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Partial-Wave Analysis of Many-Particle Scattering Amplitudes and Multi-Regge Theory*

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The multi-Regge prescription for the high-energy behavior of production amplitudes is derived starting from multiparticle partial-wave expansions and by employing analytic continuation in the total angular momentum of many-particle states.

I. INTRODUCTION

IN 1963 Kibble proposed a set of rules for determining the high-energy behavior of multiparticle production processes based on the exchange of many Regge poles.[1] Similar multi-Regge-pole prescriptions have been given independently by Ter-Martirosyan[2] and Zachariasen and Zweig[3] and developed by several other authors.[4] Recently Bali, Chew, and Pignotti,[5] amplifying the work of Toller,[6] have given the prescription for the high-energy behavior of multiparticle production processes a new basis. In their work the production amplitude is expanded in the irreducible representations of the group $O(2,1)$. This approach has the advantage that, in contrast to the usual Regge-pole type of analysis, the amplitude need never be continued into an unphysical region in order to obtain the asymptotic behavior. What is particularly appealing about this recent work is its great unity and generality, based as it is on the methods of group theory.

The purpose of this paper is to provide a formulation of the multi-Regge-pole hypotheses complementary to that of Bali, Chew, and Pignotti.[5] Here, we start not from the group $O(2,1)$ and the Toller expansion but from the group $O(3)$ and conventional partial-wave expansions. This approach leads directly to the deduction of asymptotic behavior from the assumptions of analyticity and boundedness of partial-wave scattering amplitudes in the total angular momentum j. Thus the familiarity and insight gained from the study of the three-dimensional rotation group and complex angular momentum in two-body reactions can be applied to the multi-Regge case. For example, in the present approach the so-called Toller variables just become Euler angles. For another example, when the external particles have spin, this approach leads directly to the asymptotic behavior of the convenient helicity amplitudes.

Unlike the Toller analysis, but like the usual Regge analysis, it is necessary to continue to a crossed channel to obtain the asymptotic behavior. One suspects, however, that the assumptions of analyticity and boundedness made in the Toller type of analysis are at least as strong as those made here.

The results presented here for the asymptotic behavior of the production amplitudes are identical with those implicit in the rules of Kibble[1] (at least for spinless particles), identical with those obtained explicitly by Bali, Chew and Pignotti,[5] and presumably the same as those obtained by some others authors.

The advantage of the present approach is that it makes clear the connection between multi-Regge behavior and analyticity in the total angular momentum of multiparticle partial-wave amplitudes.

II. PARTIAL-WAVE EXPANSION FOR THE MULTIPARTICLE SCATTERING AMPLITUDE

We rederive here a partial-wave expansion for a scattering amplitude with many incoming and many outgoing particles given by one of us.[7] The derivation we give here differs somewhat from that previously given by leaning heavily on the techniques of group theory.

The process we consider is N_A incoming particles with four-momenta $p_{A1}, p_{A2}, \cdots, p_{AN_A}$ scattering to produce a final state of N_B particles with four-momenta

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[1] T. W. B. Kibble, Phys. Rev. **131**, 2282 (1963).

[2] K. A. Ter-Martirosyan Zh. Eksperim. i Teor. Fiz. **44**, 341 (1963) [English transl.: Soviet Phys.—JETP **17**, 233 (1963); Nucl. Phys. **68**, 591 (1964).

[3] F. Zachariasen and G. Zweig, Phys. Rev. **160**, 1322 (1967); **160**, 1326 (1967).

[4] See Ref. 5 for other references.

[5] N. F. Bali, G. F. Chew, and A. Pignotti, Phys. Rev. **163**, 1572 (1967).

[6] M. Toller, Nuovo Cimento **37**, 631 (1965).

[7] J. B. Hartle, Phys. Rev. **134**, B612 (1964).

p_{B1}, p_{B2}, $\cdots$, p_{BN_B} (see Fig. 1). The two groups of particles will be called clusters A and B, respectively.

In what follows we work in the c.m. frame where the three-vector sum is zero.

$$\sum_{i=1}^{N_A} \mathbf{p}_{Ai} = \sum_{i=1}^{N_B} \mathbf{p}_{Bi} = 0.$$

It is useful to define a primitive plane-wave state similarly to that defined in Ref. 5. We designate it by $|B_z\rangle$ or $|A_z\rangle$, in which one designated member of the cluster has its three-momentum parallel to the z axis of a space-fixed coordinate system, and in which another designated member of the cluster has its three-momentum in the zy plane with positive y component.

It is not necessary to define the primitive states in exactly this fashion, but it is convenient to do so. Any method of specifying a body-fixed set of axes in the momenta of cluster A will yield a partial-wave expansion of the same general form as that obtained by this method (see Ref. 7). Indeed, this fact will be employed in Sec. IV.

An arbitrary plane-wave state can be brought into the above-defined primitive form by means of a rotation. Thus we write, for example,

$$|A_z\rangle = |\mathbf{p}_{A1}, \mathbf{p}_{A2}, \mathbf{p}_{A3} \cdots\rangle,$$

where $\mathbf{p}_{A1}$ is parallel to the z axis and $\mathbf{p}_{A2}$ is in the zy plane with positive y component. An arbitrary plane-wave state $|A\rangle$ can then be specified in terms of a rigid rotation $R(u)$ applied to one such primitive state:

$$|A\rangle = R(u)|A_z\rangle.$$

The basic states in terms of which we perform the partial-wave expansion are defined as follows:

$$|j_A m_A \mu_A t_{AB} a\rangle = (2j+1)^{1/2} \int du\, D_{m_A \mu_B}{}^{j_A *}(u) R(u) |A_z\rangle, \quad (2.1)$$

where the integral in (2.1) means integration over the entire rotation group. The rotations in (2.1) are parametrized as usual by means of the Euler angles:

$$R(u) = e^{-iJ_z\varphi} e^{-iJ_y\theta} e^{-iJ_z\psi}.$$

In Appendix A, the properties of the states defined by (2.1) are fully derived. Here we simply list these properties of the states $|j_A m_A \mu_A t_{AB} a\rangle$:

(1) The state has well-defined total angular momentum j_A and projection m_A on the z space-fixed axis.

(2) The quantum number μ_A is the projection of the total angular momentum along the direction of $\mathbf{p}_{A1}$, which is parallel to the z axis in the primitive state $|A_z\rangle$.

(3) The variable t_{AB} is the total four-momentum squared:

$$t_{AB} = (\sum_i p_{Ai})^2 = (\sum_i p_{Bi})^2,$$

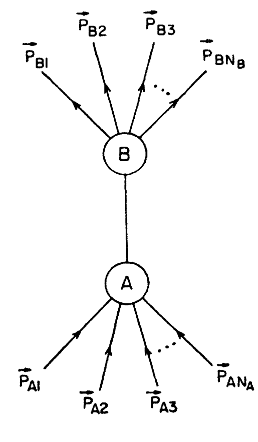


FIG. 1. Multiparticle scattering process.

and corresponds to the total c.m. energy squared, in particular $t_{AB} > 0$.

(4) The symbol a denotes the "internal" variables needed to specify the configuration of cluster A in addition to the energy which is given by t_{AB}. For example, if cluster A consists of three particles, the three-vectors $\mathbf{p}_{A1}$, $\mathbf{p}_{A2}$, and $\mathbf{p}_{A3}$ form a triangle. In this case, a consists of two variables which might be specified by two angles of the triangle (such as $\mathbf{p}_{A1} \cdot \mathbf{p}_{A2}$ and $\mathbf{p}_{A1} \cdot \mathbf{p}_{A3}$), or, alternatively, by two Lorentz invariants [such as $(p_{A1}+p_{A2})^2$ and $(p_{A1}+p_{A3})^2$]. Since t_{AB} is given, the triangle is then completely specified. Generally, a consists of $3N_A - 7$ variables.

This development is similar to the helicity formalism of Jacob and Wick,[8] which deals with two-body scattering in the presence of spin. In their case the helicity is simply the projection of the total angular momentum on the only natural body-fixed axis for the two-body problem—the relative momentum.

The next step in obtaining a partial-wave expansion comes by inverting the relationship given in (2.1), thereby obtaining an expression for the plane-wave states as a superposition of the angular-momentum states. To this end we perform the summation:

$$\sum_{\mu_A=-\infty}^{\infty} \sum_{j_A=|\mu_A|}^{\infty} \sum_{m_A=-j_A}^{j_A} (2j+1)^{1/2} D_{m_A \mu_A}{}^{j_A}(u') \times |j_A m_A \mu_A t_{AB} a\rangle.$$

Using the group property,

$$\sum_{m_A=-j_A}^{j_A} D_{m_A \mu_A}{}^{j_A}(u') D_{m_A \mu_A}{}^{j_A *}(u) = D_{\mu_A \mu_A}{}^{j_A}(u^{-1}u'), \quad (2.2)$$

and the identity,

$$\sum_{\mu_A=-\infty}^{\infty} \sum_{j_A=|\mu_A|}^{\infty} (2j_A+1) D_{\mu_A \mu_A}{}^{j_A}(u^{-1}u') = \delta(u^{-1}u'), \quad (2.3)$$

[8] M. Jacob and G. C. Wick, Ann. Phys. (N Y) 7. 404 (1959).

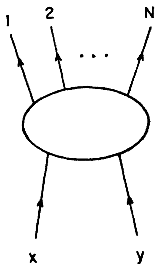


FIG. 2. Diagram for production process.

we obtain from (2.1)

$$|A\rangle = R(u)|A_z\rangle = \sum_{j_A=0}^{\infty} \sum_{\mu_A=-j_A}^{j_A} \sum_{m_A=-j_A}^{j_A} (2j_A+1)^{1/2} D_{m_A\mu_A}{}^{j_A}(u) \times |j_A m_A \mu_A t_{AB} a\rangle. \quad (2.4)$$

We can now employ (2.4) to obtain a partial-wave expansion for $\langle B|T|A\rangle$, the connected part of the multiparticle scattering amplitude. No loss of generality results by evaluating $\langle B|T|A\rangle$ in a frame in which the state $|B\rangle$ is simply $|B_z\rangle$. Thus we wish to expand the matrix element $\langle B_z|TR(u)|A_z\rangle$, where now the rotation $R(u)$ may be thought of as expressing the relative orientation of the A and B clusters with respect to their primitive states. The expansion for the state $|B_z\rangle$ is simply

$$|B_z\rangle = \sum_{j_B=0}^{\infty} \sum_{\mu_B=-j_B}^{j_B} (2j_B+1)^{1/2} |j_B \mu_B \mu_B t_{AB} b\rangle, \quad (2.5)$$

since $D_{m\mu}{}^j(I) = \delta_{m\mu}$. In (2.5), the equality of the helicity μ_B with the z projection of j_B is a special property of the primitive state $|B_z\rangle$. The state $R(u)|A_z\rangle$ is given in its expanded form by (2.4).

Using the rotational invariance of the operation T which implies the conservation of angular momentum j and its z projection m, we have finally from (2.4) and (2.5):

$$\langle B|T|A\rangle = \sum_{j=0}^{\infty} \sum_{\mu_B=-j}^{j} \sum_{\mu_A=-j}^{j} (2j+1) T_{\mu_B\mu_A}{}^j(b,t_{AB},a) D_{\mu_B\mu_A}{}^j(u)$$

$$= \sum_{\mu_B,\mu_A=-\infty}^{\infty} e^{-i(\mu_B\varphi+\mu_A\psi)} \sum_{j=\max\{|\mu_A|,|\mu_B|\}}^{\infty} (2j+1) \times T_{\mu_B\mu_A}{}^j(bt_{AB}a) d_{\mu_B\mu_A}{}^j(\cos\theta). \quad (2.6)$$

Equation (2.6) includes the definitions

$$T_{\mu_B\mu_A}{}^j(b,t_{AB},a) = \langle j m \mu_B t_{AB} b|T|j m \mu_A t_{AB} a\rangle, \quad (2.7)$$
$$D_{\mu_B\mu_A}{}^j(u) = e^{-i(\mu_B\varphi+\mu_A\psi)} d_{\mu_B\mu_A}{}^j(\cos\theta).$$

As a consequence of rotational invariance, the amplitude $T_{\mu_B\mu_A}{}^j$ has no dependence on the azimuthal quantum number m. For the case of two-body scattering, (2.6) just becomes the ordinary partial-wave expansion.

III. THE COMPLETE MULTI-REGGE FORMULA FOR SPINLESS PARTICLES

In this section we apply the partial-wave expansion (2.7) to derive a multi-Regge formula for the production process (see Fig. 2),

$$x+y \rightarrow 1+2+\cdots+N,$$

where the corresponding particle four-momenta are p_x, p_y, p_1, $p_2, \cdots$, p_N, and all the particles are spinless. What we seek is a Regge-pole expansion that is valid when the subenergies $S_{ij} = (p_i+p_j)^2$ are all large.

To begin, we consider the crossed-channel process $x+\bar{1} \rightarrow \bar{y}+2+3+\cdots+N$ (Fig. 3). According to (2.6), the partial-wave expansion for this process can now be written

$$T = \sum_{\mu_1=-\infty}^{\infty} e^{-i\mu_1\varphi_1} \sum_{j=|\mu_1|}^{\infty} (2j+1) \times T_{\mu_1}{}^j(V_1,t_1) d_{\mu_1 0}{}^j(\cos\theta_1). \quad (3.1)$$

Here V_1 denotes the variables that define the final cluster $(p_2,\cdots,p_n,p_{\bar{y}})$, and $t_1 = (p_x - p_1)^2$. The absence in (3.1) of a second μ index and a corresponding dependence on the third Euler angle ψ_1 results from the special case of the initial state being a two-particle configuration of spinless particles. The two Euler angles φ_1 and θ_1 specify the orientation of the initial state with respect to the primitive final state $|p_2,p_3,\cdots,p_{\bar{y}}\rangle$, which is analogous to the $|B_z\rangle$ state of Sec. II (for $\varphi_1 = \theta_1 = 0$, $\mathbf{p}_{\bar{1}} \| \mathbf{p}_2$).

One now makes the crucial assumption that $T_{\mu_1}{}^j$ in (3.1) possesses a continuation in j with a Regge pole at $j = \alpha_1(t_1)$ with the appropriate quantum numbers to couple to $x+\bar{1}$ which controls the asymptotic behavior.[9] Furthermore, one assumes that the residue of the Regge pole factorizes in its dependence on the variables characterizing the initial and final states. These assumptions are direct and natural generalizations of two-body Regge theory. Thus we derive the following large $\cos\theta$, behavior of (3.1):

$$T \xrightarrow[\cos\theta_1\to\infty]{} \sum_{\mu_1=-\infty}^{\infty} e^{-i\mu_1\varphi_1} A_{\mu_1}{}^{\alpha_1}(V_1,t_1) \times f_1(t_1)(\cos\theta_1)^{\alpha_1(t_1)}, \quad (3.2)$$

with φ_1, t_1, and V_1 all held fixed.

[9] The assumption that the multiparticle partial-wave amplitude T^J has a continuation in J with Regge poles which control the asymptotic behavior will be discussed in a forthcoming paper. [T. K. Gaisser and C. E. Jones (to be published)]. We only mention here that it is possible to define a generalization of the Froissart-Gribov amplitude to many-particle amplitudes. In particular, the kinematic singularities in J of the d_J functions are combined with corresponding singularities in T_J in a way analogous to that in the case of problems involving spin.

The factor $f_1(t_1)$ in (3.2) is essentially the square root of β, the two-particle Regge residue for the process $x+\bar{1}\to x+\bar{1}$, except that certain kinematic factors including $(\sin\pi\alpha_1)^{-1}$ have been absorbed. The function $A_{\mu_1}{}^{\alpha_1}$ is the other half of the factored Regge-pole residue at $j=\alpha_1$. It has the physical interpretation of being the amplitude for the decay process $(1)\to 2+3+\cdots+N+\bar{y}$, where particle (1) is a Reggeon at rest with mass $t_1{}^{1/2}$, spin α_1, and projection μ_1 of its spin along the direction $\mathbf{p}_2$. The asymptotic variable $\cos\theta_1$ in (3.2) is linearly related to s_{12} by

$$s_{12}=m_1{}^2+m_2{}^2-2[E_1E_2-(E_1{}^2-m_1{}^2)^{1/2}\times(E_2{}^2-m_2{}^2)^{1/2}\cos\theta_1], \quad (3.3)$$

where

$$E_1=\frac{t_1-m_x{}^2+m_1{}^2}{2t_1{}^{1/2}} \quad\text{and}\quad E_2=\frac{t_1-t_2+m_2{}^2}{2t_1{}^{1/2}}.$$

It is important to verify that (3.2) gives a well-defined asymptotic limit by showing that there are no variables left over. There are $3N-7$ variables in V_1 plus t_1, φ_1, and θ_1, giving a total of $3(N+2)-10$, which is the correct number of variables.

The decay amplitude $A_{\mu_1}{}^{\alpha_1}$ may be connected with a c.m. production amplitude by first making a complex Lorentz transformation in the direction $\mathbf{p}_2$ to the frame where $\mathbf{p}_{(1)}-\mathbf{p}_2$ vanishes, $p_{(1)}$ being the momentum of the Reggeon. Under this transformation, μ_1 remains the projection of the angular momentum in the direction $\mathbf{p}_2$. The amplitude may then be analytically continued in the four-vector p_2 along a path which takes $p_2\to -p_2$. The quantity μ_1 becomes the helicity of particle (1) and $A_{\mu_1}{}^{\alpha_1}$ thus represents the c.m. scattering amplitude for the process

$$(1)+\bar{2}\to 3+4+\cdots+N+\bar{y},$$

where (1) denotes the Reggeon with helicity μ_1.

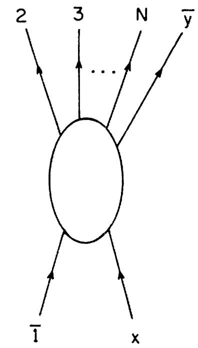
In the c.m. system of (1) and $\bar{2}$, we can apply the multiparticle partial-wave expansion (2.6) to $A_{\mu_1}{}^{\alpha_1}$:

$$A_{\mu_1}{}^{\alpha_1}=\sum_{\mu_2=-\infty}^{\infty} e^{-i(\mu_2\varphi_2+\mu_1\psi_2)}\sum_{j=|\mu_2|}^{\infty}(2j+1)\times A_{\mu_2\mu_1}{}^{j}(V_2,t_2,\alpha_1,t_1)d_{\mu_2\mu_1}{}^{j}(\cos\theta_2), \quad (3.4)$$

where $t_2=(p_x-p_1-p_2)^2$, V_2 denotes the variables in the cluster $(p_3\cdots p_N,p_y)$, and the Euler angles φ_2, θ_2, and ψ_2 specify the orientation in the c.m. system of the initial state $(1)+\bar{2}$ with respect to the primitive final state $|\mathbf{p}_3\cdots\mathbf{p}_{\bar{y}}\rangle$ (with φ_2, θ_2, and ψ_2 all zero, $\mathbf{p}_{\bar{2}}\|\mathbf{p}_3$). Note that the dependence on the third Euler angle ψ_2 must now be kept in general, because one of the two initial particles has spin.

The partial-wave amplitude $A_{\mu_2\mu_1}{}^{j}$ has a Regge pole at $j=\alpha_2(t_2)$ with quantum numbers of $(1)+\bar{2}$ and $A_{\mu_1}{}^{\alpha_1}$

FIG. 3. Crossed-channel process corresponding to Fig. 2.



has the asymptotic behavior in $\cos\theta_2$ [in analogy with (3.2)]

$$A_{\mu_1}{}^{\alpha_1}\xrightarrow[\cos\theta_2\to\infty]{}\sum_{\mu_2=\infty}^{\infty} e^{-i\mu_2\varphi_2}A_{\mu_2}{}^{\alpha_2}(t_2,V_2)f_2(t_2,t_1,\mu_1)\times e^{-i\mu_1\psi_2}(\cos\theta_2)^{\alpha_2(t_2)}. \quad (3.5)$$

As before, $\cos\theta_2$ can be related to s_{23} by a formula analogous to (3.3).

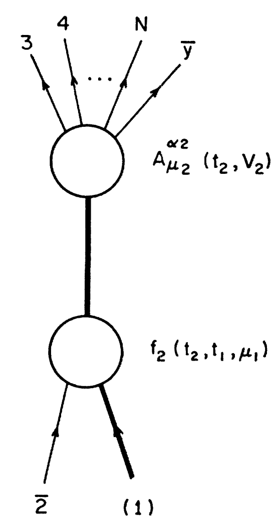
It is important to observe the form of the factorized residue of the Regge pole at $j=\alpha_2(t_2)$, as expressed by (3.5) and shown in Fig. 4. That $f_2(t_2t_1\mu_1)$ does not depend upon μ_2 is part of the factorization assumption, since μ_2 is quantum number pertaining to the cluster V_2. The function $A_{\mu_2}{}^{\alpha_2}$, just as before, is the decay amplitude

$$(2)\to 3+4+\cdots+N+\bar{y},$$

where (2) is the Reggeon α_2. This form of factorization may be shown to be valid for Feynman diagrams at physical-particle poles.

By repeating the above process $N-1$ times, we arrive at the complete multi-Regge formula for the original amplitude T, $x+y\to 1+2+\cdots+N$, where all the

FIG. 4. Factorized Regge residue.



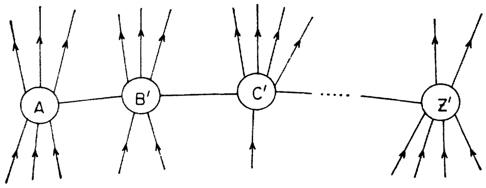


FIG. 5. More general cluster decomposition.

$\cos\theta_i$ (or, alternatively, all the subenergies s_{ij}) are large

$$T \xrightarrow[\cos\theta_i\to\infty]{} F_1(t_1)(\cos\theta_1)^{\alpha_1(t_1)}F_2(t_2,t_1,\omega_1)(\cos\theta_2)^{\alpha_2(t_2)}$$
$$\times F_3(t_3,t_2,\omega_2)\cdots(\cos\theta_{N-1})^{\alpha_{N-1}(t_{N-1})}F_N(t_{N-1}), \quad (3.6)$$

where

$$\omega_i=\psi_{i+1}+\varphi_i,$$
$$t_i=(p_x-p_1-p_2-\cdots-p_i)^2;$$
$$F_1(t_1)=f_1(t_1),$$
$$F_N(t_{N-1})=f_N(t_{N-1}),$$

and

$$F_{k+1}(t_{k+1},t_k,\omega_k)=\sum_{\mu_k=-\infty}^{\infty} e^{-i\mu_k\omega_k}f_{k+1}(t_{k+1},t,\mu_k).$$

Being held fixed as the limit is taken are the $N-2$ angles ω_i and the $N-1$ momentum transfers t_i. This, together with the $N-1$ $\cos\theta_i$, makes a total of $3N-4=3(N+2)-10$ variables, which is the correct number.

IV. MORE GENERAL CLUSTER DECOMPOSITIONS

In this section we illustrate the generality of the theory developed in the previous sections for spinless external particles by obtaining a multi-Regge formula, not for the complete decomposition of Sec. III, but for a cluster decomposition illustrated in Fig. 5. In that figure the lines connecting the clusters represent Regge poles with appropriate quantum numbers and serve to define the momentum-transfer variables which are held fixed in the multi-Regge limit. The variables which are becoming large in this limit are then the cosines of the scattering angles connecting successive clusters in the crossed channel. These will be defined more precisely in what follows.

In order to obtain the multi-Regge formula, we begin by decomposing the external particles into two clusters, as shown in Fig. 6. To make a partial-wave expansion it is convenient to cross to a region where all the particles in cluster A are incoming and all the particles in cluster B are outgoing. The partial-wave expansion then has the form of Eq. (2.6) where the Euler angles express the relative orientation of clusters A and B in the c.m. system. These will be denoted by $\varphi_{AB'}$, $\theta_{AB'}$, and $\psi_{AB'}$, and may be defined by specifying two-body fixed frames as follows. The body-fixed frame for cluster A may be chosen arbitrarily and this arbitrariness simply corresponds to the complete freedom one has in choosing the primitive plane-wave states discussed in Sec. II. In particular, one need not assume that any particle of cluster A has its three-momentum in the z direction. The body-fixed frame of cluster B is picked more specially because of the later decomposition of cluster B into the B', C', $\cdots$ of Fig. 5. The body-fixed B frame, while defined in the over-all c.m. system, must be specified in terms of only those three-vectors corresponding to the particles of cluster B'. Otherwise the frame is arbitrary.

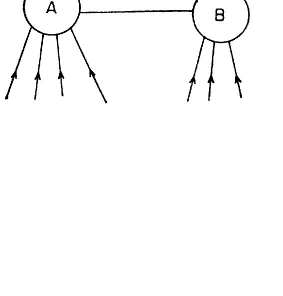


FIG. 6. Two-cluster decomposition.

With the assumption of Regge poles in the partial-wave amplitudes of (2.6) we arrive at the following asymptotic behavior in $\cos\theta_{AB'}$, in analogy with (3.2);

$$T\to\sum_{\mu_1=-\infty}^{\infty} e^{-i\mu_1\varphi_{AB'}}A_{\mu_1}{}^{\alpha_1}(b,t_{AB'})\sum_{\mu_0=-\infty}^{+\infty} e^{-i\mu_0\psi_{AB'}}$$
$$\times f_{\mu_0}{}^{\alpha_1}(a,t_{AB'})(\cos\theta_{AB'})^{\alpha_1(t_{AB'})}, \quad (4.1)$$

where $f_{\mu_0}{}^{\alpha_1}$ is essentially the complex conjugate of the decay amplitude for the Reggeon α_1 going into cluster A.

As before, the appropriately continued $A_{\mu_1}{}^{\alpha_1}$ can be identified with the amplitude for the process

$$(1)+B'\to C=C'+D'+\cdots+Z', \quad (4.2)$$

where where the primed letters in (4.2) stand for the clusters of particles in Fig. 5 and (1) is the Reggeon of mass $t_1^{1/2}$ and spin α_1. For (4.2), we pick a region where the particles in cluster B' are all incoming and those in C are all outgoing.

We now go to the c.m. system for process (4.2) and, just as before, we select an arbitrary body-fixed frame for the incoming particle cluster $(1)+B'$, and a body-fixed frame for the cluster C, which is defined purely in terms of the three-vectors of cluster C'. Again we perform an expansion of the type (2.6), now in the variable $\cos\theta_{B'C'}$. It is important to note that if cluster B' consists of more than one particle, then μ_1 is not the helicity of the Reggeon (1) in the c.m. system of process (4.2), but is a spin label specifying the state. This fact in no way invalidates the use of expansion (2.6) for $A_{\mu_1}{}^{\alpha_1}$, but it must be remembered that μ_1', the projection of the total angular momentum of the cluster $(1)+B'$ along the body-fixed z axis, is no longer the same as μ_1, and consequently the partial-wave amplitude for $A_{\mu_1}{}^{\alpha}$ will depend both on μ_1 and μ_1'. This fact has the important consequence that the final amplitude depends upon both $\psi_{B'C'}$ and $\varphi_{AB'}$, not only their sum as in Sec. III with one-particle clusters. In fact, if the end clusters in Fig. 5 have three or more particles and if each internal clusters have two or more particles, then the amplitude depends on all $3N$ Euler angles, where $N+1$ is the total number of clusters. In the case just described, the asymptotic formula for Fig. 5, analogous to (3.6) for $\cos\theta_{AB'}$, $\cdots$, all large, is

$$T \to F_1(t_{AB'},a,\psi_{AB'})(\cos\theta_{AB'})^{\alpha_1(t_{AB'})} \times F_2(t_{B'C'},t_{AB'},b',\psi_{B'C'},\varphi_{AB'})(\cos\theta_{B'C'})^{\alpha_2(t_{A'C'})}\cdots \times(\cos\theta_{Y'Z'})^{\alpha_N(t_{Y'Z'})}F_{N+1}(t_{Y'Z'},z',\varphi_{Y'Z'}). \quad (4.3)$$

Exactly which particles are incoming or outgoing in Fig. 5 can be determined by the values of the internal-cluster variables a, b', $\cdots$, z'.

V. ANALYSIS WITH SPIN

The multi-Regge analysis for amplitudes describing the scattering of particles with spin may be found by a similar method to that given above for the spinless case. The only additional complication comes from the crossing matrices.

Consider the process shown in Fig. 2, and suppose that the external particles have spins s_x, s_1, $\cdots$, s_N, s_y, helicities λ_x, λ_1, $\cdots$, λ_N, λ_y, and that the amplitude is written $T_{\lambda_x\lambda_1\cdots\lambda_N\lambda_y}$. This amplitude is written in the c.m. frame for which

$$\mathbf{p}_x+\mathbf{p}_y=\mathbf{p}_1+\cdots+\mathbf{p}_N=0. \quad (5.1)$$

It is to be related by crossing to that for the process $\bar{1}+x\to 2+\cdots+\bar{y}$, and the result is to be partial-wave-analyzed as in Sec. II. The method for finding the crossing relation has been given by Trueman and Wick[10] and consists of two steps:

(1) Make a Lorentz transformation from the c.m. frame of the xy system to a frame where the total c.m. momentum has an arbitrary value [i.e., the momenta are not restricted by Eq. (5.1)]. The amplitude $T_{\lambda_x\lambda_1\cdots\lambda_N\lambda_y}$ is related to the "generalized helicity amplitude" $T'_{\mu_x\cdots\mu_y}$ in the arbitrary frame by

$$T_{\lambda_x\lambda_1\cdots\lambda_N\lambda_y}=\Sigma_{\mu_x\mu_1\cdots\mu_N\mu_y}\eta(\lambda_x\cdots\lambda_y,\mu_x\cdots\mu_y)\,, \quad D_{\mu_x\lambda_x}{}^{s_x}(r_x)\cdots D_{\mu_y\lambda_y}{}^{s_y}(r_y)T'_{\mu_x\mu_1\cdots\mu_N\mu_y}\,,$$

where r_i is the Wigner rotation associated with the Lorentz transformation and the momentum p_i, and η is a phase which does not concern us here.

(2) An analytic continuation is performed to the c.m. frame of the crossed channel where $-p_1$ and $-p_y$ are then timelike and $T'_{\mu_x\cdots\mu_y}$ becomes the helicity amplitude for the process $\bar{1}+x\to 2+\cdots+\bar{y}$ in its c.m. frame, the continued rotation matrices becoming the crossing matrices. Alternatively, one can make a complex Lorentz transformation to the frame where $\mathbf{p}_x-\mathbf{p}_1=0$, and then reach the crossed channel by a continuation in the time components of the four-vectors.

[10] T. L. Trueman and G. C. Wick, Ann. Phys. (N. Y.) **26**, 322 (1964).

For our purposes we would follow this procedure by (a) expressing the crossing matrices in terms of the invariants, (b) a partial-wave expansion of $T_{\mu_x\cdots\mu_y}{}^1$, (c) a Regge derivation of the asymptotic behavior in the first subenergy, and (d) a continuation of the invariants back to the region of interest for the process $x+y\to 1+\cdots+N$.

All the succeeding steps in obtaining the multi-Regge formula as described in Sec. III involve considering Reggeon decay amplitudes of the form $(j-1)\to j+\cdots+\bar{y}$. As discussed in Sec. III, the crossing of particle j will not affect the helicity of the decaying Reggeon so that crossing matrices are needed only for external particles. Proceeding in this way one arrives at the complete multi-Regge formula for particles with spin:

$$T_{\lambda_x\lambda_1\cdots\lambda_N\lambda_y}\to \Sigma_{\mu_x\mu_1\cdots\mu_N\mu_y}D_{\mu_x\lambda_x}{}^{s_x}(r_x)D_{\mu_1\lambda_1}{}^{s_1*}(r_1)\cdots \times D_{\mu_y\lambda_y}{}^{s_y}(r_y)F_1(\mu_1,\mu_x,t_1)(\cos\theta_1)^{\alpha_1(t_1)} \times F_2(\mu_2 t_2 t_1 \omega_1)(\cos\theta_2)^{\alpha_2(t_2)}\cdots \times(\cos\theta_{N-1})^{\alpha_{N-1}(t_{N-1})}F_N(\mu_N\mu_y t_{N-1}).$$

Here successive rotation matrices affecting the same particle have been combined. The Wigner rotations r_x, r_1, $\cdots$, r_N, r_y are expressible as

$$r_i=h^{-1}(l_ip_i)l_ih(p_i),\quad i=x,1,\cdots,N,y \quad (5.2)$$

where $h^{-1}(q)$ is a standard Lorentz transformation to the frame in which q is at rest and l_i is the Lorentz transformation from the frame where $\mathbf{p}_x+\mathbf{p}_y$ vanishes to the following:

(a) for $i=x$ to the frame where $\mathbf{p}_x-\mathbf{p}_1$ vanishes;

(b) for $i=1$, $\cdots$, $N-1$ to the frame where $\mathbf{p}_x-\mathbf{p}_1-\cdots-\mathbf{p}_i$ vanishes;

(c) for $i=N$, y to the frame where $\mathbf{p}_x-\mathbf{p}_1-\cdots\mathbf{p}_{N-1}$ vanishes.

It is important to note that the Lorentz transformations l_i will be complex transformations since the vectors $p_x-p_1\cdots-p_i$ are in general spacelike in the physical region for the process $x+y\to 1+\cdots+N$. Even though the Lorentz transformations are complex, it is shown in Appendix B that the r_i remain real rotations. This means that the elements of the crossing matrices are each bounded by 1 so that the asymptotic behavior in the $\cos\theta_i$ remains the same as in the spinless case.

While the power-law dependence in the $\cos\theta_i$ remains the same as in the spinless case, the question naturally arises as to whether the same factorization of the residues holds. It is shown in Appendix B that if the energy of the ith particle becomes large in the multi-Regge limit ($s_{ij}\to\infty$, t_i, ω_i fixed), then the crossing angles are functions of the invariants t_i, t_{i-1} (in the case of particles x and 1 the crossing angle is a function of t_1 alone and for particles Y and N of t_{N-1} alone). Thus if the multi-Regge limit is taken so that *all* the particle energies become large, then the amplitude will factor as in the spinless case:

$$T_{\lambda_x\lambda_1\cdots\lambda_N\lambda_y} \to G_1(\lambda_1,\lambda_x,t_1)(\cos\theta_1)^{\alpha_1(t_1)}G_2(\lambda_2 t_2 t_1\omega_1) \times(\cos\theta_2)^{\alpha_2(t_2)}\cdots(\cos\theta_{N-1})^{\alpha_{N-1}(t_{N-1})}F_N(\lambda_N\lambda_y,t_{N-1}).$$

It is not necessary, however, to take the multi-Regge limit in a way in which all the particle energies become large. It is possible (as is shown in Appendix B and Refs. 3) to have at most one of the particle energies small and still have all the s_{ij} large, and the t_i and ω_i fixed. If this is the case, the amplitude will not factor in the way described above.

VI. CONCLUSION

The multi-Regge analysis for production amplitudes has been developed using partial-wave analysis techniques. While proceeding in this way leads to the same general prescriptions as those obtained by previous workers using different techniques, this approach does have some advantages. At each stage the asymptotic behavior in a subenergy is derived from an analyticity and boundedness assumption of a scattering amplitude in an angular-momentum variable. This allows the intuition built up about complex angular momentum to be applied directly and may provide a somewhat simpler kinematic understanding of the multi-Regge decomposition than that provided by other derivations.

APPENDIX A: MULTI-PARTICLE ANGULAR-MOMENTUM STATES

In this appendix, we wish to derive the properties of the states

$$|j_A m_A \mu_A t a\rangle = (2j+1)^{1/2}\int du\, D_{m_A\mu_A}{}^{j_A}(u)R(u)|A_z\rangle. \quad \text{(A1)}$$

We wish to prove that they are eigenstates with (i) total angular momentum $j_A(j_A+1)$, (ii) z projection of angular momentum m_A, and (iii) projection of angular momentum along p_{1A} of μ_A.

Properties (i) and (ii) can be proved by operating on both sides of (A1) with an arbitrary rotation $R(\bar{u})$

$$R(\bar{u})|j_A m_A \mu_A t a\rangle = (2j_A+1)^{1/2}\int du\, D_{m_A\mu_A}{}^{j_A*}(u)R(\bar{u}u)|A_z\rangle$$

$$= (2j_A+1)^{1/2}\int du\, D_{m_A\mu_A}{}^{j_A*}(\bar{u}^{-1}u)R(u)|A_z\rangle. \quad \text{(A2)}$$

We state here for reference the invariance property of group integration which we shall use repeatedly in the arguments that follow. If $f(u)$ is a continuous function defined on the manifold of the rotation group, then

$$\int du\, f(u) = \int du\, f(\bar{u}u) = \int du\, f(u\bar{u}),$$

where the integration is carried out over the entire group manifold and $\bar{u}$ is some element of the group.

The second equality in (A2) follows from the invariance of group integration. However, since

$$D_{m_A\mu_A}{}^{j_A*}(\bar{u}^{-1}u) = \sum_\alpha D_{m_A\alpha}{}^{j_A*}(\bar{u}^{-1})D_{\alpha\mu_A}{}^{j_A*}(u)$$

$$= \sum_\alpha D_{\alpha m_A}{}^{j_A}(\bar{u})D_{\alpha\mu_A}{}^{j_A*}(u), \quad \text{(A3)}$$

it follows that

$$R(\bar{u})|j_A m_A \mu_A t a\rangle = \sum_\alpha D_{\alpha m_A}{}^{j_A}(\bar{u})|j_A \alpha \mu_A t a\rangle. \quad \text{(A4)}$$

The behavior of the state $|j_A m_A \mu_A t a\rangle$ under rotations, as expressed by (A4) proves that this state has a total angular momentum j_A and z projection m_A.

To establish property (iii), we first form a state $|\mu_A\rangle$, defined as follows:

$$|\mu_A\rangle = \frac{1}{2\pi}\int_0^{2\pi} d\varphi\, e^{i\mu_A\varphi}e^{-i\varphi J_z}|A_z\rangle. \quad \text{(A5)}$$

We now show that $|\mu_A\rangle$ is an eigenstate of J_z with eigenvalue μ_A, and consequently an eigenstate of $\mathbf{p}_{1A}\cdot\mathbf{J}$. To see this, we transform the state $|\mu_A\rangle$ as follows:

$$e^{i\beta J_z}|\mu_A\rangle = \frac{1}{2\pi}\int_0^{2\pi} d\varphi\, e^{i\mu_A\varphi}e^{-i(\varphi-\beta)J_z}|A_z\rangle$$

$$= e^{i\beta\mu_A}\frac{1}{2\pi}\int_0^{2\pi} d\varphi\, e^{i\mu_A\varphi}e^{-i\varphi J_z}|A_z\rangle, \quad \text{(A6)}$$

again using the invariance property of the group integration. Thus,

$$e^{i\beta J_z}|\mu_A\rangle = e^{i\beta\mu_A}|\mu_A\rangle, \quad \text{(A7)}$$

and $|\mu_A\rangle$ is an eigenstate of J_z. It is also an eigenstate of $\mathbf{p}_{A1}\cdot\mathbf{J}$ since $\mathbf{p}_{1A}$ is along the z axis in the state $|A_z\rangle$ and, according to (A5), only rotations around the z axis are involved.

Now we wish to form states

$$(2j_A+1)^{1/2}\int du\, D_{m_A\mu_A}{}^{j_A*}(u)R(u)|\mu_A\rangle. \quad \text{(A8)}$$

Since the operator $\mathbf{p}_{1A}\cdot\mathbf{J}$ commutes with the rotation $R(u)$, these states (A8) are clearly eigenstates of $\mathbf{p}_1\cdot\mathbf{J}$, corresponding to a projection of the total angular momentum along $\mathbf{p}_1$ of μ_A. We now show that the states (A8) are the same as the states (A1), which completes the demonstration. We have from (A5)

$$(2j_A+1)^{1/2}\int du\, D_{m_A\mu_A}{}^{j_A*}(u)R(u)|\mu_A\rangle$$
$$=(2j_A+1)^{1/2}\frac{1}{2\pi}\int_0^{2\pi}d\varphi\int du\, D_{m_A\mu_A}{}^{j_A*}(u)\times e^{i\mu_A\varphi}R(u)e^{-i\varphi J_z}|A_z\rangle$$
$$=|j_Am_A\mu_A t a\rangle, \qquad \text{(A9)}$$

where again we have used the invariance property of the group integration.

APPENDIX B: CROSSING MATRICES IN THE MULTI-REGGE LIMIT

We first give the argument that the operator defined by Eq. (5.2) still is a rotation even when the Lorentz transformation l_i is complex. Denote by q_i the momentum-transfer variable (typically $p_x-p_1-\cdots-p_i$), which l_i transforms to rest. Consider Eq. (5.2) as a 4×4 matrix equation first for timelike q_i, where r_i is then a genuine Wigner rotation. As such it is real and orthogonal, $(\tilde{r}_i)r_i=1$. Now, continue the components of q_i until the vector assumes its spacelike value. The r_i continues to be orthogonal, and operates only in the 3×3 spatial subspace. If the matrix elements are real at the end of the continuation, it will be a rotation. To see that this is the case, we note that we can pick l_i to be the boost in the direction of $\mathbf{q}_i$ which transforms q_i to rest. The only two Lorentz transformations in Eq. (5.2) which involve q are then l_i and $h^{-1}(l_ip)$, and it suffices to verify that their product is real. A consequence of rotational invariance is that this can be done in the frame in which $\mathbf{q}_i$ points in the z axis and $\mathbf{p}_i$ lies in the xz plane. This is sufficient because in continuing q_i from a timelike to a spacelike vector only the component q^t need be varied through real values and $\mathbf{q}$ may be left unchanged. It is then an easy matter to write out the matrices for the two boosts and one rotation involved and verify explicitly that the product is real.

We now consider the crossing matrices for a complete multi-Regge decomposition (Fig. 2) in which all the external particle energies become large. In this case, $q_i=(q_i^t,\mathbf{q}_i)$ is the momentum-transfer four-vector such that $q_i^2=t_i$. From the conservation of energy it is clear that all the q_i^t must become large in this multi-Regge limit with the possible exception of one which we denote q_a. Let us consider a typical vertex with momenta p, q, $p+q$, in which all of the vectors are becoming infinite with p^2, q^2, and $(p+q)^2$ finite. Then

$$\mathbf{p}\cdot\mathbf{q}=(1/|\mathbf{p}||\mathbf{q}|)[p^2+q^2-(p+q)^2+p^tq^t]\to 1\,.$$

One concludes, therefore, that in the multi-Regge limit where all the particle energies become large, all the vectors $(\mathbf{p}_x,\mathbf{p}_i,\mathbf{p}_y,\mathbf{q}_i)$ become collinear or anticollinear except for the q_a corresponding to the one q_a which may remain finite. Thus in the infinite-energy multi-Regge limit, all the crossing matrices as described in Sec. V, except that for the particle a, become trivial since no directions need be changed in the crossing process. For particle a, the Wigner rotation associated with the crossing is in the direction $\mathbf{q}_a\times\mathbf{p}_a$, through an angle obtained by appropriately continuing a formula given by Trueman and Wick[10]:

$$\cos\Omega=\frac{\cosh\sigma\cosh\sigma'-\cosh\rho}{\sinh\sigma\sinh\sigma'}\,.$$

Here,

$$\tanh\sigma=p_a{}^t/|\mathbf{p}_a|\,,\quad \tanh\sigma'=p_a{}^{t\prime}/|\mathbf{p}_a{}'|\,,\quad \tanh\rho=q_a{}^t/|\mathbf{q}_a|\,,$$

and $(p_a{}^{t\prime},\mathbf{p}_a{}')$ are the components of the four-vector p_a in the frame in which $\mathbf{q}_a=0$. In the limit as $p^t\to\infty$, one finds

$$\cos\Omega=\frac{q_a\cdot p_a}{[q_a\cdot p_a-q_a{}^2m^2]^{1/2}}=\frac{t_{a-1}-t_a-m^2}{[(t_{a-1}-t_a-m^2)^2-4m^2t_a]^{1/2}}\,.$$

Since without loss of generality we may take $\mathbf{q}_a$ in the z direction and $\mathbf{p}_a$ in the xz plane, the rotation becomes one about the y axis by an angle which depends only on the invariants t_{a-1} and t_a. This is enough for the factorization of Sec. V to hold.

It is not necessary for all the particle energies to become large in the multi-Regge limit ($s_{ij}\to\infty$, t_i, ω_i fixed). It is easy to see that one can have p_a, q_a, and q_{a-1} finite for some a and still satisfy the multi-Regge constraints. In fact, at most one p_a can remain finite, since this condition implies that q_a, q_{a-1} are finite and by conservation of energy this can happen for only a single a. In these cases it is no longer possible to express the angle ω in terms of t_a and t_{a-1} so that the factorization will not hold.