Use of Nonsignatured Partial-Wave Amplitudes in Regge-Pole Theory*

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It is shown that a Froissart-Gribov-type partial-wave amplitude without signature can be defined for both two-body and multiparticle processes. Such an amplitude permits a Sommerfeld-Watson transformation, and its poles govern asymptotic behavior in the usual way, although unitarity properties of the amplitude are complicated. The relation to Toller amplitudes is discussed.

I. INTRODUCTION

HERE, we show how to define a Froissart-Gribov-type partial-wave amplitude that may be continued into the complex l plane without introducing signature. The crucial point of our argument is the observation that while nonsignatured partial-wave amplitudes do not in general satisfy Carlson's theorem, they can nonetheless be used in a Sommerfeld-Watson transform with their poles controlling the asymptotic behavior.

For the sake of simplicity and so that the basic idea of the method will be clear, we condiser the example of elastic scattering of equal-mass, spinless particles. We shall also explicitly indicate how the results may be generalized to multiparticle amplitudes, because it is desirable to have a definition of the analytically continued partial-wave amplitude for application (e.g., in the multiperipheral model of production processes), where details of the cut structure of the amplitude are unknown.[1]

Specifically, we examine the region of convergence in the $z=\cos\theta$ plane of the background integral which appears in the Sommerfeld-Watson transform of the scattering amplitude $A(s,z)=A(z)$. For this purpose it is necessary to know the asymptotic behavior of the partial-wave amplitude in the right-half complex l plane, including the case in which Iml becomes large in either direction while Rel remains fixed. We find that the partial-wave amplitude without signature $a(l)$ is Carlsonian[2] only when the left-hand side singularities of $A(z)$ are displaced, however slightly, from the real axis. Otherwise, the bound $a(l)<e^{\pi|l|}$, Re$l\geqslant$ const fails. Nevertheless, it is possible to find a line in the z plane that intersects the Lehmann ellipse in which the Sommerfeld-Watson background integral for the nonsignatured amplitude converges, even if all singularities are on the real axis so that the partial-wave amplitude without signature no longer satisfies the conditions of Carlson's theorem. Singularities in the nonsignatured partial-wave amplitude therefore determine the asymptotic behavior of the amplitude in this region of the z plane in the usual way.

Since the right-hand side of the partial-wave unitarity relation contains, roughly speaking, the square of the amplitude, its high-l behavior will not, in general, be such as to admit a Sommerfeld-Watson transformation. Thus, the unitarity properties of the nonsignatured amplitudes are, at least, more obscure than for the signatured amplitudes.

In order to illustrate the method of analysis, we begin in Sec. II by studying the partial-wave amplitude and the Sommerfeld-Watson transformation of a function $A(z)$ with singularities in the finite z plane not on the real axis. Each singularity determines a hyperbola with focus at $+1$ or -1, which passes through the singularity. The basic result is that the background integral which arises in the Sommerfeld-Watson transformation of

$$A(z)=\sum_{l=0}^{\infty}(2l+1)a_lP_l(z) \tag{1a}$$

converges to the left of the hyperbola through the leftmost singularity. By transforming the partial-wave expansion of $A(-z)=B(z)$,

$$B(z)=\sum_{l=0}^{\infty}(2l+1)b_lP_l(z)\,, \tag{1b}$$

we find a representation of $A(z)$ which converges to the right of the hyperbola through the rightmost singularity.

The generalization to the case of an amplitude with singularities at arbitrarily large z and with singularities on the real axis is easily made. In particular for a realistic amplitude with branch points on the entire z axis excluding the physical region $-1\leqslant z\leqslant 1$, we find one representation of $A(z)$ that converges for $z\leqslant -1$ on the real axis and another representation that converges for $z\geqslant +1$ on the real axis.

Having established regions of convergence of the background integrals of nonsignatured partial-wave amplitudes, we show in Sec. III how Regge asymptotic behavior in z appears when the background integral is displaced to the left. It is interesting to note that the background integral that converges for $z\geqslant 1$ on the real axis is closely related to the Toller expansion, which

* Work supported in part through funds provided by the U. S. Atomic Energy Commission under Contract No. At (30-1) 2098.
[1] J. B. Hartle and C. E. Jones, Phys. Rev. 184, 1564 (1969).
[2] E. C. Titchmarsh, *Theory of Functions* (Oxford University Press, New York, 1939), 2nd ed., p. 186.

is usually obtained[3] by expanding directly in representations of $O(2,1)$ in which case $z \geqslant 1$ throughout the analysis. Boyce[4] has derived the $O(3)$ partial-wave expansion from the Toller expansion by an inverse Sommerfeld-Watson transformation, using two different partial-wave amplitudes that treat right and left z-plane singularities separately and which, therefore, correspond to the introduction of signature. We state the precise relation between the Toller amplitude and the nonsignatured Froissart-Gribov partial-wave amplitude in Sec. III.

Finally, in Sec. IV we indicate how the analysis outlined above is to be generalized to provide a definition of multiparticle partial-wave amplitudes, whose poles govern the asymptotic behavior of, for example, production amplitudes.

II. PARTIAL-WAVE AMPLITUDE AT COMPLEX l

The partial-wave amplitude in Eq. (1a) is defined as usual by

$$a_l = \frac{1}{2}\int_{-1}^{1} dz\, A(z) P_l(z) = \frac{1}{2\pi i}\int_C dz\, Q_l(z) A(z)\,, \quad (2)$$

where C is a simple closed contour which encloses $z=+1$ and $z=-1$ in the positive sense in such a way that all singularities of $A(z)$ lie outside the contour. We assume that asymptotically $A(z) \lesssim z^x$, so that for $\mathrm{Re}l > x$ the contour C may be deformed into the contour C' shown in Fig. 1. All singularities of $A(z)$ have been displaced from the real axis, and we can write $C' = \sum_i C_i$. Note that each C_i may enclose several singularities and that the directions in which they go to infinity are arbitrary, all directions being equivalent. For $\mathrm{Re}l > x$,

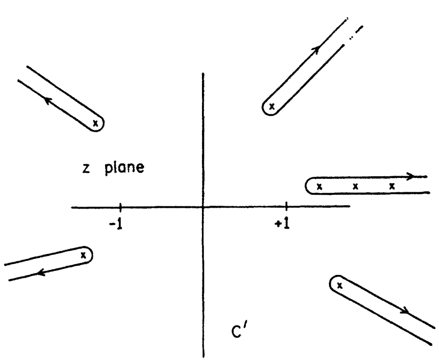

FIG. 1. Contour C' used to define $a(l)$ in Eq. (3). Crosses indicate the location of singularities of the amplitude $A(z)$.

[3] M. Toller, Nuovo Cimento 37, 631 (1965).
[4] J. F. Boyce, J. Math. Phys. 8, 675 (1967).

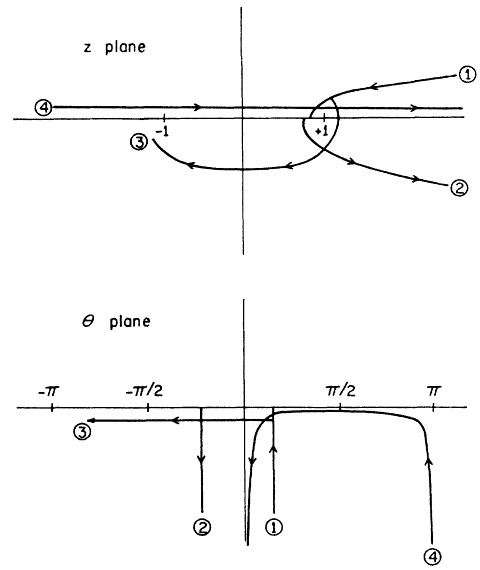

FIG. 2. Mapping between the $z = \cos\theta$ plane and the θ plane for z on the principal branch of $Q_l(z)$. Corresponding paths are labeled by the same number in both planes. Hyperbolas with foci at ± 1 in the z plane map into vertical lines in the θ plane. Ellipses with foci at ± 1 in the z plane map into horizontal lines in the θ plane. Arrows indicate path direction.

we define the analytic continuation of a_l by

$$a(l) = \frac{1}{2\pi i}\int_{C'} dz\, Q_l(z) A(z)$$
$$= \frac{1}{\pi}\sum_i \int_{z_i}^{\infty} dz\, Q_l(z) D_i(z) = \sum_i a_i(l)\,, \quad (3)$$

where $2iD_i(z)$ is the discontinuity across the ith cut in $A(z)$. Similar expressions define the analytic continuation $b(l)$ of b_l in terms of $B(z) = A(-z)$.

In order to perform the Sommerfeld-Watson transformation on the expansions in Eqs. (1), we need to know the asymptotic behavior of $a(l)$ and $b(l)$ for large l. This is obtained by inserting the asymptotic form of $Q_l(z)$ into Eq. (3). For large[5] l ($|\arg l| \neq \pi$)

$$Q_l(z) \sim \frac{e^{-i\lambda\theta}}{\lambda^{1/2}}\left(\frac{\pi}{2i\sin\theta}\right)^{1/2}$$
$$= (\tfrac{1}{2}\pi)^{1/2} \frac{\exp\{-\lambda \ln[z + (z^2-1)^{1/2}]\}}{\lambda^{1/2}(z^2-1)^{1/4}}\,, \quad (4)$$

where $z = \cos\theta$, $\lambda = l + \frac{1}{2}$, and the positive sign of the square root is taken when z is real and greater than one. (The last proviso means that on the principal sheet of $Q_l(z)$, $\theta_I \leqslant 0$, and $|\theta_R| \leqslant \pi$, where R and I denote the real and imaginary parts of subscripted variables.)

[5] *Higher Transcendental Functions*, edited by A. Erdélyi (McGraw-Hill Book Co., New York, 1953), Vol. 1, p. 132, Eqs. (37), p. 77, Eq. (16); or p. 157, Eq. (12).

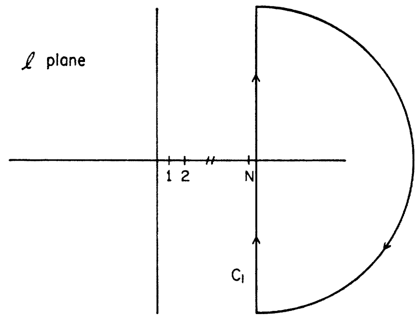

FIG. 3. Sommerfeld-Watson contour C_1 for Eq. (7). N is the largest integer less than x, where the scattering amplitude $A(z)\lesssim z^x$ for large z.

Thus,

$$a_i(l)\sim\frac{1}{(2\pi)^{1/2}}\int_{z_i}^{\infty}dz\frac{e^{-i\lambda\theta}}{\lambda^{1/2}}\frac{D_i(z)}{(z^2-1)^{1/4}}$$

$$=\frac{1}{(2\pi\lambda)^{1/2}}\int_{z_i}^{\infty}dz\,\exp[-i(\lambda_R\theta_R-\lambda_I\theta_I)+\lambda_I\theta_R+\lambda_R\theta_I]\frac{D_i(z)}{(z^2-1)^{1/4}}. \quad (5)$$

In order to extract the leading λ dependence from Eq. (5), it is useful to consult Fig. 2, which displays the relation between the θ plane and the $z=\cos\theta$ plane. We find that it is always possible to choose the contour from z_i to ∞ in such a way that both $\lambda_I\theta_R$ and $\lambda_R\theta_I$ decrease monotonically from their values at $\theta=\theta_i=\cos^{-1}z_i$. Along such a contour, the factor $\exp(\lambda_R\theta_I+\lambda_I\theta_R)$ damps the integrand relative to its value at $z=z_i$. The damping effect increases as $|\lambda|\to\infty$ in the right-half plane including the case $\lambda_I\to+\infty$ with λ_R fixed. Therefore, if $\lambda_R>\text{Re}x+\frac{1}{2}$ so that the integral converges, the leading λ behavior is determined entirely by the behavior of $D_i(z)$ in the neighborhood of $z=z_i$. In particular, if $D_i(z)\propto(z-z_i)^\alpha$ near $z=z_i$, we have from an examination of Eq. (5) that

$$a_i(l)\underset{|\lambda|\to\infty}{\sim}\frac{e^{-i\lambda\theta_i}}{\lambda^{3/2+\alpha}}. \quad (6)$$

The asymptotic form (6) holds for $|\lambda|\to\infty$ in the right-half λ plane, including a direction parallel to the imaginary axis. We see from (6) that when $\alpha=-1$, the case of a simple pole at $z=z_i$, we just find the asymptotic behavior to be that of $Q_l(z_i)$, as it must be for a simple pole.

We are now in a position to examine the validity of the Sommerfeld-Watson transformation procedure applied to the expansion in Eqs. (1). For definiteness we assume $D_i(z)\propto(z-z_i)^\alpha$ near $z=z_i$ with $\alpha_i>-1$. Corresponding to Eq. (1a), we have the equation

$$A(z)-\sum_{l=0}^{N}(2l+1)a_lP_l(z)=-\frac{1}{2i}\int_{C_1}dl\frac{(2l+1)a(l)P_l(-z)}{\sin\pi l}=-\frac{1}{2i}\sum_i\int_{C_1}dl\frac{(2l+1)a_i(l)P_l(-z)}{\sin\pi l}, \quad (7)$$

where the contour C_1 is shown in Fig. 3, and N is the largest integer less than $\text{Re}x$. The asymptotic form of $P_l(-z)$ is[6]

$$\frac{P_l(-z)}{\sin\pi l}\propto\frac{\{\cos[(\theta\mp\pi)\lambda]-\frac{1}{4}\pi\}}{\lambda^{1/2}\cos\pi\lambda}f(z),\quad \text{Im}z\gtrless 0. \quad (8)$$

By substituting the asymptotic forms (6) and (8) into Eq. (7), we conclude that the contribution to the integral from the contour at infinity vanishes when $|\theta_I|<|\theta_{mI}|$, i.e., inside the Lehman ellipse, provided the remaining background integral converges. ($z_m=\cos\theta_m$ is the location of that singularity which defines the smallest ellipse with foci at ±1.)

The background integral contours in Eq. (7) run parallel to the imaginary l axis from $l_0-i\infty$ to $l_0+i\infty$, where l_0 is real and greater than $\text{Re}x$. The convergence of these background integrals depends on the relative values of θ_{iR} and θ_R as follows: We find by using Eqs. (6) and (8) that the integral over $a_i(l)$ in Eq. (7) converges in the shaded region of Fig. 4 and on the boundary of the region for $\alpha>-1$ (except at $z=z_i$ when $-1<\alpha_i<0$). The representation (7) for $A(z)$ therefore converges to the left of the hyperbola, which passes through the singularity z_j which has the largest $|\theta_{iR}|$ of all the singularities of $A(z)$.

By a similar analysis, we can establish the region of convergence of the background integral obtained by

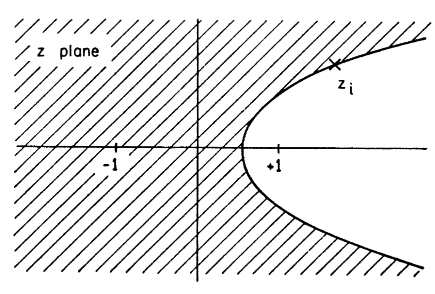

FIG. 4. Shaded region to the left of the hyperbola with focus at $+1$ that passes through z_i is the region of convergence of the background integral in Eq. (7) associated with the singularity of $A(z)$ at $z=z_i$. The expression (7) for $A(z)$ thus converges to the left of the leftmost of the hyperbolas defined by the singularities of $A(z)$.

[6] Reference 5, p. 162, Eq. (2). See also E. J. Squires, *Complex Angular Momenta and Particle Physics* (W. A. Benjamin, Inc., New York, 1964), p. 5.

transforming the expansion (1b), where $B(z)=A(-z)$, and

$$b(l)=\frac{1}{2\pi i}\sum_i \int_{\bar{C}_i} dz\, Q_l(z)B(z)\,. \tag{9}$$

The contours $\bar{C}_i$ look just like those shown in Fig. 1, except that a singularity of $A(z)$ at $z=z_i$ corresponds to a singularity of $B(z)$ at $z=-z_i$. The leftmost singularity of $B(z)$ is therefore the rightmost singularity of $A(z)$. The background integral for $B(z)$ thus converges to the left of the hyperbola through $-z_k$, where z_k is the location of the singularity of $A(z)$ with the smallest value of $|\theta_{kR}|$. Therefore,

$$A(z)=B(-z)=-\frac{1}{2i}\int_{l_0-i\infty}^{l_0+i\infty} dl\,\frac{(2l+1)b(l)P_l(z)}{\sin\pi l} +\sum_{l=0}^{N}(2l+1)a_lP_l(z) \tag{10}$$

converges to the right of the hyperbola through z_k and on the hyperbola for $\alpha>-1$ (except at $z=z_k$ when $-1<\alpha<0$).

The behavior of the background integral corresponding to a realistic amplitude $A(z)$ with singularities extending to infinity on the real z axis can be obtained as a limiting case of the preceding analysis. We find that

$$A(z)-\sum_{l=0}^{N}(2l+1)a_lP_l(z) =-\frac{1}{2i}\int_{l_0-i\infty}^{l_0+i\infty} dl\,(2l+1)\frac{a(l)P_l(-z)}{\sin\pi l} \tag{11a}$$

converges on the negative z axis with $z\leqslant -1$. The partial-wave amplitude in Eq. (11a) is given by

$$a(l)=\frac{1}{2\pi i}\int_{C'} dz\, Q_l(z)A(z) =\frac{1}{\pi}\int_{z_0}^{\infty} dz\, Q_l(z)D_R(z) +\frac{1}{\pi}\int_{-z_1}^{-\infty} dz\, Q_l(z)D_L(z)\,, \tag{11b}$$

where the contour C' is defined in Fig. 5, and D_R and D_L are right and left discontinuities of $A(z)$. On the other hand,

$$A(z)-\sum_{l=0}^{N}(2l+1)a_lP_l(z) =-\frac{1}{2i}\int_{l_0-i\infty}^{l_0+i\infty} dl\,(2l+1)\frac{b(l)}{\sin\pi l}P_l(z) \tag{12a}$$

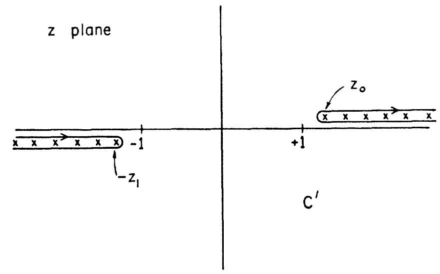

FIG. 5. Contour C' used to define $a(l)$ in Eq. (11b). Crosses indicate the location of singularities of the amplitude $A(z)$.

converges for z on the real axis with $z>1$. In Eq. (12a)

$$b(l)=\frac{1}{2\pi i}\int_{C''} dz\, Q_l(z)A(-z)\,, \tag{12b}$$

with C'' enclosing the singularities of $A(-z)$.

We note that the branch of the function represented by Eqs. (11) and (12) is determined by the way in which the singularities are moved off the real axis and by the corresponding definitions of C' and C''. In particular, it is not necessary to move all right-hand side singularities or all left-hand side singularities either all above or all below the real axis. Rather, each choice corresponds to a different branch of the function $A(z)$.

III. ASYMPTOTIC BEHAVIOR

We now wish to extract the consequences of the assumption that $a(l)$ and $b(l)$ are meromorphic in the complex l plane. It is sufficient to examine Eq. (12b), since the analysis is identical for Eq. (11a). The amplitude $b(l)$ has poles of residue γ_i at $l=\alpha_i$. Thus from from Eq. (12a), we obtain

$$A(z)=-\frac{1}{2i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dl(2l+1)\frac{b(l)}{\sin\pi l}P_l(z) -\pi\sum_i\frac{(2\alpha_i+1)\gamma_iP_{\alpha_i}(z)}{\sin\pi\alpha_i}\,, \tag{13}$$

where the summation extends only over α_i such that $\mathrm{Re}\alpha_i>-\frac{1}{2}$. As a result of the symmetry of the integrand about $\mathrm{Re}l=-\frac{1}{2}$,

$$A(z)=\frac{1}{2i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dl(2l+1)\,\frac{f(l)}{\tan\pi l}P_l(z) -\pi\sum_i\frac{(2\alpha_i+1)\gamma_iP_{\alpha_i}(z)}{\sin\pi\alpha_i}\,, \tag{14}$$

where

$$f(l)=-[b(l)-b(-l-1)]/2\cos\pi l=f(-l-1)\,. \tag{15}$$

Since[7]

$$P_l(z)/\tan\pi l=\pi^{-1}[Q_l(z)-Q_{-l-1}(z)]\,, \tag{16}$$

[7] Reference 5, p. 140, Eq. (8).

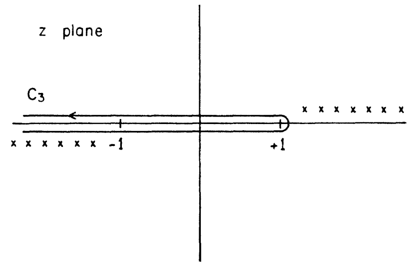

FIG. 6. Contour C_3 of Eq. (22). When $b(l)$ is expressed as a contour integral along C_3 its relation to the Toller amplitude $f(l)$ of Eq. (21) becomes particularly transparent.

we can again invoke the behavior of the integrand under reflection about $\mathrm{Re}\,l=-\frac{1}{2}$ to obtain

$$A(z)=-\frac{1}{\pi i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dl(2l+1)f(l)Q_{-l-1}(z) -\pi\sum_i \frac{(2\alpha_i+1)\gamma_i P_{\alpha_i}(z)}{\sin\pi\alpha_i}, \quad (17)$$

which is the appropriate form for moving the background contour to the left of $l=-\frac{1}{2}$. If we now assume that $b(l)$ has the Mandelstam symmetry[8] so that $b(\frac{1}{2}n)=b(-\frac{1}{2}n-1)$ for $n=-3, -5, \cdots$, then $f(l)$ has only poles of $b(l)$ and of $b(-l-1)$. In moving the contour of Eq. (17) to the left, we encounter poles of $b(l)$ of residue γ_j at $l=\alpha_j$ where $\mathrm{Re}\,\alpha_j<-\frac{1}{2}$, and poles of $b(-l-1)$ of residue $-\gamma_i$ at $l=-\alpha_i-1$ where $\mathrm{Re}\,\alpha_i>-\frac{1}{2}$. Thus,

$$A(z)=\sum_j \frac{(2\alpha_j+1)\gamma_j Q_{-\alpha_j-1}(z)}{\cos\pi\alpha_j} +\sum_i \frac{[2(-\alpha_i-1)+1]\gamma_i Q_{\alpha_i}(z)}{\cos\pi(-\alpha_i-1)} -\pi\sum_i \frac{(2\alpha_i+1)\gamma_i P_{\alpha_i}(z)}{\sin\pi\alpha_i} -\frac{1}{\pi i}\int_{-L-i\infty}^{-L+i\infty} dl\,(2l+1)f(l)Q_{-l-1}(z), \quad (18)$$

where the j summation extends over poles for which $-L<\mathrm{Re}\,\alpha_j<-\frac{1}{2}$ and $-L<-\mathrm{Re}\,x-1$. Combining the coefficients of $(2\alpha_i+1)$ for $\mathrm{Re}\,\alpha_i>-\frac{1}{2}$ with the help of Eq. (16), we then find that

$$A(z)=\sum_{\mathrm{Re}\,\alpha_i>-L} \frac{(2\alpha_i+1)\gamma_i Q_{-\alpha_i-1}(z)}{\cos\pi\alpha_i} -\frac{1}{\pi i}\int_{-L-i\infty}^{-L+i\infty} dl\,(2l+1)f(l)Q_{-l-1}(z). \quad (19)$$

The representation (19) holds for $z\geqslant+1$ and provides the desired asymptotic behavior: For $z\to+\infty$, the leading term in the expansion is given by the term in Eq. (19) from the rightmost Regge pole.

Similarly, we find from Eq. (11a) for $z\leqslant-1$ that

$$A(z)=\sum_i \frac{(2\alpha_i+1)\delta_i Q_{-\beta_i-1}(-z)}{\cos\pi\beta_i}+\text{background}, \quad (20)$$

where the sum extends over all poles of $a(l)$ of residue δ_i at $l=\beta_i$, $\mathrm{Re}\,\beta_i>-L$.

Equation (14) is the Toller expansion[9] of $A(z)$, provided $A(z)$ is bounded by $z^{-1/2}$ at large z so that the discrete sum is absent (i.e., all Regge poles lie to the left of $\mathrm{Re}\,l=-\frac{1}{2}$). The inversion formula for the Toller expansion is[10]

$$f(l)=\frac{1}{2}\int_1^\infty dz\,P_l(z)A(z). \quad (21)$$

We can check our results by demonstrating that the right-hand side of Eq. (21) agrees with the definition (15) of $f(l)$. By deforming the contour C'' which appears in the definition (12a) of $b(l)$, we find (see Fig. 6 for the definition of the contour C_3) that

$$b(l)=\frac{1}{2\pi i}\int_{C_3} dz\,Q_l(z)A(-z) =-\frac{1}{2\pi i}\int_{-\infty}^{+1} dz[Q_l(z+i\epsilon)-Q_l(z-i\epsilon)]A(-z), \quad (22)$$

and, therefore, after evaluating the discontinuity of $Q_l(z)$,[11] we find that

$$b(l)=-\frac{\sin\pi l}{\pi}\int_1^\infty dz\,Q_l(z)A(z) +\frac{1}{2}\int_{-1}^{1} dz\,P_l(z)A(-z). \quad (23)$$

Finally, as a result of Eq. (16),

$$-\frac{1}{2}\frac{b(l)-b(-l-1)}{\cos\pi l}=\frac{1}{2}\int_1^\infty dz\,P_l(z)A(z). \quad (24)$$

If $A(z)\lesssim z^x$, then the integral (12a) which defines $b(l)$ converges for $\mathrm{Re}\,l>x$. Thus, the integral that appears in Eqs. (21) and (24) converges for l in the strip

$$x<\mathrm{Re}\,l<-x-1,$$

a result which may be obtained directly from the high-z behavior of $P_l(z)$.

[8] S. Mandelstam, Ann. Phys. (N. Y.) **19**, 254 (1962); the extension of Mandelstam symmetry to S-matrix theory is discussed by S. M. Roy, Phys. Rev. **161**, 1575 (1967).

[9] Reference 3, Eq. (74).

[10] Reference 3, Eq. (73).

[11] Reference 5, p. 140, Eqs. (11) and (12).

IV. APPLICATION TO MANY-PARTICLE AMPLITUDES

In order to carry out the Regge analysis of production amplitudes,[1] it is necessary to have a satisfactory continuation of the relevant partial-wave amplitude $A_{\mu\nu}{}^{j}$ from physical j to arbitrary complex j. In this section we discuss the analytic continuation $A_{\mu\nu}(j)$ of the partial-wave amplitude and the region of convergence of the Sommerfeld-Watson transformation applied to the partial-wave expansion of a multiparticle amplitude for spinless particles. (It can be noted that formally the following analysis is very similar to that for helicity amplitudes for two-particle scattering with spin.)

The expansion of a multiparticle amplitude is given in Ref. 1:

$$F(s,a,b,\varphi,\psi,z)=\sum_{\mu,\nu=-\infty}^{\infty} e^{-i(\mu\varphi+\nu\psi)}F_{\mu\nu}(s,a,b,z)\,, \quad (25)$$

where

$$F_{\mu\nu}(z)=\sum_{j=M}(2j+1)A_{\mu\nu}{}^{j}d_{\mu\nu}{}^{j}(z)\,. \quad (26)$$

Here,

$$M=\max(|\mu|,|\nu|)\,.$$

The total c.m. energy squared is s; a and b denote the internal variables of the initial and final clusters; and φ and ψ are the $O(3)$ analogs of certain $O(2,1)$ "Toller variables," and $z=\cos\theta$, where θ is the angle between "body-fixed" axes in the initial and final clusters (see Ref. 1). In writing Eq. (26), dependence on variables other than z has been suppressed. For multiscalar-particle amplitudes, μ and ν are both integers.

The inverse of Eq. (26) is

$$A_{\mu\nu}{}^{j}=\frac{1}{2}\int_{-1}^{1} dz\, F_{\mu\nu}(z)d_{\mu\nu}{}^{j}(z)\,. \quad (27)$$

In order to define a Froissart-Gribov-type partial-wave amplitude suitable for continuation to complex j, it is necessary to exercise some care, since the factors in the integrand of Eq. (27) may have kinematical singularities at $z=\pm 1$ not present in the spinless case. In particular, it is clear from Eq. (27) and the properties of $d_{\mu\nu}{}^{j}(z)$[12] and $F_{\mu\nu}(z)$ may have square-root singularities at $z=\pm 1$. We therefore define a kinematical singularity-free amplitude $\tilde{F}_{\mu\nu}(z)$ by

$$F_{\mu\nu}(z)=(1+z)^{(\mu+\nu)/2}(1-z)^{(\mu-\nu)/2}\tilde{F}_{\mu\nu}(z)\,. \quad (28)$$

Defining[13]

$$h_{\mu\nu}{}^{j}(z)\equiv(1+z)^{(\mu+\nu)/2}(1-z)^{(\mu-\nu)/2}d_{\mu\nu}{}^{j}(z) \quad (29)$$

and

$$g_{\mu\nu}{}^{j}(z)\equiv(1+z)^{(\mu+\nu)/2}(1-z)^{(\mu-\nu)/2}e_{\mu\nu}{}^{j}(z)\,, \quad (30)$$

we have

$$A_{\mu\nu}{}^{j}=\frac{1}{2}\int_{-1}^{1} dz\,\tilde{F}_{\mu\nu}(z)h_{\mu\nu}{}^{j}(z)\,. \quad (31)$$

Since[13] the discontinuity of $g_{\mu\nu}{}^{j}(z)$ is $-i\pi h_{\mu\nu}{}^{j}(z)$ for z between -1 and $+1$, and since for integer $j-\mu$ the function $g_{\mu\nu}{}^{j}(z)$ is cut from $+1$ to -1 only, we may write

$$A_{\mu\nu}{}^{j}=\frac{1}{2\pi i}\int_{C} dz\, g_{\mu\nu}{}^{j}(z)\tilde{F}_{\mu\nu}(z)\,. \quad (32)$$

The contour C is defined like that of Eq. (2) for the spinless case and may be deformed into the contour C' enclosing the singularities of $\tilde{F}(z)$ as in Fig. 1. Thus, we define the analytic continuation of $A_{\mu\nu}{}^{j}$ by

$$A_{\mu\nu}(j)=\frac{1}{2\pi i}\int_{C'} dz\, g_{\mu\nu}{}^{j}(z)\tilde{F}_{\mu\nu}(z)\,. \quad (33)$$

In Eqs. (28)–(33) and throughout the rest of this section we assume that $\mu\geqslant|\nu|\geqslant 0$. Corresponding equations for other regions of the indices are obtained from the symmetry relations among the d's and the e's given in Ref. 12.

Analysis of the background integral of $A_{\mu\nu}(j)$ proceeds in analogy with Secs. II and III for the spinless case. Throughout this section, however, we assume for simplicity that we are in a region of the variables such that $F_{\mu\nu}(z)\geqslant(1/z)^{1/2}$ for large z, then $A_{\mu\nu}(j)$ as defined in Eq. (33) converges for $j\geqslant-\frac{1}{2}$, i.e., the Regge poles lie to the left of $-\frac{1}{2}$. At the end of the analysis one can continue in the energy of the Regge pole to the region of interest; Regge poles may then move to the right of $\mathrm{Re}\,j=-\frac{1}{2}$. Alternatively, one could carry out the whole analysis with Regge poles to the right of $-\frac{1}{2}$, as we did above for elastic scattering, at the expense of carrying along a discrete sum, e.g., in Eqs. (7) and (17).

Following the steps of Sec. II, we find that the background integral

$$F_{\mu\nu}(z)=-\frac{1}{2i}\int_{-i\infty+M-\epsilon}^{+i\infty+M-\epsilon} dj\,(2j+1) \times\frac{A_{\mu\nu}(j)d_{\mu-\nu}{}^{j}(-z)}{\sin\pi(j-\mu)} \quad (34)$$

converges for z on the real axis $\leqslant -1$. The region of convergence may be larger depending on the location of singularities in z. Similarly, we may expand $F_{\mu-\nu}(-z)\equiv G_{\mu\nu}(z)$ as follows:

$$G_{\mu\nu}(z)=\sum_{j=M}^{\infty}(2j+1)B_{\mu\nu}{}^{j}d_{\mu\nu}{}^{j}(z)\,, \quad (35)$$

where $B_{\mu\nu}{}^{j}$ and its continuation $B_{\mu\nu}(j)$ are defined by

[12] The functions $d_{\mu\nu}{}^{j}(z)$ and corresponding functions of the second kind, $e_{\mu\nu}{}^{j}(z)$, are discussed by M. Andrews and J. Gunson, J. Math. Phys. 5, 1391 (1964).

[13] W. Drechsler, Nuovo Cimento 53, 115 (1968).

analogy to $A_{\mu\nu}{}^j$ and $A_{\mu\nu}(j)$ above; namely,

$$B_{\mu\nu}(j)=\frac{1}{2\pi i}\int_{C''}dz\, g_{\mu\nu}{}^j(z)\tilde{F}_{\mu-\nu}(-z)\,, \tag{36}$$

where C'' encloses the singularities of $\tilde{F}_{\mu-\nu}(-z)$. The background integral

$$F_{\mu\nu}(z)=-\frac{1}{2i}\int_{-i\infty+M-\epsilon}^{+i\infty+M-\epsilon}dj\,(2j+1)\times\frac{B_{\mu-\nu}(j)d_{\mu\nu}{}^j(z)}{\sin\pi(j-\mu)} \tag{37}$$

converges for z on the real axis $\geqslant 1$.

The representation (37) is related to the Toller expansion for $F_{\mu\nu}(z)$ in a way analogous to that discussed in Sec. III. In order to make the connection with the Toller expansion explicit, it is necessary to move the background integral contour left to $\mathrm{Re}j=-\frac{1}{2}$. In the process one encounters singularities of the integrand in j at integer values of $j-\mu<0$. The contributions of these so-called nonsense singularities are related to the discrete representation terms of the Toller expansion. Similar singularities in j are encountered when the contour in Eq. (34) is shifted to the left. The result of displacing the background contour in Eq. (37) to $-\frac{1}{2}$ is

$$F_{\mu\nu}(z)=-\frac{1}{2i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}d_j\,(2j+1)\frac{B_{\mu-\nu}(j)d_{\mu\nu}{}^j(z)}{\sin\pi(j-\mu)}+\sum_{k=0}^{|\nu|-1}(2k+1)b_{\mu\nu}{}^k d_{\mu\nu}{}^k(z)\,, \tag{38a}$$

where

$$b_{\mu\nu}{}^k=\frac{1}{2}\int_1^\infty dz\, h_{\mu\nu}{}^k(z)\tilde{F}_{\mu\nu}(z)\,. \tag{38b}$$

We note that although Eq. (38) is valid for $\mu\geqslant|\nu|\geqslant 0$, both $b_{\mu\nu}{}^k$ and $d_{\mu\nu}{}^k(z)$ are zero in this region of indices when $k<|\nu|$ and $\nu<0$. For $k-\mu=$integer with $0\leqslant k<\nu$. $d_{\mu\nu}{}^k(z)\sim z^{-k-1}$, so there is no conflict between Eq. (38) and the assumed high-z behavior of $F_{\mu\nu}(z)$.

We now exploit the symmetry properties of the integrand, as we did in Sec. III, to rewrite Eq. (38) in the form

$$F_{\mu\nu}(z)=\frac{1}{2i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}d_j\,(2j+1)\frac{f_{\mu\nu}(j)}{\tan\pi(j-\mu)}d_{\mu\nu}{}^j(z)+\sum_{k=0}^{|\nu|-1}(2k+1)b_{\mu\nu}{}^k d_{\mu\nu}{}^k(z)\,, \tag{39a}$$

where

$$f_{\mu\nu}(j)\equiv-\frac{1}{2}\frac{B_{\mu-\nu}(j)-(-1)^{\mu+\nu}B_{\mu-\nu}(-j-1)}{\cos\pi(j-\mu)}=(-1)^{\mu-\nu}f_{\mu\nu}(-j-1)\,. \tag{39b}$$

An integral representation for $F_{\mu\nu}(j)$ is found by comparing the definition (39b) with the integral representation of $B_{\mu-\nu}(j)$. The latter is the generalization of Eq. (23) of Sec. III. We find that

$$f_{\mu\nu}(j)=\frac{1}{2}\int_1^\infty dz\, h_{\mu\nu}{}^j(z)\tilde{F}_{\mu\nu}(z)\,. \tag{40}$$

Equations (39) and (40) constitute the Toller expansion and its inverse for $F_{\mu\nu}(z)$.

We employ the formula[12]

$$\frac{d_{\mu\nu}{}^j(z)}{\tan\pi(j-\mu)}=\frac{1}{\pi}[e_{\mu\nu}{}^j(z)-(-1)^{\mu-\nu}e_{\mu\nu}{}^{-j-1}(z)]$$

together with the symmetry properties of the integrand, to rewrite Eq. (39) as

$$F_{\mu\nu}(z)=-\frac{2}{2\pi i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}d_j\,(2j+1)f_{\mu\nu}(-j-1)e_{\mu\nu}{}^{-j-1}(z)+\sum_{k=0}^{|\nu|-1}(2k+1)b_{\mu\nu}{}^k d_{\mu\nu}{}^k(z)\,. \tag{41}$$

The background integral is now in a form appropriate for moving the background contour to the left, because $e_{\mu\nu}{}^{-j-1}(z)\sim z^j$ for large z. If $B_{\mu-\nu}(j)-(-1)^{\mu+\nu}B_{\mu-\nu}(-j-1)$ vanishes for $j-\mu=\frac{1}{2}$-integer (Mandelstam symmetry), then the integrand of Eq. (41) has singularities only from Regge poles of $B_{\mu-\nu}(j)$ for $j<-\frac{1}{2}$ and from singularities of $e_{\mu\nu}{}^{-j-1}(z)$ at "nonsense-nonsense" points (i.e., for $j=k$, where $-k-1-\mu$ is an integer and $0\leqslant -k-1<\nu$. The contributions from the nonsense singularities exactly cancel the discrete representation sum leaving only contributions from poles of $B_{\mu-\nu}(j)$ of residue $\gamma_{\mu\nu}{}^i$ at $j=\alpha_i$. Thus,

$$F_{\mu\nu}(z)=\sum_i\frac{(2\alpha_i+1)\gamma_{\mu\nu}{}^i(-1)^{\mu-\nu}e_{\mu\nu}{}^{-\alpha_i-1}(z)}{\cos\pi(\alpha_i-\mu)}+\text{background integral} \tag{42}$$

holds for $z\geqslant+1$ with the rightmost Regge pole providing the leading asymptotic behavior as $z\sim+\infty$. A similar expression valid for $z\leqslant-1$ may be found.