$
$$+(-1)^{\mu_1}\sinh(2\mu_1\xi)\delta_{\lambda_2\mu_1}\delta_{\lambda_1\mu_5}\delta_{-\lambda_5,\mu_2}]$$
$$=\tfrac{1}{2}(3\sqrt{3}/8)f_{\lambda_1\mu_1}{}^{v(s)}f_{\lambda_2\mu_2}{}^{v(s)}f_{\lambda_5\mu_5}{}^{v(s)}$$
$$\times(\delta_{\lambda_1\mu_2}\delta_{\lambda_2,-\mu_5}\delta_{\lambda_5\mu_1}+\delta_{\lambda_2\mu_1}\delta_{\lambda_1\mu_5}\delta_{-\lambda_5\mu_2}),\quad(4.13)$$

since $\cosh\xi=\frac{1}{2}$ when $s=\mu^2$. Again, the expression (4.13) is just the $SU(6)_W$ vertex.

We can now collect together our results for the state vectors of the different Reggeons. We shall use the symbols $A$ and $S'$ to denote the axial vector and scalar trajectories with $C=-(-1)^j$, and $B$ and $S$ to denote the axial-vector and scalar trajectories with $C=(-1)^j$. Then

$$f^V{}_{\lambda\mu,cd}=2^{-1/2}\{(s/\mu^2)+1\}^{-1/2}f_{\lambda\mu}{}^v$$
$$\times[(s^{1/2}/\mu)(\delta_{c1}\delta_{d1}+\delta_{c,-1}\delta_{d,-1})$$
$$+(\delta_{c1}\delta_{d,-1}+\delta_{c,-1}\delta_{d1})],\quad(4.14\text{a})$$

$$f^{V'}{}_{\lambda\mu,cd}=2^{-1/2}\{(s/\mu^2)+1\}^{-1/2}f_{\lambda\mu}{}^v[(\delta_{c1}\delta_{d1}+\delta_{c,-1}\delta_{d,-1})$$
$$-(s^{1/2}/\mu)(\delta_{c1}\delta_{d,-1}+\delta_{c,-1}\delta_{d1})],\quad(4.14\text{b})$$

$$f^{\Pi}{}_{\lambda\mu,cd}=2^{-1/2}\{(s/\mu^2)+1\}^{-1/2}f_{\lambda\mu}{}^s$$
$$\times[(s^{1/2}/\mu)(\delta_{c1}\delta_{d,-1}+\delta_{c,-1}\delta_{d1})$$
$$+(\delta_{c1}\delta_{d1}+\delta_{c,-1}\delta_{d,-1})],\quad(4.14\text{c})$$

$$f^{\Pi'}{}_{\lambda\mu,cd}=2^{-1/2}\{(s/\mu^2)+1\}^{-1/2}f_{\lambda\mu}{}^s[(\delta_{c1}\delta_{d,-1}+\delta_{c,-1}\delta_{d1})$$
$$-(s^{1/2}/\mu)(\delta_{c1}\delta_{d1}+\delta_{c,-1}\delta_{d,-1})],\quad(4.14\text{d})$$

$$f^{A(S')}{}_{\lambda\mu,cd}=(1/\sqrt{2})f_{\lambda\mu}{}^{v(s)}(\delta_{c1}\delta_{d,-1}-\delta_{c,-1}\delta_{d1}),\quad(4.14\text{e})$$

$$f^{B(S)}{}_{\lambda\mu,cd}=(1/\sqrt{2})f_{\lambda\mu}{}^{v(s)}(\delta_{c1}\delta_{d1}-\delta_{c,-1}\delta_{d,-1}).\quad(4.14\text{f})$$

Our use of the symbols $A$ and $B$ is not meant to imply that the trajectories correspond to the physical $A_1$ and $B$. They do have the same quantum numbers, but our present $A_1$ and $B$ "particles" are the $l=0$ members of a repulsive trajectory, whereas the physical $A_1$ and $B$ are the $l=1$ members of an attractive trajectory.

It is now a straightforward matter to substitute (4.14) in (4.11) and thereby to obtain the vertex functions for two external $V$ or $\Pi$ particles and any Reggeon. Since the vertex (4.11) differs from the nonrelativistic vertex (4.12) only by the factors $(-1)^{\mu_2}$ or $(-1)^{\mu_1}$ and by factors depending on $\lambda_5$ and $\mu_5$, it is most convenient to express the result in terms of the nonrelativistic vertex. One can take into account the factor $(-1)^{\mu_2}$ or $(-1)^{\mu_1}$ by making the usual $SU(6)_W$ transformation:

$$V_1\leftrightarrow V_1,\quad(4.15\text{a})$$

$$V_0\leftrightarrow \Pi,\quad(4.15\text{b})$$

$$V_{-1}\leftrightarrow -V_{-1},\quad(4.15\text{c})$$

where the subscript denotes the helicity. On examining (4.14) and (4.11), we find that we have to make the transformation (4.15) for the initial particles 1 and 2, and for the intermediate Reggeon 5 associated with the trajectories $V$, $\Pi$ and $V'$, $\Pi'$. No transformation is necessary for the trajectories $A$, $S'$, $B$, and $S$. In addition we obtain the following factors, which depends only on the intermediate Reggeon:

$$V_1 \text{ or } \Pi:\ \sqrt{2}[(s/\mu^2)+1]^{-1/2}[(s^{1/2}/\mu)=\cosh\xi]\sinh\xi,$$
$$V_1' \text{ or } \Pi':\ \sqrt{2}[(s/\mu^2)+1]^{-1/2}[1-(s^{1/2}/\mu)\cosh\xi]\sinh\xi,$$
$$V_0:\ \sqrt{2}[(s/\mu^2)+1]^{-1/2}[(s^{1/2}/\mu)\cosh\xi+1]\sinh\xi,$$
$$V_0':\ \sqrt{2}[(s/\mu^2)+1]^{-1/2}[\cosh\xi-(s^{1/2}/\mu)]\sinh\xi,$$
$$A_1:\ i\sqrt{2}\sinh^2\xi,\quad A_0 \text{ or } S':\ 0,$$
$$B_1:\ 0,\quad B_0 \text{ or } S:\ i\sqrt{2}\sinh^2\xi.\quad(4.16)$$

The factors (4.16) must be applied after the transformation (4.15).

The phase factors $i$ in front of two of the expressions (4.16) require explanation, since they do not arise by combining (4.14) with (4.11). If we had simply combined these two equations, we would have obtained a vertex function which is not invariant under time reversal, since outgoing and incoming quarks require inverse boosts. One could work with such a vertex if one wished, and the theory as a whole would remain invariant under time reversal. Nevertheless, it is more convenient to work with vertex functions which are invariant under time reversal. Since those expressions in (4.16) which are odd in $\xi$ are also odd under time reversal, we can remove the apparent noninvariance by multiplying all such odd vertices by $i$; the incoming vertices would be multiplied by $-i$. In addition, we have multiplied all the expressions quoted in (4.16) by a factor $i$. Such an operation is permissible, as it merely alters the sign of all scattering amplitudes and will not affect the validity of the crossing relation. We have chosen to apply the extra factor $i$ in order that the $V$ and $\Pi$ trajectories be attractive, the $A$, $B$, and $S$ trajectories repulsive.

The nonrelativistic vertex (4.12) can be treated in the same way as the corresponding $SU(3)$ vertex, or it can very easily be treated directly. If the coefficients of $v^{vvv}$ are represented by the appropriate Clebsch-Gordan coefficients it turns out that the components of the vertex $v^{vvs}$, as well as the vertex $v^{sss}$, will be equal to $1/\sqrt{2}$. In addition, one must apply an extra factor $[(2l+1)/(2j+1)]^{1/2}$ to the vertices $v^{vvv}$ and $v^{svv}$. In this case the factor will be $\{[2\alpha(s)-1]/[2\alpha(s)+1]\}^{1/2}$. All the vertices must finally be multiplied by the orbital vertex in the Veneziano formula. At this point, and only at this point, we must make use of the full content of our assumption that the nonleading terms are unimportant.

### D. Toller Quantum Number

It remains to justify the assertion that our Lorentz-group quantum number $M$ is identical to the Toller quantum number of the leading trajectory at $s=0$. We begin with the simplified case in which the trajectories of the Veneziano function $W$ pass through $\alpha=0$ at $s=0$, so that there is no orbital angular momentum. We shall work on the basis of the treatment of Freeman and Wang.[19]

Let us examine the first term of (4.9) for particular values of the $c$'s and $d$'s where the Kronecker deltas do not vanish, and let us isolate the term where the particles 1 and 2 have channel spin equal to $S$, while particle 5 has angular momentum $J$ and helicity $\nu$. The two quarks of helicity $\lambda_1$ and $\mu_2$ have already been combined into a spinless particle and may be ingored. To obtain the term in question we have to multiply by the Clebsch-Gordan coefficients $C(\frac{1}{2},\frac{1}{2},S,\mu_1,\lambda_2)$ and $C(\frac{1}{2},\frac{1}{2},J,\lambda_5,\mu_5)$ by the Kronecker delta $\delta_{\nu,\lambda_5+\mu_5}$ and by the phase factor $(-1)^{\lambda_2}$. The last factor is necessary in order to take into account the Freedman-Wang definition of the channel spin. We then have to sum over the helicities. The term is thus equal to

$$\tfrac{1}{2}i\sum_{\lambda_5} C(\tfrac{1}{2},\tfrac{1}{2},S,\lambda_5,\ \nu-\lambda_5)C(\tfrac{1}{2},\tfrac{1}{2},J,\lambda_5,\ \nu-\lambda_5) \times e^{i[c_5\lambda_5-d_5(\nu-\lambda_5)-2(\nu-\lambda_5)]\pi/2}, \quad (4.17)$$

since $\xi=-i\pi/2$ when $s=0$. If $c_5$ and $d_5$ are both equal to 1, the expression (4.18) reduces to $e^{i\nu\pi/2}\delta_{JS}$, which is the same as the Freedman-Wang function $d_{JS\nu}{}^{*(J,J)}(\pi/2)$. If on the other hand $c_5=-1$ and $d_5=1$, the expression (4.18) becomes $d_{JS\nu}{}^{*(1,0)}(\pi/2)$. In either case our result is equal to the Freedman-Wang vertex function which corresponds to the Toller quantum numbers we have quoted.

We can easily extend our results to the case where the orbital trajectories do not pass through $\alpha=0$ at $s=0$. The above argument still applies to the spin degrees of freedom. We then have to combine the representations (1,0), (0,1), (0,0), or $(\frac{1}{2},\frac{1}{2})$ with the orbital representation $(l/2,l/2)$. The $M$-value of the leading trajectory remains unchanged by the combination, and our assertion is proved.

The value of $S$, the Freedman-Wang channel spin, can easily be found for all our trajectories. Combining the results with those already found for $M$, we obtain the following assignments:

$V$: $M=0$, $S=0$;

$A$: $M=0$, $S=1$, $n=0$; $\Pi'$: $M=0$, $S=1$, $n=1$;

$V'$ and $B$: $M=1$;

$\Pi$: $M=0$, $S=1$;

$S$: $M=0$, $S=0$.

Apart from the trajectory $\Pi'$, which is the daughter of the $A$, all our trajectories have the daughter number $n$ equal to zero. Note that the trajectory $S'$, which would have been the daughter of the $V$, has a residue which vanishes identically.

[19] D. Z. Freedman and J. M. Wang, Phys. Rev. **160**, 1560 (1967).

## 5. ORDINARY SPIN AND UNITARY SPIN

The dynamics of the system with unitary spin and ordinary spin is no more complicated than the dynamics of the system with ordinary spin alone. We can repeat the reasoning of the previous section line for line, with the difference that the indices $a$ and $b$, or $\lambda$ and $\mu$, run over the six states of the quark instead of over the two spin states.

Our trajectories will now become nonets in unitary space. The prescription for finding the vertex functions will again be to use the nonrelativistic vertex, make the transformation (4.15), and apply the factors (4.16). The nonrelativistic vertex will have the form (4.12), with the indices $\lambda$ and $\mu$ running over all six values. It is thus an $SU(6)$ vertex. All our trajectories except the (pseudo-) scalar $SU(3)$ singlets are to be regarded as parts of 35-dimensional representations of $SU(3)$, while the (pseudo-) scalar $SU(3)$ singlets will be $SU(6)$ singlets.

Within the 35-dimensional representation, we can determine the ratio between the vertices associated with the different spin and $SU(3)$ multiplets by using the $SU(6)$ Clebsch-Gordan coefficients.[20] We can then determine the ratios between the different components of the spin or $SU(3)$ multiplets by using the Clebsch-Gordan coefficients for $SU(2)$ and $SU(3)$. Corresponding to Eq. (3.10), the eigenvector of the $\mathbf{35}\times\mathbf{35}$ crossing matrix is

$$16\ \mathbf{35}_{DD}+18\ \mathbf{35}_{FF}+35\ \mathbf{1}. \quad (5.1)$$

Thus, if over-all factors of 4 and $3\sqrt{2}$ are associated with the vertices $(V_{35,35,35})_D$ and $(V_{35,35,35})_F$, the components of the vertex $V_{35,35,1}$, as well as the vertex $V_{111}$, will be equal to unity. The vertices associated with intermediate vector trajectories will also have a factor $\{[2\alpha(s)-1]/[2\alpha(s)+1]\}^{1/2}$, and all vertices will be multiplied by the orbital Veneziano vertex.

It is important to notice that the transformation (4.15) must be made after working out the nonrelativistic vertex, but before applying the factors (4.16). These factors will of course modify the nonrelativistic $SU(6)$ predictions, except at $s=\mu^2$. The transformation (4.15) indicates that the vertex function at $s=\mu^2$ is given by the predictions of $SU(6)_W$.

As in Sec. 3, the $\mathbf{35}_D$ and the singlet representations are to be combined with trajectories of even orbital signature, the $\mathbf{35}_F$ with trajectories of odd orbital signature. The intrinsic spin will reverse the signature of the vector and axial-vector trajectories.

As in the model without unitary spin, the particles of the system will lie on the $V$ and $\Pi$ trajectories of both signature. The $\mathbf{35}_D$ and singlet trajectories have their lowest particles at the point where the leading orbital Veneziano trajectory passes through $l=0$, the $\mathbf{35}_F$ at the point where the leading orbital trajectory

[20] C. L. Cook and G. Murtaza, Nuovo Cimento **39**, 531 (1965).

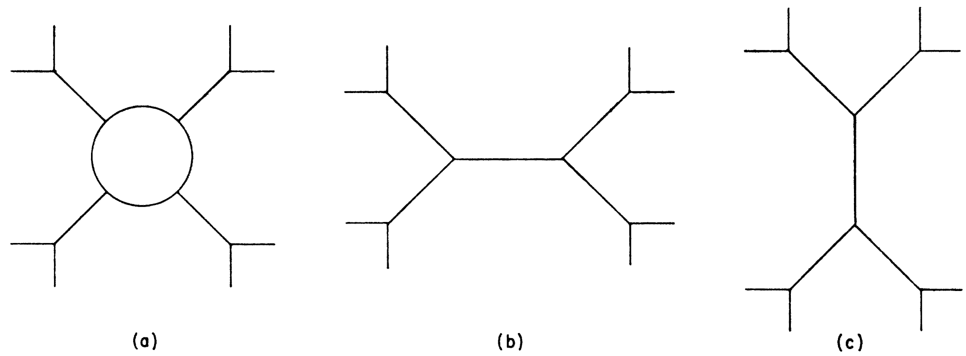


FIG. 2. Factorization of the eight-point Veneziano amplitude.

passes through $l=1$. The orbital angular momentum in our model corresponds to that of the nonrelativistic quark model.

## 6. EXTERNAL MESONS WITH ANY SPIN

We now wish to combine our model with the Bardakci-Halpern model[3] in order to obtain a consistent bootstrap scheme where any particle on the leading trajectory may also play the role of an external particle. Bardakci and Halpern propose to take the higher members of a trajectory as external particles by examining the eight-point amplitude. They then isolate the four-particle pole term shown in Fig. 2(a). From this amplitude one can isolate further pole terms, as in Fig. 2(b) and 2(c). If all the poles are on leading trajectories, Bardakci and Halpern point out that the internal vertices are symmetric in their three particles. It follows that the pole term shown in Fig. 2(a) represents a consistent solution of the two-to-two bootstrap problem, where any particle on the leading internal trajectory may be an external particle.

It is now evident that one may combine the two models by assigning quarklike spin and unitary-spin indices to the external particles of Fig. 2. To determine the manner in which the indices are combined, we recall the fact that the contractions in Eq. (3.7) are in the order 12431 and 13421. The particular $R$ matrix (3.7) is to be used in conjunction with the $st$ Veneziano term and, if the box diagram for that term is drawn as a planar diagram, the order of the external particles is again 12431 and 13421. Thus, if a diagram such as Fig. 3(a) is drawn with no lines crossed, the individual lines should simultaneously represent the particles in the box diagram and the $\delta$-function contraction of the quark indices.

Such rules can easily be generalized to the eight-point diagram. The external particles of Fig. 3(b) and (c) are to be drawn in the order corresponding to the multiparticle Veneziano term under consideration.[2] The individual lines must also represent the $\delta$-function contractions of the quark indices. No lines are allowed to cross. Each multiparticle Veneziano diagram gives rise to two terms, where the lines go round in a clockwise or an anticlockwise direction, and the sum of all diagrams is to be taken.

A graphical representation similar to Fig. 2 has been used by Harari[8] to treat the unitary-spin problem in the quark model.

The four-point amplitude is obtained by projecting out the pole term as before. If the four poles all correspond to the lowest members of the trajectory, the projection of the orbital Veneziano terms gives us our original four-point amplitude. Thus, when the spin and unitary-spin factors are included, we obtain precisely the amplitudes calculated in Secs. 3–5.

As in the simpler bootstrap problem, the multiparticle Veneziano amplitude is not unique. We can add nonleading terms whose coefficients cannot be determined unless we bootstrap the lower trajectories. Again our philosophy will be to neglect such terms in the first approximation, though results which depend sensitively on their magnitude should not be regarded as accurate.

Diagrams such as Figs. 2 and 3 provide a pictorial representation of the role of quarks in the Veneziano bootstrap model. A meson can be regarded as composed

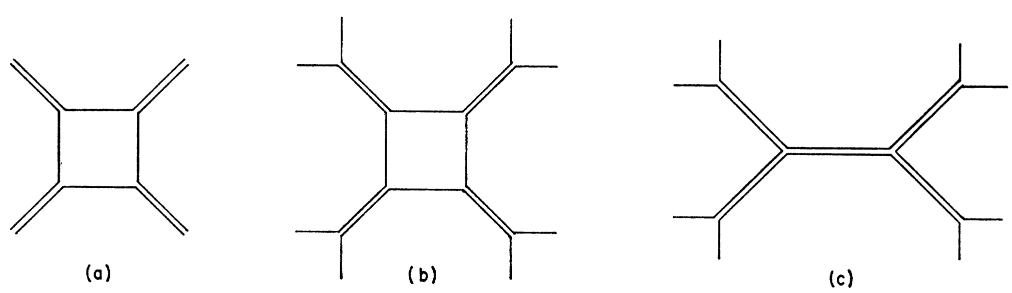


FIG. 3. Contraction of quark indices in scattering amplitudes.

of two quarks which have space, spin and unitary-spin degrees of freedom, and the meson scattering amplitudes may be constructed from the quark scattering amplitudes. Nevertheless, the quarks need not exist as real particles.

## 7. CONCLUDING REMARKS

The trajectories discussed in the previous two sections do have the properties which were outlined in the introduction and in Sec. 2. When they pass through $l=0$ there will be a single multiplet of 36 particles which we identify with the $\rho$ and $\pi$ nonets. For higher values of $l$ the spin and orbital angular momentum of the vector Reggeon can combine to produce three values of $j$, and the number of multiplets will be correspondingly increased. The nonets associated with the $\Pi$ and the three $V$ trajectories at $l=1$ are those which contain the $B$, $A_2$, $A_1$, and $\pi_N$ particles.

The positive-parity trajectories $A$, $B$, and $S$ are repulsive, as may be seen from the phase factors in (4.16). No particles are associated with these trajectories. The attractive trajectories with the physical $A_1$, $B$, and $\pi_N$ particles have the same quantum numbers as the leading repulsive trajectories, but they are one unit lower, and the particles in question are $l=1$ quark-model states.

The trajectories $V'$ and $\Pi'$ are attractive. Examination of (4.16) and (4.8) shows that the vertex functions for the $V'$ and $\Pi'$ trajectories are $\frac{1}{3}$ as large as those for the $V$ and $\Pi$ trajectories at $s=\mu^2$, so that the residues will be $\frac{1}{9}$ as large. We have already treated the interpretation of the trajectories in Secs. 1 and 2. Any particles on the trajectories would duplicate the quantum numbers of those on the $V$ and $\Pi$ trajectories, though the trajectories themselves have different values of $M$. The temptation to associate one member of the $A_2$ doublet with the $V'$ trajectory should perhaps be resisted, as one would expect some splitting between the $V$ and $V'$.

The spectrum of particles in our theory corresponds to representations of $SU(6)$ or, more accurately, of $SU(6)\times O(3)$. The lowest multiplet is a degenerate **35+1**. The spectrum of trajectories corresponds to representations of $SU(12)\times O(3)$, and the leading trajectories form a degenerate **143+1**. As long as we are only classifying the spectrum, we need not distinguish between $SU(12)$ and $SU(6,6)$. In view of our treatment of the repulsive trajectories, $SU(12)$ or $SU(6,6)$ is in no sense a symmetry of the system. The vertex functions are those predicted by the lower symmetry $SU(6)_W$, but, since $SU(6)_W$ has no meaning except for collinear processes, we cannot say that $SU(6)_W$ is a symmetry of the system either.

In Eq. (4.16) we arbitrarily multiplied the positive-parity vertex functions by the phase factor $i$, whereas we could equally well have applied the factor $i$ to the negative-parity vertex functions. The axial-vector and scalar trajectories would then have been attractive, the vector and pseudoscalar trajectories repulsive. One can repeat our calculations in such a model, but the positive-parity wave functions now have to be associated with the external particles. If one calculates the vertex functions at $s=\mu^2$, one finds that the vertices associated with the negative-parity intermediate particles are typically $\sqrt{3}$ times as large as those associated with the positive-parity intermediate particles. The factor $\sqrt{3}$ comes from the ratio $\sinh\xi/\cosh\xi$. Thus, the residues associated with the repulsive trajectories are three times as large as those associated with the attractive trajectories, whereas in our model they are $\frac{2}{3}$ as large. For reasons explained in Sec. 2, our model is thus to be preferred.

The choice of $\lambda_\Pi$ and $\lambda_V$ equal to 1, where $\lambda_\Pi$ and $\lambda_V$ are the parameters in Eq. (4.10), has the effect that none of our repulsive residues dominates over the attractive residues. This is not generally true for all values of $\lambda$. To take an example, we may examine the scheme to which we referred briefly in Sec. 4, where the nonrelativistic quark-model states are parity-doubled but where there is no further doubling. That model is obtained by taking $\lambda_V=0$, $\lambda_\Pi=\infty$ in (4.10). We then find that the scattering of the pseudoscalar particles takes place entirely through the repulsive trajectories. The net effect of the repulsive trajectories and the $V'$ and $\Pi'$ trajectories relative to that of the $V$ and $\Pi$ trajectories is not less than in our model, in spite of the extra doubling of trajectories which we have introduced. We have not examined the complete range of $\lambda_V$ and $\lambda_\Pi$, but we suspect that the two parameters must be roughly equal to 1 if the repulsive trajectories are not to dominate in any subchannel.

The main results of our model are of course those concerning the qualitative nature of the spectrum of mesons and trajectories. Another result is that all coupling constants involving three mesons are given in terms of one overall constant. The ratios between the vertex functions are those given by $SU(6)_W$, and they have been discussed in the papers dealing with that symmetry.[21] Most of the experimental predictions of $SU(6)_W$ involve the baryon as well as the meson coupling constants, but one prediction which involves the meson coupling constants alone is the relation

$$g_{\omega\rho\pi}^2=(4/m_\rho^2)g_{\rho\pi\pi}^2, \tag{7.1}$$

where $g_{\omega\rho\pi}$ and $g_{\rho\pi\pi}$ are defined as in Ref. 21. If the constant $g_{\rho\pi\pi}^2$ is taken from $\rho$ decay, the predicted value of $g_{\omega\rho\pi}^2$ is about one-half to three-quarters of that given by the Gell-Mann–Sharp–Wagner or peripheral models.

One might also attempt to fit high-energy data with the Regge trajectories suggested by the present model. Since all residues are predicted in the present paper and in a paper we hope to write on the baryon system,

[21] See, for instance, B. Sakita and K. C. Wali, Phys. Rev. **139**, B1355 (1965).

the fits could in principle be made without introducing any new parameters. In the real world one would expect some breaking of the degeneracy between the trajectories and, since the predictions depend sensitively on the positions of the trajectories, it may be necessary to take the $\alpha$'s as phenomenological parameters. It would probably not make much difference whether one obtained the dependence of the Regge residues on $t$ from the present paper or simply used a function with the correct singularities, and whatever difference there is may depend on our assumption regarding non-leading terms. Our model also predicts the ratio between the residues for different trajectories, however.

High-energy photoproduction experiments provide some evidence for two of our extra trajectories. If it is assumed that the presently available energies are asymptotic, and that the amplitude is dominated by Regge poles rather than Regge cuts, the experiments lead to the unambiguous result that a pair of trajectories forming an $M=1$ conspiracy pass through $t=0$ near $\alpha=0$. No such trajectories are predicted by the nonrelativistic quark model, but our $V'$-$B$ pair has the necessary properties. The trajectories would have to be displaced about half a unit below the $\rho$ trajectory. Such a displacement is not unreasonable in view of the observed splittings of $SU(6)$ multiplets, and, in fact, the trajectory would be displaced by roughly the same amount as the $\eta'$.

The $SU(6)$ results of our theory must of course be broken in higher approximations. An interesting question is whether $SU(6)$ properties will continue to exist as long as we use the narrow-resonance approximation, or whether the symmetry will break down when a sufficiently large number of trajectories is included. Once we go beyond the narrow-resonance approximation we certainly cannot have $SU(6)$ symmetry.

The role of the pion as a Goldstone boson is not manifest in our model, and cannot be exhibited in any $SU(6)$-invariant model, owing to the contradiction between the two requirements $m_\pi=m_\rho$ and $m_\pi=0$.

One can treat the baryon system by methods very similar to those of the present paper. The lowest particles on the leading trajectory form a **56** representation of $SU(6)$, whereas the trajectories form a **364** representation of $SU(12)$. The exchange partner has particles which form a **70** representation of $SU(6)$, and the trajectories form a **572** representation of $SU(12)$. With an assumption similar to assumption 3 of the present paper, the vertex functions are those given by $SU(6)_W$, and all $MMM$ and $BBM$ coupling constants are determined to within a scale factor. The intercept of the baryon trajectories cannot be determined except by applying the generalized Veneziano supplementary condition. It is hoped to present the model in a later paper.

*Note added in proof*: K. Bardakci and the present author have shown that the factorization properties of the multi-particle Veneziano amplitude can be extended to nonleading Regge trajectories. The degeneracy is greater if we adopt a Veneziano amplitude with nonleading terms. We may therefore drop our assumption regarding nonleading terms and, instead, we ask for that solution with the minimum number of lower trajectories. Our solution will be subject to the usual ambiguity regarding the spin and multiplicity of quarks, but otherwise the solution we have given is *exact* (apart from the fundamental approximation of narrow resonances) and *unique*.

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