High-Energy Contributions to Current-Algebra Sum Rules. II

KASHYAP V. VASAVADA
Department of Physics, University of Connecticut, Storrs, Connecticut 06268
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A method, discussed in a previous paper, of evaluating the high-energy contributions to current-algebra sum rules is applied to several well-known sum rules. These include (i) the Adler-Weisberger sum rule for the π-π system, using π-π phase shifts, (ii) the Cabibbo-Radicati sum rule, (iii) π-N sum rules, and (iv) the π-N spin-flip sum rule. In the case of the π-π Adler-Weisberger sum rule, a substantial high-energy contribution is obtained which makes the sum rule inconsistent with a very large width of the σ meson (>400 MeV) for $m_\sigma=730$ MeV. The π-ρ sum rules favor a small width for the ρ ($\approx$100–120 MeV) and the A_1 meson (<100 MeV).

I. INTRODUCTION

IN a recent paper (hereafter referred to as I)[1] we discussed a method of evaluating the high-energy contributions[1] to some of the current-algebra sum rules. This method relies on the possibility of assuming a Regge-like form for the high-energy part of the integrals and eliminating unknown functions by writing some finite-energy sum rules for the same amplitudes. We applied this procedure to sum rules of the Adler-Weisberger type for the π-π and the K-π systems in the narrow-resonance approximation and to several sum rules for the pion-photoproduction amplitudes. These lead to some interesting results. One can, equivalently, look upon this procedure as a study of consistent solutions of the current-algebra and the finite-energy sum rules. In the present work, we apply these ideas to (i) the Adler-Weisberger sum rules for the π-π system (with inclusion of a scalar resonance and the continuum), (ii) the Cabibbo-Radicati sum rule, (iii) a number of sum rules for the π-ρ system, and (iv) some π-N sum rules. For the π-π sum rule we find results similar to those in I. Detailed results for the other sum rules are discussed in the following sections.

II. π-π SUM RULES

A. Adler-Weisberger-Type Sum Rule

Let $\nu=\frac{1}{2}(s-m_\pi^2)$ and $T_\pm(\nu,0)$ be the forward invariant amplitude for scattering of a zero mass $\pi^\pm$ by a physical π^+ meson. Then the Adler-Weisberger type sum rule for the π-π system is given by

$$\pi^{-1}\int d\nu\,\frac{\mathrm{Im}[T_-(\nu,0)-T_+(\nu,0)]}{\nu^2}=\frac{4}{f_\pi^2},\qquad(1)$$

where f_π (experimental) = 135 MeV. The Goldberger-Treiman relation gives $f_\pi^2=2g_A^2m_n^2/g^2K^2(0)$, $K(0)$ being the pionic form factor of the nucleon. The last relation gives $K(0)=0.86$. However, in addition to the possibility that $K(0)\approx1$, there are other corrections to the Goldberger-Treiman relation. We prefer to use the experimental value of f_π obtained from the pion-decay lifetime.

In terms of the $I=0$, 1, and 2 total cross sections for the π-π scattering, Eq. (1) becomes

$$\frac{2}{\pi}\int\frac{d\nu}{\nu}\,[\tfrac{2}{3}\sigma_0(\nu)+\sigma_1(\nu)-(5/3)\sigma_2(\nu)]=\frac{4}{f_\pi^2}.\qquad(2)$$

In I, we estimated the left-hand side of Eq. (2) in a narrow-resonance approximation (retaining only the ρ- and the f^0-meson contributions) and evaluated the high-energy contributions. Recently, however, some results from the phase-shift analysis of the pion-production data have appeared.[2] There are a number of ambiguities in the analysis, and the errors are large, but an s-wave resonance around the ρ-mass region is strongly indicated.[2] Hence, we have reconsidered this problem using the S-, P-, and D- wave phase shifts for all the isospin states. Instead of directly using various experimental results, we have used phase-shift expressions obtained by Arnowitt, Friedman, Nath, and Suitor in their work on the application of hard-pion methods to the π-π scattering problem.[3] Their results agree with the experimental results within the experimental errors. At any rate, for our purpose, we can look upon these expressions as convenient parametrizations of the data. Our results will have no bearing on the fact that the expressions come from a particular theory for the π-π scattering. Since these expressions are long, we will not give them here, but will just refer to the equations in Ref. 3, where the various symbols mentioned below are defined by the authors.

We use their Eqs. (4.3), (4.4), and (4.5a) for (i) $I=0$, $J=0$, (ii) $I=2$, $J=0$, and (iii) $I=1$, $J=1$ phase shifts,

[1] K. V. Vasavada, Phys. Rev. **178**, 2350 (1969). As in I, by "high-energy contribution" we will mean a contribution from the energy region above the low-energy resonant or nonresonant region for which experimental information is available, and not just the asymptotic region.

[2] W. D. Walker, J. Carroll, A. Garfinkel, and B. Y. Oh, Phys. Rev. Letters **18**, 630 (1967); E. Malamud and P. E. Schlein, *ibid.* **19**, 1056 (1967); L. J. Gutay, D. D. Carmory, P. L. Czonka, F. J. Loeffler, and F. T. Meiere, Phys. Rev. (to be published). On theoretical grounds, this resonance was predicted by M. M. Islam and R. Pinon [Phys. Rev. Letters **12**, 310 (1964)]; S. H. Patil [*ibid.* **13**, 261 (1964)]; and L. Durand and Y. T. Chiu, [*ibid.* **14**, 329 (1965)].

[3] R. Arnowitt, M. M. Friedman, P. Nath, and R. Suitor, Phys. Rev. **175**, 1820 (1968).

respectively. Values for the various parameters used are the following: $x=y=z=1$, $\lambda_A=0.5$ (corresponding to $\Gamma_\rho=108$ MeV) or $\lambda_A=0.2$ ($\Gamma_\rho=127$ MeV), $m_\sigma=730$ MeV with $\lambda^2=\lambda_1^2=0.75$ ($\Gamma_\sigma=246$ MeV), or $m_\sigma=930$ MeV with $\lambda^2=\lambda_1^2=0.9$ ($\Gamma_\sigma=611$ MeV). These phase shifts are used up to 1 BeV, beyond which they are taken to be zero (or π). This is the region of validity of of phase-shift formulas according to Ref. 3. If we continue to use these above 1 BeV, we get large unphysical phase shifts, especially in the case of the P wave. This may be because the Breit-Wigner form gives too large a contribution away from the resonance. While considering the solution corresponding to $m_\sigma=930$ MeV, we use the S-wave, $I=0$ phase shift up to the final cutoff (mentioned below) in order to take into account the full contribution of the scalar-meson resonance. According to Gutay *et al.*,[2] however, the other solution ($m_\sigma=730$ MeV, $\Gamma_\sigma=246$ MeV) is preferable to this one.

For $I=0$, $J=2$ we use the Breit-Wigner form (f_0 resonance)[4]

$$\sigma^{2,0}=\frac{20\pi\gamma_f{}^2q^8/(q^2+m_\pi{}^2)}{(s_f-s)^2+\gamma_f{}^2q^{10}/(q^2+m_\pi{}^2)}\,, \tag{3}$$

where

$$\gamma_f{}^2=[(q_f{}^2+m_\pi{}^2)/q_f{}^{10}]s_f\Gamma_f{}^2,\quad q^2=\tfrac{1}{4}s-m_\pi{}^2,$$
$$s_f=m_f{}^2=81.5m_\pi{}^2,$$

and $\Gamma_f=140$ MeV.

Other phase shifts are taken to be zero.

Following Adler[5] we also rewrite the integrals in Eq. (2) with the threshold-correction factors for zero-mass pions and obtain

$$\frac{2}{\pi}\int\frac{d\nu}{\nu}\Bigg[\sum_{l=0,2}\left(\frac{(s-m_\pi{}^2)^2}{s(s-4m_\pi{}^2)}\right)^l(\tfrac{2}{3}\sigma^{l,0}-(5/3)\sigma^{l,2})$$
$$+\left(\frac{(s-m_\pi{}^2)^2}{s(s-4m_\pi{}^2)}\right)\sigma^{l,1}\Bigg]\frac{[s(s-4m_\pi{}^2)]^{1/2}}{(s-m_\pi{}^2)}=\frac{4}{f_\pi{}^2}, \tag{4}$$

$$\sigma^{l,I}=(4\pi/q^2)(2l+1)\sin^2\delta_{l,I}. \tag{5}$$

With these cross sections, the low-energy integral is evaluated up to the cutoff value $N=55m_\pi$ (which corresponds to $W_{\text{c.m.}}\approx1470$ MeV). This is the same as the cutoff chosen in I and lies about half-way between the positions of the f and $g(1650)$ mesons.

TABLE I. Left-hand side[a] of Eq. (4) for various values of the parameters. The right-hand side is 4.28. I_L = low-energy contribution; I_H = high-energy contribution; I_T = total contribution.

m_σ (MeV)	λ^2 (Γ_σ (MeV))	λ_A (Γ_ρ (MeV))	I_L	I_H	I_T
730	0.75 (246)	0.5 (108)	2.88[b]	1.03	3.91
730	0.75 (246)	0.2 (127)	3.20	1.09	4.29
930	0.9 (611)	0.5 (108)	3.01	1.22	4.23
930	0.9 (611)	0.2 (127)	3.34	1.32	4.66

[a] See remarks in Ref. 5.
[b] The $I=0$, $J=0$ and the $I=1$, $J=1$ contributions in this case are 1.03 and 1.82, respectively.

As described in I, the high-energy correction term (I_H) to the left-hand side of (1) is given by

$$I_H=-\frac{\alpha+1}{\alpha-1}N^{-2}\pi^{-1}\int_0^N d\nu\,\mathrm{Im}[T_-(\nu,0)-T_+(\nu,0)] \tag{6}$$

in a one-trajectory (ρ) approximation, where α is the $t=0$ intercept of the trajectory. This integral is evaluated in the same manner as above. The results are shown in Table I for different values of λ_A (Γ_ρ) and m_σ, λ^2 (Γ_σ). The scalar-meson contribution is also shown. It can be seen that the scalar contribution is substantial (about 35% of the low-energy contribution), but it is not large enough to saturate the sum rule. A quite large high-energy contribution is obtained, which brings the sum rule into reasonable agreement with experiment.

In the above we have taken $\Gamma_\sigma=246$ MeV for $m_\sigma=730$ MeV. Taking $\Gamma_\sigma=400$ MeV and 650 MeV gives $I_T=4.5$ and 5.4, respectively, for $\Gamma_\rho=108$ MeV. These values, especially the latter, are too large for the sum rule to be consistent. Hence, a very large value of the σ width (>400 MeV) is excluded by our considerations. This is interesting in view of the fact that Gilman and Harari[6] are led to $\Gamma_\sigma=650$ MeV for $m_\sigma\approx m_\rho$ when they neglect the high-energy contributions. The value of width is quite sensitive to the mass and phase-space factors, so that when $m_\sigma=930$ MeV, such a large value of Γ_σ is permissible (see Table I).

To check the validity of the method, one can go one step further and attempt to determine α from the same data by using the two higher-moment finite-energy sum rules:

$$I_1=\pi^{-1}\int_0^N d\nu\,\mathrm{Im}[T_-(\nu,0)-T_+(\nu,0)]=\frac{\beta N^{\alpha+1}}{\alpha+1}, \tag{7}$$

$$I_3=\pi^{-1}\int_0^N d\nu\,\nu^2\,\mathrm{Im}[T_-(\nu,0)-T_+(\nu,0)]=\frac{\beta N^{\alpha+3}}{\alpha+3}, \tag{8}$$

$$I_1/I_3=N^{-2}(\alpha+3)/(\alpha+1). \tag{9}$$

[4] The experimental numbers used here are mostly taken from: Particle Data Group, Rev. Mod. Phys. **41**, 109 (1969). All quantities will be in units of $\hbar=c=m_\pi=1$, unless explicitly mentioned otherwise.

[5] S. L. Adler, Phys. Rev. **140**, B736 (1965); **149**, 1294(E) (1966); and **175**, 2224(E) (1968). In contrast to this reference, we have not multiplied the cross sections on the left-hand side of Eq. (4) by $K^2(0)$, where $K^2(0)$ is given by the Goldberger-Treiman relation. Unlike the case of π-N scattering, where the nucleon-exchange term gives the closest singularity, π-π scattering has no such term. Hence multiplication by this factor, in extrapolating from zero-pion mass to physical mass, may not be realistic. In any case, since there may be corrections to the Goldberger-Treiman relation itself, we may take $K^2(0)=1$. Taking $K^2(0)=0.74$, and multiplying the left-hand side of Eq. (4) by it would change the values given in Table I by this factor. This would lead to disagreement with the sum rule. Thus, our results indicate that for extrapolating π-π scattering cross sections, multiplication only by the threshold factors may be more reasonable.

[6] F. J. Gilman and H. Harari, Phys. Rev. **165**, 1803 (1968).

This procedure gives $\alpha=0.52$ for $\lambda^2=0.75$, $m_\sigma=730$ MeV, and $\Gamma_\rho=108$ MeV. This is in excellent agreement with the usually accepted value of the α for the ρ trajectory and is very encouraging in view of the crudeness of the input data. Thus the conclusions for the Adler-Weisberger π-π sum rule remain the same as in I, even with the inclusion of a scalar resonance and the continuum. The high-energy contribution is both substantial and necessary for the validity of the sum rule.

B. $I=2$ Exchange Sum Rule

There is one sum rule for the π-π system which corresponds to $I=2$ exchange in the t channel. This follows from the assumption that

$$[\partial_\mu A_\mu^+(t), Q_5^+(t)]=0\,, \tag{10}$$

where A_μ is the axial-vector current, and Q_5 is the corresponding charge.

This is consistent with most of the Lagrangian models for $SU(2)\otimes SU(2)$ current algebra and also with the absence of any evidence for $I=2$ particles. Taking the matrix element of the commutator between a π^+ and a π^- state, one obtains a sum rule

$$\pi^{-1}\int d\nu\, \frac{\operatorname{Im}T_2(\nu,0)}{\nu}=0\,, \tag{11}$$

where $T_2(\nu,0)$ is the forward-scattering invariant amplitude for the π-π system with the t channel in a pure $I=2$ state. The sum rule has been considered by Furlan and Rossetti[7] and by Gilman and Harari.[6]

In terms of isospin $I=0$, 1, and 2 total cross sections, Eq. (11) becomes

$$\pi^{-1}\int d\nu\, (\tfrac{1}{3}\sigma^0-\tfrac{1}{2}\sigma^1+\tfrac{1}{6}\sigma^2)=0\,. \tag{12}$$

Thus, there is a cancellation between the $I=0$ and 1 cross sections. Evaluating the integrals in the same manner as above, we find that $I_L=4.6$ for $\Gamma_\rho=108$ MeV and 1.3 for $\Gamma_\rho=127$ MeV. The $I=1$ (ρ) contribution in these cases are -22.3 and -25.6, respectively. Thus, because of the large cancellations, we conclude that the sum rule is approximately satisfied by the low-energy data alone and that there is no particular need for requiring significant high-energy contributions. This is consistent with the fact there is no known $I=2$ trajectory and that the cut contributions are likely to be small at $t=0$.

III. CABIBBO-RADICATI SUM RULE

This sum rule and the original Adler–Fubini–Dashen–Gell-Mann sum rule from which it can be derived are regarded as crucial tests of the local current algebra.[8] The current commutator used here is

$$[\mathfrak{F}_{1+i2}{}^0(x),\mathfrak{F}_{1-i2}{}^0(y)]_{x_0=y_0}=2\delta^3(x-y)\mathfrak{F}_3{}^0(y)\,, \tag{13}$$

where the $\mathfrak{F}$'s are the various isovector vector-current densities.

From this commutator, one obtains the Adler–Fubini–Dashen–Gell-Mann sum rule, and on differentiating with respect to q^2 (the mass squared associated with the current) at $q^2=0$, the Cabibbo-Radicati sum rule can be derived. On isospin rotation, one obtains a convenient form for the sum rule in terms of total cross sections for the scattering of isovector photons off protons:

$$\begin{aligned}\frac{2dF_1{}^V(q^2)}{dq^2}\bigg|_{q^2=0}&+\left(\frac{\mu_V}{2m_n}\right)^2+\frac{2}{\pi e^2}\int_{m_\pi+m_\pi{}^2/2m_n}\frac{d\nu}{\nu}\\&\times[2\sigma_T(\gamma^V+p\to I=\tfrac{1}{2})-\sigma_T(\gamma^V+p\to I=\tfrac{3}{2})]\\&=0\,,\end{aligned} \tag{14}$$

where now $\nu=(s-m_n{}^2)/2m_n$; $F_1{}^V$ and μ^V are the isovector form factor and the nucleon-isovector anomalous magnetic moment, respectively, and $\sigma_T(\gamma^V+p\to I=\frac{1}{2})$ and $\sigma_T(\gamma^V+p\to I=\frac{3}{2})$ are, respectively, the total cross sections leading to the $I=\frac{1}{2}$ and $\frac{3}{2}$ states.

Numerical evaluations of this sum rule have been considered by Adler and Gilman[9] and by Gilman and Schnitzer,[9] using the pion-photoproduction data. We have reconsidered evaluation of this integral, using the recent pion-photoproduction-multipole fits by Walker.[10] We will also discuss the high-energy contributions. These multipoles include contributions from several πN resonances [$N^*(1236)$, $N(1519)$, $N(1672)$, $N(1561)$, $N(1471)$, and $N(1652)$], the nucleon pole, and the continuum.

In terms of single-pion-photoproduction multipoles, we have

$$\begin{aligned}\sigma_T=\frac{2\pi(p_\pi)_{\text{c.m.}}}{(q)_{\text{c.m.}}}&\sum(4|M_{1+}|^2+18|M_{2+}|^2+2|E_{0+}|^2\\&+12|E_{1+}|^2+36|E_{2+}|^2+12|M_{2-}|^2+36|M_{3-}|^2\\&+4|E_{2-}|^2+18|E_{3-}|^2+2|M_{1-}|^2+\cdots)\,.\end{aligned} \tag{15}$$

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[7] G. Furlan and C. Rossetti, Phys. Letters **23**, 499 (1966).

[8] S. L. Adler, Phys. Rev. **143**, 1144 (1966); R. Dashen and M. Gell-Mann, in *Proceedings of the Third Coral Gables Conference on Symmetry Principles at High Energy, University of Miami* (W. H. Freeman and Co., San Francisco); S. Fubini, Nuovo Cimento **43A**, 475 (1966); N. Cabibbo and L. Radicati, Phys. Letters **19**, 697 (1966); J. D. Bjorken (unpublished).

[9] S. L. Adler and F. J. Gilman, Phys. Rev. **156**, 1598 (1967); F. J. Gilman, *ibid.* **167**, 1365 (1968); F. J. Gilman and H. J. Schnitzer, *ibid.* **150**, 1362 (1966).

[10] R. L. Walker, Phys. Rev. **182**, 1729 (1969), and private com munications. The author wishes to thank Professor Walker for communications regarding these multipole fits.

The notation here is the same as that of Chew, Goldberger, Low, and Nambu (CGLN).[11] $(p_\pi)_{\text{c.m.}}$ and $(q)_{\text{c.m.}}$ are the c.m. momenta of pion and photon, respectively. Fits for these multipoles are available in Ref. 10, up to a photon-lab energy of about 1.2 BeV, for the reactions $\gamma+p\to\pi^++n$, $\gamma+p\to\pi^0+p$, and $\gamma+n\to\pi^-+p$. The isovector amplitudes leading to the $I=\frac{1}{2}$ and $\frac{3}{2}$ states can be extracted from these fits. We have ignored the fact that pions have nonzero mass in these considerations since the zero-mass approximation will affect the amplitudes only in the region close to the threshold.

The cutoff N is chosen to be 1.2 BeV. From the first two terms on the left-hand side of (14) we obtain $-200\,\mu$b. The integral term gives $-73\ \mu$b. Thus, the discrepancy is $-273\ \mu$b, whereas in Ref. 9 it is found to be about $-300\ \mu$b, which is quite close. The major contribution to the low-energy integral comes from single-pion-photoproduction cross sections. This may explain the closeness of our estimate to that of Ref. 9, where some attempt is made to take into account inelastic contributions.

The high-energy part of the integral will be dominated by the multipion-photoproduction cross sections. If we estimate this by writing a first-moment finite-energy sum rule (FESR), assuming dominance by the ρ trajectory, we find the correction term to be

$$-\frac{\alpha+1}{\alpha-1}\frac{1}{N^2}\frac{2}{\pi e^2}\int_0^N \nu d\nu[2\sigma_T(\tfrac{1}{2})-\sigma_T(\tfrac{3}{2})]. \tag{16}$$

Taking $\alpha=0.5$, this gives 537 μb. Although this is of the right sign,[12] it is quite large. We can, of course, attempt to determine α by writing two FESR:

$$I_1=\frac{2}{\pi e^2}\int_0^N \nu d\nu[2\sigma_T(\tfrac{1}{2})-\sigma_T(\tfrac{3}{2})]=\frac{\beta N^{\alpha+1}}{\alpha+1}, \tag{17}$$

$$I_3=\frac{2}{\pi e^2}\int_0^N \nu^3 d\nu[2\sigma_T(\tfrac{1}{2})-\sigma_T(\tfrac{3}{2})]=\frac{\beta N^{\alpha+3}}{\alpha+3}. \tag{18}$$

Using the above-mentioned data, we find that $\alpha\approx3$, which is entirely unacceptable. This is somewhat surprising in view of the fact that such sum rules have themselves yielded reasonable results for the relevant trajectories in the case of single-pion-photoproduction amplitudes.[13] In the present case, this result may be interpreted as due to the non-Regge behavior of the amplitudes under consideration. The fixed pole demanded by the current-algebra considerations in this case has a vanishing residue because we are dealing with a derivative sum rule. However, it has been suggested on dynamical grounds that there may be fixed poles in Compton scattering amplitudes.[14] Although this is possible, it seems that our result is more likely due to large cancellations between the $I=\frac{1}{2}$ and $\frac{3}{2}$ terms and inaccuracies in the data. Indeed, several $I=\frac{3}{2}$ πN resonances, which are not included in our cross sections,[15] have been suggested to exist near the upper limit of the integral. Being highly inelastic, they will contribute significantly to the double-pion-photoproduction cross section and will particularly affect the higher-moment sum rules. It can be seen that increasing the $I=\frac{3}{2}$ cross section will lower α considerably. Hence, we adopt the following procedure: We demand that $\alpha=0.5$ be obtained from the ratio $(I_1/I_3)N^2$, and we determine N from this. This gives $N\approx1.5$ BeV. Remembering that the original cutoff was $N\approx1.2$ BeV, this appears to be very reasonable. Thus, because of the delicate cancellations the value of α obtained from the sum rule is extremely sensitive to cutoff. With $N=1.5$ BeV and $\alpha=0.5$, we find that the correction term is 344 μb, which is in fair agreement with the value required to satisfy the sum rule. Thus, in spite of the incompleteness of the data, one may at least state that the magnitudes and signs of the low-energy and high-energy contributions appear to satisfy the sum rule reasonably well.

IV. π-ρ SUM RULES

Current-algebra and superconvergence sum rules for the π-ρ system have been discussed by several authors. In particular, Gilman and Harari have given a fairly complete discussion of the sum rules corresponding to the $I=1$ and 2 states in the t channel.[16] There are two sum rules for the $I=1$ state:

$$\pi^{-1}\int_0^\infty \frac{d\nu}{\nu^2}\,\mathrm{Im}\bar{f}_{10,10}{}^{(1)}(\nu,0)=\frac{4}{f_\pi{}^2}, \tag{19}$$

and

$$\pi^{-1}\int_0^\infty \frac{d\nu}{\nu^2}\,\mathrm{Im}\bar{f}_{00,00}{}^{(1)}(\nu,0)=\frac{4}{f_\pi{}^2}. \tag{20}$$

[11] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1345 (1957).

[12] Sign of the high-energy contribution is important in view of the recent controversy on modification of this sum rule. Barry, Gounaris, and Sakurai [Phys. Rev. Letters **21**, 941 (1968)] have proposed additional q^2-dependent terms in the Adler–Fubini–Dashen–Gell-Mann sum rule. This would require the contribution from the high-energy part of the integral to be about $-350\ \mu$b. I. S. Gerstein [*ibid.* **21**, 1465 (1968)], on the other hand, has argued that these modifications are inconsistent. With the Regge-behavior assumption, we find that the high-energy contribution has positive sign.

[13] See, e.g., K. V. Vasavada and K. Raman, Phys. Rev. Letters **21**, 577 (1968); and K. Raman and K. V. Vasavada, Phys. Rev. **175**, 2191 (1968).

[14] M. J. Creutz, S. D. Drell, and E. A. Paschos, Phys. Rev. **178**, 2300 (1969). Previous theoretical references are mentioned in this work.

[15] Phase-shift analyses of the CERN group. See Ref. 4.

[16] F. J. Gilman and H. Harari, Phys. Rev. Letters **18**, 1150 (1967); Phys. Rev. **165**, 1803 (1968). Previous references are given in these papers.

Similarly for $I=2$, we have

$$\pi^{-1}\int_0^\infty \frac{d\nu}{\nu}\,\mathrm{Im}\bar{f}_{10,10}{}^{(2)}(\nu,0)=0\,, \tag{21}$$

and

$$\pi^{-1}\int_0^\infty \frac{d\nu}{\nu}\,\mathrm{Im}\bar{f}_{00,00}{}^{(2)}(\nu,0)=0\,, \tag{22}$$

where $\nu=\frac{1}{2}(s-m_\rho{}^2)$.

Here $\bar{f}_{\lambda 0,\lambda 0}{}^{(I)}$ is the s-channel, kinematical-singularity-free helicity amplitude with the initial and final helicities of ρ being λ and the t-channel isospin I. The first two sum rules are the Adler-Weisberger-type sum rules, and the last two follow from the commutator involving $\partial_\mu A_\mu$, as in the π-π case.

In Ref. 16 these sum rules have been studied by using the resonance-saturation approximation, retaining ω, π, A_1, and A_2 contributions, and consistent solutions have been suggested. We can calculate the high-energy contributions to the $I=1$ sum rules by assuming ρ-trajectory exchange. Instead, we follow an equivalent procedure and consider the following sets of equations by writing two more higher-moment FESR for each of Eqs. (19) and (20) and by studying consistent solutions. In the resonance-saturation approximation for the low-energy integrals, we have the equations

$$g_{\omega\rho\pi}{}^2+\frac{\nu_A{}^2}{m_A{}^4}g_T{}^2+\frac{g_{A_2\rho\pi}{}^2\nu_{A_2}{}^2}{m_{A_2}{}^2}-\frac{8}{f_\pi{}^2}=\frac{\beta_1 N_1{}^{\alpha-1}}{\alpha-1}\,, \tag{23}$$

$$\nu_\omega{}^2 g_{\omega\rho\pi}{}^2+\frac{\nu_A{}^4}{m_A{}^4}g_T{}^2+\frac{g_{A_2\rho\pi}{}^2\nu_{A_2}{}^4}{m_{A_2}{}^2}=\frac{\beta_1 N_1{}^{\alpha+1}}{\alpha+1}\,, \tag{24}$$

$$\nu_\omega{}^4 g_{\omega\rho\pi}{}^2+\frac{\nu_A{}^6}{m_A{}^4}g_T{}^2+\frac{g_{A_2\rho\pi}{}^2\nu_{A_2}{}^6}{m_{A_2}{}^2}=\frac{\beta_1 N_1{}^{\alpha+3}}{\alpha+3}\,, \tag{25}$$

$$\frac{4g_{\rho\pi\pi}{}^2}{m_\rho{}^2}+\frac{\nu_A{}^2 g_L{}^2}{m_\rho{}^2 m_A{}^2}-\frac{8}{f_\pi{}^2}+\cdots=\frac{\beta_2 N_2{}^{\alpha-1}}{\alpha-1}\,, \tag{26}$$

$$\frac{4g_{\rho\pi\pi}{}^2}{m_\rho{}^2}\nu_\pi{}^2+\frac{\nu_A{}^4}{m_\rho{}^2 m_A{}^2}g_L{}^2+\cdots=\frac{\beta_2 N_2{}^{\alpha+1}}{\alpha+1}\,, \tag{27}$$

$$\frac{4g_{\rho\pi\pi}{}^2}{m_\rho{}^2}\nu_\pi{}^4+\frac{\nu_A{}^6}{m_\rho{}^2 m_A{}^2}g_L{}^2+\cdots=\frac{\beta_2 N_2{}^{\alpha+3}}{\alpha+3}\,, \tag{28}$$

$$x\nu_\omega g_{\omega\rho\pi}{}^2-(\nu_A{}^3/m_A{}^4)g_T{}^2=0\,, \tag{29}$$

$$4y\nu_\pi g_{\rho\pi\pi}{}^2+(\nu_A{}^3/m_A{}^2)g_L{}^2=0\,. \tag{30}$$

Here the notation is the same as in Ref. 16: g_T and g_L define the $A\rho\pi$ couplings; $g_{\omega\rho\pi}$ and $g_{\rho\pi\pi}$ describe the $\omega\rho\pi$ and $\rho\pi\pi$ couplings. The factors on the right-hand side of Eqs. (23)–(28) represent the high-energy contributions; α, $\beta_{1,2}$, and $N_{1,2}$ are, respectively, the trajectory parameter, the residues, and the cutoffs for the ρ-trajectory contribution.

We assume that there is no leading $I=2$ trajectory or cut giving significant contributions to Eqs. (21) and (22). However, the $I=0$ and 1 states in the s channel for these $I=2$ sum rules occur with alternating signs because of the crossing matrix. Hence, there are delicate cancellations between various resonant contributions. In order to take these into account, we have multiplied the left-hand sides of Eqs. (29) and (30) by unknown constants x and y. We will not use these equations in our analysis. Their content has been discussed in Ref. 16, where it is found that x, $y\approx 1$. If we take $x=y=1$, Eqs. (29) and (30) imply that

$$\frac{g_T{}^2}{g_L{}^2}=-\frac{\nu_\omega g_{\omega\rho\pi}{}^2 m_A{}^2}{4\nu_\pi g_{\rho\pi\pi}{}^2}\,. \tag{31}$$

Now from the Gell-Mann–Sharp–Wagner[17] model of $\omega\to\pi+\gamma$, we have $g_{\omega\rho\pi}=17\pm3$ BeV^{-1}. With $90\leq\Gamma_\rho\leq125$ MeV, Eq. (31) gives

$$0.12\leq g_T{}^2/g_L{}^2\leq0.23\,.$$

Because this approximately agrees with the preliminary experimental results on A_1 decay parameters[18] which give $g_T{}^2/g_L{}^2=0.16\pm0.08$, the assumptions leading to Eqs. (29) and (30) may be reasonable. Now we look into the possibility of consistent solutions of Eqs. (23)–(28). These are very stringent conditions, especially in a saturation scheme with a few resonances.

Eliminating β_1 and N_1 between Eqs. (23), (24), and (25), we find that

$$(\alpha+1)^2/(\alpha-1)(\alpha+3)=AC/B^2\,, \tag{32}$$

where A, B, and C are the left-hand sides of the three equations, respectively. Since the ρ trajectory is involved, we demand that even in such an approximate treatment, $0<\alpha<1$ should be obtained. Since this requires that A must be negative, it places restrictions on the acceptable values of $g_{\omega\rho\pi}{}^2$, $g_T{}^2$, and $g_{A_2\rho\pi}{}^2$. At present there is considerable confusion regarding resonances near the A_2 mass region. Two or more resonances, with perhaps different quantum numbers, appear to be in the region. First we consider the unsplit $A_2(1300)$, and take $\Gamma_{A_2}=80$ MeV. With $g_{\omega\rho\pi}=17$ BeV^{-1} and $0<g_T{}^2<27$, values of α between 0 and 1 are obtained. For $g_{\omega\rho\pi}=20$ BeV^{-1}, no value of $g_T{}^2$ can give α in this range, whereas for $g_{\omega\rho\pi}=14$ BeV^{-1}, $0<g^2<51$ is acceptable. If $\Gamma_{A_2}=30$ MeV, then the three values of $g_{\omega\rho\pi}$, respectively, give acceptable values of α for $0<g^2<33$, $0<g_T{}^2<6$, and $0<g_T{}^2<57$.

Thus, depending on the values assigned to the width of the A_2 and to $g_{\omega\rho\pi}$, there is a rather wide range of values of $g_T{}^2$ which give reasonable results. The corresponding transverse widths for the A_1 meson [$\Gamma_t=(g_T{}^2/6\pi)(q^5/m_A{}^2)$, where q is the c.m. momentum

[17] M. Gell-Mann, D. Sharp, and W. G. Wagner, Phys. Rev. Letters **8**, 261 (1962).

[18] J. Ballam *et al.*, Phys. Rev. Letters **21**, 934 (1968).

of the pion at the position of the resonance] depend sensitively upon whether $m_\pi=0$ is taken for the phase-space factor. (The current-algebra sum rules are written for the zero-mass pions.) Then $0<g_T^2<57$ corresponds to $0<\Gamma_t(m_\pi=0)=3.03g_T^2<173$ MeV or to $0<\Gamma_t(m_\pi=1)=1.65g_T^2<94$ MeV.

In view of the uncertainties in the values of the other coupling constants, precise values of g_T^2 satisfying the sum rules cannot be given. However, it is satisfying that for a range of these values reasonable values of α can be obtained; hence, the sum rules seem to be satisfied.

Similar consideration of the sum rules (26)–(28) lead to interesting results. We find that

$$(\alpha+1)^2/(\alpha-1)(\alpha+3)=A'C'/B'^2\,, \tag{33}$$

A', B', C' being the left-hand sides of Eqs. (26), (27), and (28), respectively. Again A' should be negative in order to obtain $0<\alpha<1$. This requires that $g_{\rho\pi\pi}^2$ and g_L^2 not be very large. The $I=0$ and 1 states both enter with the same sign on the left-hand side of these equations. Hence, the contribution of the additional resonant states cannot amend the situation. These states will contribute much more to the sum rules (27) and (28) than to (26). Thus, they will affect the actual value of α obtained. As some typical cases, one may consider both resonances with $J^P=1^+$ or 2^-, and relatively large longitudinal couplings around the A_2 mass region or near the 1640-MeV resonance ($\tilde{A}$). Then, we let $a=4g_{\rho\pi\pi}^2/m_\rho^2+\nu_A^2g_L^2/m_\rho^2m_A^2$ and b be the contribution of this last resonance. We now have

$$\frac{(\alpha+1)^2}{(\alpha-1)(\alpha+3)}=\frac{(a+b-8/f_\pi^2)(a\nu_\pi^4+b\nu_{\tilde{A}}^4)}{(a\nu_\pi^2+b\nu_{\tilde{A}}^2)^2}\,. \tag{34}$$

This simplification occurs because $\nu_\pi=-\nu_A$ when one takes $m_A^2=2m_\rho^2-m_\pi^2$.

For various values of a and b, α can be determined. For the $\tilde{A}$ resonance (1640 MeV) we find that values of b between 0 and 1 lead to acceptable values of α for $a<7$. For A near the A_2 mass, however, only values of $a<6$ are acceptable. For a given value of a and the ρ width, we can find the corresponding g_L^2. Thus $a=6$ leads to $g_L^2=17$ ($\Gamma_\rho=125$ MeV) or 26 ($\Gamma_\rho=90$ MeV), while $a=6.5$ or 7 lead to $g_L^2=21$ or 30 and $g_L^2=25$ or 35 in the two cases. Smaller values of a, of course, are acceptable as far as output values of α are concerned, but they lead to very small values of g_L^2. As mentioned above, the experimental indications are that g_T^2/g_L^2 is quite small. Hence, small values of g_L^2 imply quite a small width of the $A_1\to\rho\pi$ decay.

The corresponding values of the longitudinal widths $\Gamma_L=(g_L^2/12\pi)q^5/m_\rho^2$, are the following: $0<g_L^2<35$ corresponds to $0<\Gamma_L(m_\pi=0)=3.03g_L^2<104$ MeV and $0<\Gamma_L(m_\pi=1)=1.65g_L^2<57$ MeV.

Thus it appears that the sum rules show definite preference for smaller ρ width ($\Gamma_\rho\approx100$ MeV) and not too large $A_1\to\rho\pi$ width (<100 MeV). Finally, Eq. (30) with $y=1$ gives $g_L^2/g_{\rho\pi\pi}^2=1.1$. This is not inconsistent with the above values. To conclude the discussion of the present set of π-ρ sum rules, we note that they seem to lead to reasonably consistent results, although a definite conclusion regarding the actual magnitudes cannot be reached unless experimental results become more precise.

V. π-N SUM RULES

In this section we reexamine two of the π-N sum rules: (i) the Adler-Weisberger sum rule[5] and (ii) the corresponding spin-flip sum rule.[19]

In I we made some remarks regarding the Adler-Weisberger sum rule for the π-N system which we now write as

$$g_A^2=1-\frac{2f_\pi^2}{\pi}\int_{m_\pi}^{\infty}d\nu\,\frac{\mathrm{Im}[A^-+\nu B^-]}{\nu^2}\,. \tag{35}$$

The off-mass-shell contributions are significant, but for the purpose of our consideration we will ignore them. They have been considered extensively earlier. Here $\nu=(s-m_n^2-m_\pi^2)/2m_n$ and A^- and B^- are the usual π-N amplitudes corresponding to the $I=1$ eigenstate in the t channel.

With the high-energy correction, this equation reads

$$g_A^2=1-\frac{2f_\pi^2}{\pi}\int_{m_\pi}^{N}d\nu\,\frac{\mathrm{Im}[A^-+\nu B^-]}{\nu^2}+\frac{2f_\pi^2}{\pi N^2}\frac{\alpha+1}{\alpha-1}\int_{m_\pi}^{N}d\nu\,\mathrm{Im}[A^-+\nu B^-]\,, \tag{36}$$

where N is chosen to be sufficiently high to make the dominance by the ρ trajectory (α) reasonable. If we wish to determine α from the same data, we can use the two FESR

$$I_1=\pi^{-1}\int_0^N d\nu\,\mathrm{Im}[A^-+\nu B^-]=\frac{\beta N^{\alpha+1}}{\alpha+1}\,, \tag{37}$$

$$I_3=\pi^{-1}\int_0^N \nu^2d\nu\,\mathrm{Im}[A^-+\nu B^-]=\frac{\beta N^{\alpha+3}}{\alpha+3}\,. \tag{38}$$

The above integrals can be readily expressed in terms of the π-N total cross sections, and reasonably accurate data are available up to very high energies. To illustrate our point, however, we have considered the narrow-resonance-saturation approximation. This will be used also for the spin-flip sum rule, which cannot be expressed in terms of the total cross sections.

The $N^*(1236)$ resonance itself gives rather large value of g_A when the high-energy contribution is neglected,

[19] This sum rule has been considered by several authors. Some of these references are I. S. Gerstein, Phys. Rev. **161**, 1631 (1967); K. Raman, *ibid.* **159**, 1501 (1967); H. Goldberg and F. Gross, *ibid.* **162**, 1350 (1967); E E. Radescu, *ibid.* **171**, 1655 (1968); L. Maiani and G. Preparata (unpublished).

and gives contributions to the correction term in the wrong direction. It was guessed in I that the higher resonances will change the sign of the correction term to bring it in the right direction to satisfy the sum rule. In this work, we have indeed verified that this is the case.

In the narrow-resonance approximation we have

$$I=\pi^{-1}\int_0^N d\nu\,\frac{\mathrm{Im}[A^-+\nu B^-]}{\nu^2}=\frac{2\pi}{m_n^2}\sum_r\frac{C_r\eta_rW_r^2\Gamma_r(J_r+\frac{1}{2})}{q_r\nu_r^2},\quad(39)$$

where W_r, Γ_r, η_r, J_r, and q_r are, respectively, the values of energy, width, inelasticity, total angular momentum, and outgoing-pion three-momentum of the resonance. $C_r=\frac{1}{3}$ for $I=\frac{1}{2}$ and $C_r=-\frac{1}{3}$ for $I=\frac{3}{2}$ states. The question of the choice of N is difficult here because of the large cancellations between the $I=\frac{1}{2}$ and $\frac{3}{2}$ resonances. This fact has been discussed by Dolen, Horn, and Schmid,[20] who suggest that a suitable value of N for this amplitude may be around 2.5 BeV. With this choice of N and with data from Ref. 4, we find that $I=-0.4$, $I_1=7$, and $I_3=827$. First of all, this shows that I_1 and I have opposite signs. Hence, the correction term will be in the right direction. Values of I_1 and I_3 give $\alpha=0.17$, which is rather low for the ρ trajectory, though understandable, because of the cancellations and the narrow-resonance approximation. Also, it should be noted that there may be several resonances making important contributions to the higher-moment sum rules, which are not included in our evaluation (see Ref. 15). These are automatically included when one uses the total cross-section data. These will change the results for α significantly. Now, when the high-energy part is neglected we find that $g_A=1.32$. The correction term is found to be $2f_\pi^2(0.031)$, leading to $g_A=1.30$. Thus, the contribution from the region above 2.5 BeV is about 8% of the low-energy integral. As mentioned in I, the values of g_A are not sensitive to rather appreciable changes in the values of the integrals. Now if we demand that an $\alpha=0.5$ should be obtained from the values of I_1 and I_3 and adjust N accordingly, we have $N=2.7$ BeV. This clearly shows the sensitivity of the results for α on the cutoff value. With this cutoff, the correction term becomes $2f_\pi^2(0.056)$, leading to $g_A=1.28$. The negative sign of the correction term persists even for such low cutoffs as 1.5 BeV. In the latter case we have $I=-0.42$ and $I_1=3.17$, leading to $g_A=1.34$. If only $N^*(1236)$ is retained, it is found that $I=-0.66$ and $I_1=-3.75$. Without the high-energy contribution, this leads to $g_A=1.49$, and as mentioned above, the high-energy contribution increases the value of g_A.

[20] R. Dolen, D. Horn, and C. Schmid, Phys. Rev. **166**, 1768 (1968).

We may add that when the total cross-section data are available and usable for the sum rules, they should be considered in preference to the narrow-resonance approximation. However, the above discussion shows that even in the case of the original Adler-Weisberger relation our method yields a correction term of the right sign and of a reasonable magnitude.

Now we consider the corresponding spin-flip sum rule given by

$$1+\mu_p-\mu_n-g_A^2-\frac{2f_\pi^2}{\pi}\int_{m_\pi}^\infty d\nu\,\frac{\mathrm{Im}B^-(\nu)}{\nu}=4m_n\tilde{D}^-(0),\quad(40)$$

where μ_p and μ_n are the anomalous magnetic moments of the nucleons and $\tilde{D}^-(0)$ is the coefficient of the term $\bar{u}i\sigma_{\mu\nu}\times\frac{1}{2}[\tau^a,\tau^b]u$ in the expansion of the axial-vector-current–nucelon-scattering amplitude in a complete set of independent kinematical-singularity-free covariants. The nucleon-pole term is understood to be already extracted from this. As has been discussed by previous authors,[19] the appearance of the axial-vector-current–nucleon-scattering term makes the sum-rule model-dependent, and there is no way of eliminating this term. Goldberg and Gross,[19] in particular, rewrite this sum rule by combining the amplitudes $\tilde{D}$ and B in terms of a single amplitude for the axial-vector-current–nucleon scattering. They find rather large high-energy contributions by making some estimates of the residue $\beta(0)$ for the ρ trajectory. We have studied this sum rule using the resonance approximation for the B amplitude.

In terms of resonances, we have

$$I=\pi^{-1}\int_{m_\pi}^N d\nu\,\frac{\mathrm{Im}B^-}{\nu}=\frac{\pi}{m}\sum_{\substack{r\\ \bar{l}=l+1,\,-l}}\frac{C_r\eta_r\Gamma_r}{q_r^3}\times\{\bar{l}[(W_r-m_n)^2-m_\pi^2]-2W_rm_nl(l+1)\}.\quad(41)$$

Here

$$\bar{l}=\binom{l+1}{-l},$$

according as $J=l\pm\frac{1}{2}$, where l is the orbital angular momentum. The two higher-moment FESR are given by

$$I_1=\pi^{-1}\int_0^N \nu d\nu\,\mathrm{Im}B^-(\nu)=\frac{\beta N^{\alpha+1}}{\alpha+1},\quad(42)$$

$$I_3=\pi^{-1}\int_0^N \nu^3 d\nu\,\mathrm{Im}B^-(\nu)=\frac{\beta N^{\alpha+3}}{\alpha+3}.\quad(43)$$

Here we find that most of the resonances add constructively; hence, the cutoff difficulties may not be serious. This is confirmed by taking $N=1.5$ BeV, which leads to $I=3.9$, $I_1=62.9$, and $I_3=3150$. This gives $\alpha=0.55$, in excellent agreement with the usually accepted value of the ρ-trajectory parameter. Data up to $N=1.5$ BeV may be quite accurate, and the result gives us confidence in using this cutoff.

With these values the B contribution to the left-hand side of Eq. (40) becomes

$$\frac{-2f_\pi^2}{\pi}\int_{m_\pi}^{\infty} d\nu \frac{\mathrm{Im}B^-(\nu)}{\nu} = -2f_\pi^2 I - \frac{2f_\pi^2}{N^2}\frac{\alpha+1}{\alpha-1}I_1 = -10.9\,. \tag{44}$$

This is larger than the value (≈ 8) obtained by Hohler and Strauss,[21] who used the available π-N phase shifts. However, the narrow-resonance approximation is known to lead to some overestimation of the contributions. It should be pointed out that in the present case, this procedure appears to be necessary. In order to estimate the right-hand side, one is forced to use the resonance approximation in any case. Hence it will not be correct to evaluate the B term exactly, since there are cancellations between these terms. This fact has been emphasized by Goldberg and Gross.[19] The value quoted by Radescu[19] for the B-integral contribution ($=4.9$) is much smaller than either our value or that given in Ref. 21.

The first three terms in Eq. (40) give 3.3. If we crudely simulate the off-mass-shell correction by multiplying the results of the B integral by $K^2(0)=0.74$, we find its contribution to be -8. This makes the left-hand side of Eq. (40) equal to -4.7.

Evaluation of resonance contributions to the right-hand side of Eq. (40) brings in considerable model dependence. Inclusion of $N^*(1236)$ and $N^{**}(1538)$ leads to $4m\tilde{D}(0)=-2.8H_1^2(0)-0.8$, where $H_1(0)$ is a parameter giving one of the couplings of the axial-vector-current N-N^* vertex. The second term is the contribution of N^{**} obtained from the known πNN^{**} coupling constant and the hypothesis of partially conserved axial-vector current (PCAC). (We have followed Radescu's notation. His convention for the sign of the B term, however, is opposite to ours.[19]) Satisfaction of the sum rule then requires that $H_1(0)^2\approx 1.4$. This is roughly consistent with some of the values obtained by Albright,[22] Schnitzer,[22] and Raman.[22]

Finally, one could argue that since only the N^* and N^{**} are included in the A-N scattering amplitude, the same thing should be done for the B integral for consistency. This gives $I=3.2$ and $I_1=26.2$. In this case we adjust N, in a one-trajectory approximation, to give $\alpha=0.5$ from the sum rules for I_1 and I_3. This leads to $N\approx 800$ MeV (corresponding to a c.m. energy ≈ 1550 MeV), which is entirely reasonable. Then we find the contribution from the B integral equal to 7.7. This value is quite close to the previous value obtained when other resonances were included and leads to $H_1(0)^2\approx 1.28$. The high-energy contribution to $\tilde{D}(0)$ is likely to be small, since there are indications that $D(\nu)\to 0$ at least as fast as $1/\nu$ as $\nu\to\infty$.[19] Thus we conclude that the sum rule would seem to be satisfied if the ANN^* coupling is large. The high-energy contributions are quite substantial. However, owing to the strong model dependence of the low-energy amplitude, no definite statements can be made until more is known about the coupling of the axial-vector current to the nucleon resonances.

[21] G. Hohler and R. Stauss (unpublished).

[22] C. H. Albright, Phys. Letters **24B**, 100 (1967); H. Schnitzer, Phys. Rev. **158**, 1471 (1967); K. Raman, *ibid.* **159**, 1501 (1967).

VI. CONCLUDING REMARKS

In I and in the present work, we have considered several examples of estimating the high-energy contributions to the current-algebra sum rules by using the finite-energy sum rules. In all the cases we found that the correction terms were of the right sign and of reasonable magnitudes. The main advantage of the method is that one uses the same low-energy data for all of the sum rules, and that there is no need to get residue functions from the high-energy fits. This is especially important when the cutoff energy is so low that one-trajectory dominance is approximate at best. In this case, one can evaluate contributions of the effective one-trajectory parameter with an effective-residue function. Of course in practical cases, when there are large cancellations between various amplitudes and where the experimental information is inadequate, a certain amount of caution is necessary. But in general, one can attempt to have a completely consistent dynamical scheme including both the current-algebra sum rules and the finite-energy sum rules, which *a priori* depend on entirely different assumptions.

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