High-Energy *pp* Elastic Diffraction Scattering from Inelastic States*

M. G. Gundzik

High Energy Physics Division, Argonne National Laboratory, Argonne, Illinois 60439†

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Small-angle high-energy elastic proton-proton scattering, i.e., the Pomeranchukon contribution, is determined through unitarity in terms of a model for the inelastic intermediate states. These are represented by two contributions: (i) quasielastic production of free pions, described in terms of a c-number source current $\beta(k)$ for the pions, assuming factorization of the production amplitude into a part proportional to the elastic pp amplitude times a part depending on $\beta(k)$ and the proton momenta; (ii) peripheral production of a spectrum of N^* states by the longest range possible force–single-pion exchange. The combination of these two contributions leads to an inhomogeneous integral equation for the elastic amplitude $F(s,\theta)$ of the form

$$2\,\mathrm{Im}F(s,\theta_0)=\frac{M^2}{(2\pi)^2}P^2\int d\Omega(\theta,\phi)\;F^*(s,\theta')F(s,\theta)Z(\theta,\theta')+\Delta(s,\theta_0)$$

which becomes the condition for elastic unitarity when $Z(\theta,\theta')=(P\sqrt{s})^{-1}$ and $\Delta(s,\theta_0)=0$, i.e., when no pions are produced. The inhomogeneous term is generated by the sum over peripheral N^* states, while the integral arises from the quasielastic contribution. The assumption of a c-number current implies that the pions are produced in a coherent state. The unitarity sum over quasielastic states can be replaced by an integral over coherent states, and their properties, familiar in quantum optics, as well as the factorization assumption, are used to obtain an analytic form for $Z(\theta,\theta')$ in the high-energy limit. The kernel $Z(\theta,\theta')$ depends upon the average multiplicity $\bar{n}(P,\theta)$ of pions in phase space at c.m. proton momentum P and scattering angle θ, as well as on more detailed characteristics of the current $\beta(k)$. However, $\bar{n}(P,\theta)$ can be determined from inelastic pp interactions and is used to establish the general behavior of $\beta(k)$, since

$$\bar{n}(P,\theta)=\int\frac{d^3k}{(2\pi)^3 2\omega}|\beta(k)|^2.$$

The above integral equation for $F(s,\theta)$ is simplified under Fourier-Bessel transformation to its impact-parameter representation when the high-energy, small-angle limit is taken. The resulting nonlinear, inhomogeneous integral equation is solved numerically, and used to determine the elastic differential cross section $d\sigma/dt$. Normalizing the amplitude to the total cross section σ_{tot} in the forward direction, one finds that the iterative solution can be considered to depend on σ_{tot}, the elastic cross section σ_{el}, a parameter α determined by $\bar{n}(P,\theta)$, and a free parameter λ that parametrizes the phase of $\beta(k)$. With experimentally determined values of σ_{tot}, σ_{el}, and α, the parameter λ can be fitted to give $d\sigma/dt$ in agreement with elastic data at 25 GeV/c for t in the range $0\leqslant -t\leqslant 1.0$ (GeV/c)^2.

I. INTRODUCTION

A NUMBER of interesting approaches have been made to the problem of calculating the elastic scattering amplitude from unitarity, by imposing suitable models for the important inelastic contributions. On the one hand, there are the rather detailed approaches based on Regge formulations such as those of Fubini and collaborators,[1] recently generalized in terms of multi-Regge contributions by Chew, Goldberger, and Low.[2] A recent approach of Freund,[3] though less ambitious, considers the Pomeranchukon as generated by a sum over all multiperipheral graph contributions via unitarity. This sum is related to a sum over two-body intermediate states by duality and then restrictions are placed on the slope and intercept of the Pomeranchukon at $t=0$.

On the other hand, there are the more phenomenological approaches. One is that of Van Hove,[4] which is characterized by the separation of the unitarity integral into an elastic contribution plus an overlap function, the latter characterizing the inelastic contributions. This formulation, however, proved to be inconvenient, since simple models for the overlap function with parameters determined in terms of inelastic data could not adequately explain the elastic shadow scattering.[5] Another phenomenological approach,[6] with the object of determining the elastic diffraction peak or the Pomeranchukon contribution as shadow scattering, was to neglect all but two-body intermediate states, peripherally produced by single-particle exchange. This proved to be of limited value: For example, in pp scattering, where the dominant contribution for two-body final states proceeds via single-pion exchange for small t, the resultant elastic differential cross section $d\sigma/dt$ agrees with the experimental diffraction peak only for $0\lesssim(-t)\lesssim 0.1\,(\mathrm{GeV}/c)^2$.

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† Present address: Center for Particle Theory, University of Texas, Austin, Tex. 78712.

[1] L. Bertocchi, S. Fubini, and M. Tonin, Nuovo Cimento 25, 626 (1962); D. Amati, A. Stanghellini, and S. Fubini, *ibid.* 26, 6 (1962).

[2] G. F. Chew, M. L. Goldberger, and F. E. Low, Phys. Rev. Letters 22, 208 (1969).

[3] P. G. O. Freund, Phys. Rev. Letters 22, 565 (1969).

[4] L. Van Hove, Nuovo Cimento 28, 798 (1963).

[5] L. Michejda, Nucl. Phys. B4, 113 (1967); K. Zalewski and L. Van Hove, Nuovo Cimento 46A, 807 (1966).

[6] R. C. Arnold, Phys. Rev. 136, B1388 (1964); A. Bialas, Th. W. Ruijgrok, and L. Van Hove, Nuovo Cimento 37, 608 (1965).

Here, we consider a simple phenomenological model for producing the Pomeranchukon contribution, i.e., the elastic diffraction peak, in high-energy proton-proton scattering as shadow scattering, saturating unitarity with two-body intermediate states plus the contribution of inelastic production of free pions. The general ideas of this approach can also be applied to large-angle elastic scattering.[7]

In this model, as with previous approaches, spin and isospin are neglected to avoid prohibitive complexities. However, spin effects are believed to be unimportant at high energies. Furthermore, based on the dominance of pion production in pp processes over other inelastic channels, we assume that at high energies only pion production which is either free—phase space distribution—or a part of an N^* is important.

Our model for the inelastic pp states consists of two contributions. One is the *quasielastic* or *phenomenological soft-meson*[8] contribution whose major assumption is that the pions are produced by a c-number current. In such a model the recoil of the mesons on the protons is negligible. A natural extension of this property is the assumption that the inelastic amplitude for production of free pions by protons factorizes into a pion part characterized by a c-number current and a quasielastic proton part that depends only on the proton momenta. Numerous attempts at calculating the secondary particle distributions based on this model for the inelastic states have been made.[9] From the results, it seems reasonable to conclude that the model is quite useful provided that an adequate model for the pion source current is used.[10] Two aspects of this model are particularly useful in simplifying the net contribution to unitarity: (i) The norm of the c-number pion source current is directly related to the average pion multiplicity for free pions, which has been inferred by Anderson and Collins[11] from an analysis of pp interactions. (ii) This method of treating the pions, i.e., as being produced in a statistically indedependent manner—phase space—by a c-number current, leads to the useful result that the pions in this model are produced in a *coherent state*.[12,13] Making use of this characteristic of the model provides a novel approach that simplifies the relevant unitarity integral.

The other contribution to the inelastic pp states consist of the sum over all two-body final states where at least one generalized N^* state is produced.[14] This sum corresponds to the peripheral contribution and is dominated by the longest-range force, i.e., one-pion exchange (OPE), near the forward direction. Similar contributions have been considered by Arnold, Van Hove,[6] and Freund.[3]

The impact-parameter or Fourier-Bessel representation of the scattering amplitudes, introduced for dispersion theory by Blankenbecler and Goldberger,[15] will be used to formulate our final equations. Not only the physical content but also the form of the expressions is most convenient in this representation. The scattering amplitude $F(s,t)$ which has the partial-wave decomposition

$$F(s,t)=\sum_l (2l+1)P_l(z)\eta_l(s),$$

where z is the cosine of the scattering angle, becomes in the Fourier-Bessel representation

$$F(s,t)=\int_0^\infty 2P^2 b db\, J_0(b\sqrt{-t})\eta(b,s). \qquad (1.1)$$

Here P is the appropriate c.m. three-momentum, and b is the impact parameter; in the high-energy limit $Pb\simeq l$.

When the two contributions dicussed above are used to saturate unitarity, one obtains an integral equation for the elastic pp amplitude where the kernal depends on the c-number source current and the inhomogeneous term is formed by the sum over the two-body N^* contribution. The equation is solved in the Fourier-Bessel representation in the high-energy limit, where it becomes particularly simple and where the real part of the elastic amplitude is negligible. The solution, expressed in terms of the elastic differential cross section, depends on the total and elastic pp cross sections σ_{tot} and σ_{el}, and a free parameter λ. For reasonable asymptotic values of σ_{tot} and σ_{el}, λ can be fitted to provide the characteristic form of the elastic diffraction peak from $t=0$ to $t=-1.0$ (GeV/c)^2, where the cross section decreases through four orders of magnitude. The resulting value of λ can be interpreted in terms of a simple model.

The development of this work will proceed as follows. Section II will be devoted to a discussion of the two-body N^* contribution to unitarity. Section III follows with a discussion of the assumptions involved in the quasielastic contribution and the relation of the pion source current to experiment. The details concerning the calculation of the quasielastic contribution to unitarity in terms of coherent states is presented in

[7] A more detailed model for the meson current than that considered here is necessary in the large-angle scattering case.

[8] See H. A. Kastrup, Nucl. Phys. **B1**, 309 (1967), and references therein.

[9] Z. Chylinski, Nucl. Phys. **44**, 58 (1963); A. Bialas and T. Ruijgrok, Nuovo Cimento **39**, 1061 (1965); R. C. Arnold and P. E. Heckman, Phys. Rev. **164**, 1822 (1967).

[10] The applicability of c-number meson source currents at high energies is discussed in Sec. III as well as Refs. 8 and 9.

[11] E. W. Anderson and G. B. Collins, Phys. Rev. Letters **19**, 201 (1967).

[12] R. J. Glauber, Phys. Rev. Letters **10**, 84 (1963); Phys. Rev. **131**, 2766 (1963).

[13] J. R. Klauder and E. C. G. Sudarshan, *Fundamentals of Quantum Optics* (W. A. Benjamin, Inc., New York, 1968).

[14] We consider a spectrum of N^*'s of various masses and spins that may be produced at a given energy and which may lie on Regge trajectories. We assume that the number of such states that are possible increases with energy.

[15] R. Blankenbecler and M. L. Goldberger, Phys. Rev. **126**, 766 (1962).

Sec. IV. The results of Secs. II and IV are combined in Sec. V, where the full nonlinear integral equation and its solution are discussed. The work is then summarized in Sec. VI. An Appendix follows which is devoted to properties of coherent states that are important to the body of the paper.

II. N^* CONTRIBUTION TO UNITARITY

The production of N^*'s in inelastic pp scattering forms our peripheral contribution. In the high-energy, small-t region it is reasonable to assume that this contribution is produced via OPE; the longest-range force dominates near the forward direction. We include all such diagrams, summing over all N^* intermediate states in unitarity up to the total energy available for their formation. That is, we take the sum over all box diagrams as given in Fig. 1.[16] Freund[3] considers a similar sum over N^* states.

Following Arnold,[6] we note that the t dependence, and thus the b dependence, of each term in the sum factors out of the sum in the high-energy limit. To see this, consider, neglecting spin contributions, a single term of Fig. 1 where an N^* and nucleon with masses M^* and M, respectively, are in the intermediate state; pions of mass μ are exchanged. This single term has a contribution to the imaginary part of the amplitude which we denote by $\mathrm{Im}F^{(4)}(s,t)$. We then have

$$2\,\mathrm{Im}F^{(4)}(s,t)=\frac{q_i}{(4\pi)^2W}\int F_{\mathrm{in}}^*(t_{\mathrm{in}})F_{\mathrm{fn}}(t_{\mathrm{fn}})d\Omega,\quad (2.1)$$

where $t_{\mathrm{in}}=(p_1-p_{\mathrm{in}})^2$, $t_{\mathrm{fn}}=(p_{\mathrm{in}}-p_1')^2$, $s=W^2=(p_1+p_2)^2$, and q_i is the c.m. three-momentum in the initial state. In our case, F_{in} and F_{fn} correspond to the single-pion-exchange contributions, i.e.,

$$F_{\mathrm{in}}=G_{NN^*\pi}G_{NN\pi}/(t_{\mathrm{in}}-\mu^2)$$

and, similarly, for F_{fn}. Using these forms in (2.1) and making a change of variables, we have[17]

$$\mathrm{Im}F^{(4)}(s,t)=\frac{D}{2\pi}\int\frac{dt'}{(t'-t)}\frac{\theta(\Delta(M^*,M,\mu,s,t'))}{[t'\Delta(M^*,M,\mu,s,t')]^{1/2}},\quad (2.2)$$

where

$$\Delta(M^*,M,\mu,s,t')=[s-(M^*-M)^2]\{(t'-4\mu^2)\times[s-(M+M^*)^2]-4(M^{*2}-M^2+\mu^2)\mu^2\}-4(M^*-M)^2[(M^*+M)M-\mu^2]^2$$

and

$$D\equiv|G_{NN^*\pi}G_{NN\pi}|^2.$$

[16] In addition to the sum of box diagrams of Fig. 1 there is also a contribution to pp elastic scattering by the longest-range force from the single meson exchange (tree) graph. This contribution, however, is $O(s^{-1})$ and is negligible in the high-energy limit under consideration.

[17] S. Mandelstan, Phys. Rev. **115**, 1741 (1959).

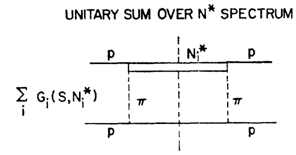

FIG. 1. Unitarity contribution to pp elastic scattering of the sum over two-body, peripherally produced N^* states.

Taking the Fourier-Bessel transform, we have

$$\mathrm{Im}\eta^{(4)}(b,s)=\frac{D}{\pi}\times\frac{[K_0(\mu b(1+\xi)^{1/2})]^2}{(s-4M^2)\{[s-(M^*-M)^2][s-(M^*+M)^2]\}^{1/2}},\quad (2.3)$$

where $K_0(\cdots)$ is a modified Bessel function,[18] and

$$\xi=\frac{(M^{*2}-M^2+\mu^2)}{s-(M+M^*)^2}+\frac{(M^*-M)^2[(M^*+M)M-\mu^2]^2}{\mu^2[s-(M^*-M)^2][s-(M^*+M)^2]}.$$

We see that, in the limit of large s,

$$\mathrm{Im}\eta^{(4)}(b,s)\sim(1/s^2)K_0(\mu b).\quad (2.4)$$

Now we consider the sum over N^*'s of Fig. 1. Using (2.3), we can write the total contribution of N^*'s as

$$\mathrm{Im}\eta(b,s)|_{N^*}=\sum_i H_i(N_i^*)\times\frac{\{K_0(\mu b[1+\xi(s,N_i^*)]^{1/2})\}^2}{s^2},\quad (2.5)$$

where the $H_i(N_i^*)$ correspond to the appropriate weighting factors for each N_i^* contribution. Taking the limit of large s in going from (2.3) to (2.4), we obtain

$$\mathrm{Im}\eta(b,s)|_{N^*}=C(s)K_0^2(\mu b).\quad (2.6)$$

After factoring out the Bessel function from each term, we are left with a sum over $H_i(N_i^*)$. We shall see below that the actual s dependence of this sum is strongly restricted by experiment. In fact, even here it is clear that

$$C(s)=O(s^{-n}),$$

where $n\geq 0$ in order that the total cross section σ_{tot} *not* increase with increasing s.

In order that $C(s)$ fall off less rapidly than the $O(1/s^2)$ of each term of (2.5), one can appeal to a model of infinitely rising N^* Regge trajectories such that at each mass value, $M_n=\sqrt{s_n}$, there are j_n N^* resonances with the same mass but possibly different

[18] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (U. S. Department of Commerce, National Bureau of Standards, Washington, D. C., 1964), Appl. Math. Ser. 55.

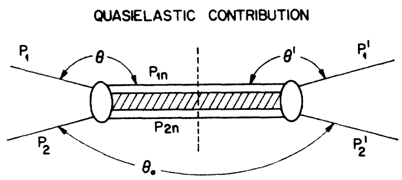

FIG. 2. Schematic version of the unitarity contribution to pp scattering of the phenomenological soft pion or quasielastic pp states.

spins where j_n increases with energy.[3] In this case, the sum on i in (2.5) becomes

$$\sum_i H_i(N_i^*)=\sum_n \sum_{j=1}^{j_n} H_{j,n}(M_n)\equiv\sum_n G_n(M_n).$$

Suppose, e.g., that we have

$$G_n(M_n)\sim(M_n)^2$$

and the trajectories involved have a common slope α', with $M_n{}^2$ given by

$$(M_n)^2=M_0{}^2+n/\alpha',$$

where n is an integer. Then the sum of G_n over n up to some integer $n=N$ is given by

$$\sum_n^N G_n(M_n)\sim\sum_n^N (M_n)^2=NM_0{}^2+\frac{1}{\alpha'}\frac{N(N+1)}{2}.$$

Since $N\sim s$ for increasing s, such a dependence on the mass of the intermediate N^* states could produce a $C(s)$ of Eq. (2.6) that is slowly varying with s.

In our model, which includes the above N^* contribution, i.e., Eq. (2.6), as well as the quasielastic contribution to be discussed next, we shall see that $C(s)$ must be a slowly varying function of s in order that the predicted falloff of the elastic diffraction peak agree with experiment.

III. QUASIELASTIC PART—ASSUMPTIONS

The important assumptions concerning the quasielastic part are[8] (i) that the pion source current is a c number and (ii) the factorization of the inelastic amplitude referring to the production of free pions. Further, we neglect the pseudoscalar character of the pions here. A possible justification of doing this as well as our neglect of isospin is to say that we consider only neutral, scalar combinations of pions.

We consider a boson field $\phi(x)$ with a current source $\beta(x)$ satisfying, as usual,

$$(\Box+\mu^2)\phi(x)=\beta(x).$$

Our first assumption, that $\beta(x)$ is a c number, permits us to solve exactly for the unitary operator S_β that connects the asymptotic in and out states

$$S_\beta{}^{-1}\phi_{\rm in}S_\beta=\phi_{\rm out}.$$

In this case $\phi_{\rm in}$ and $\phi_{\rm out}$ differ by a c number and we obtain[19]

$$S_\beta=\exp\left[\int\frac{d^3k}{2\omega(2\pi)^3}\beta(k)a_{\rm in}{}^\dagger(k)\right]\times\exp\left[-\int\frac{d^3k}{2\omega(2\pi)^3}\beta^*(k)a_{\rm in}(k)\right]\times\exp\left[-\frac{1}{2}\int\frac{d^3k}{2\omega(2\pi)^3}|\beta(k)|^2\right], \quad (3.1a)$$

where $a_{\rm in}{}^\dagger(k)$ and $a_{\rm in}(k)$ are the momentum-space creation and annihilation operators, respectively, of the meson field; $\beta(k)$ is the Fourier transform of $\beta(x)$, and $\omega\equiv k_0=[\mathbf{k}^2+\mu^2]^{1/2}$.

When we compare S_β above with the unitary operator $U(\beta)$ of Eq. (A13) that generates the coherent state $|\beta\rangle$ from the vacuum, we see that

$$S_\beta=U(\beta),$$

i.e.,

$$S_\beta|0\rangle=|\beta\rangle. \quad (3.1b)$$

Our second assumption, factorization, corresponds to a particular identification of the external source producing the pion field. With the over-all S and T matrices related by $S=1+iT$, the T matrix element between an initial state of two protons with momentum P_1 and P_2 and a final state of two protons of momenta P_3, P_4 and a collection of pions of total momentum q, we have

$$\langle\alpha(q)P_3P_4|T|P_1P_2\rangle=(2\pi)^4\delta^4(P_1+P_2-(P_3+P_4+q))M_{12,34q}. \quad (3.2)$$

Our assumption is that the amplitude M factorizes, i.e.,

$$M_{12,34q}=\langle\alpha(q)|S_\beta|0\rangle t_{12,34}, \quad (3.3)$$

where $t_{12,34}$ equals the elastic pp amplitude when $q=0$ and varies slowly with energy in an energy region including that taken off by the pions. Below, the combinations of pions with momentum q will correspond to the coherent state of definite energy momentum[20] which we use to saturate the intermediate states for this contribution. The meson factor $\langle\alpha(q)|S_\beta|0\rangle\equiv\langle\alpha(q)|\beta\rangle$ depends on the proton momenta through the meson source current $\beta(k)$.

The factorization assumption is equivalent to treating the over-all Hilbert space for the proton and pion fields as some kind of semidirect product of proton and pion parts. That is, we have

$$\langle{\rm out}|=\langle\alpha(q)\ {\rm out}|\otimes\langle P_3P_4\ {\rm out}|,$$

$$\langle{\rm in}|=|0\ {\rm in}\rangle\otimes|P_1P_2\ {\rm in}\rangle,$$

[19] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill Book Co., New York, 1964), 1st ed., p. 202.

[20] See Part C of the Appendix for the definition of coherent states of definite energy-momentum.

where the semidirect product multiplication is such as to give the amplitude factorization of Eq. (3.3).

Both the assumption regarding the c-number character of the meson source current as well as factorization require that the pion production occur with negligible recoil to the nucleons. This seems to be supported by high-energy cosmic-ray data[8] on nucleon-nucleon interactions. Another justification of these assumptions is that properties of high-energy scattering seem to be dependent on the transverse momenta of the primary particles.[11,21] This would suggest that the inelastic pion production amplitude (3.3) including the proton part $t_{12,34}$ is not very sensitive to losses in longitudinal momentum attributed to the final pions.

Descriptions of inelastic processes incorporating these two assumptions have been attempted by many authors[8,9] with varying degrees of success, depending upon their ingenuity in constructing an appropriate meson source current. Arnold and Heckman[9] used a detailed model for the current based on classical synchrotron radiation and obtained good fits to inelastic data for single pion production.

Here we have attempted to restrict the form of this current as much as possible from experiment. In this model the average pion multiplicity $\bar{n}(P,\theta)$ at a given c.m. incident momentum P and scattering angle θ of the primaries is given by

$$\bar{n}(P,\theta)=\int \frac{d^3k}{2\omega(2\pi)^3}|\beta(k,P,\theta)|^2, \qquad (3.4)$$

where we have explicitly put in the P, θ dependence of the current β. Since our quasielastic contribution refers to the production of pions in phase space, the multiplicity which interests us is the corresponding pion phase-space multiplicity. Though direct measurements of this quantity have not been made, Anderson and Collins[11] have inferred it from an analysis of $p+p\to p$ +anything, using the above phenomenological soft-pion model for large angles where peripheral contributions were believed unimportant. To do this they fit the whole momentum spectrum $d^2\sigma/d\Omega dP$ in the c.m. for fixed θ and P using the differential phase space for two nucleons and $\bar{n}$ pions. They found that

$$\bar{n}(P,\theta)=2\alpha P\sin\theta, \qquad (3.5)$$

where

$$\alpha=1.25\pm0.16\ (\text{GeV}/c)^{-1}.$$

The contribution of the quasielastic part to unitarity depends directly on $\bar{n}(P,\theta)$. However, further characteristics of the current $\beta(k,P,\theta)$ are required. In what follows, we assume that the current factorizes as follows:

$$\beta(k,P,\theta)=(P\sin\theta)^{1/2}\hat{\beta}(k,P)e^{ig(\theta)}, \qquad (3.6)$$

where $\hat{\beta}(k,P)$ depends on the magnitude of P and the phase factor is such that $g(\theta)\sim\theta$ for small θ. A reasonable model for $\hat{\beta}(k,P)$ is derived by considering the pions to have a Gaussian distribution in configuration space of width A. Further, the Gaussian will be Lorentz-contracted along the direction of the nucleon's motion. The corresponding form for $\hat{\beta}(k,P)$ in momentum space is given by[22]

$$\hat{\beta}(k,P)\sim\exp[-(\tfrac{1}{2}A^2k_\perp^2+A^2k_{\parallel}^2/2\gamma^2)], \qquad (3.7)$$

where $k_\perp$ and $k_{\parallel}$ correspond to the meson momenta in the directions perpendicular and parallel to the c.m. nucleon momentum. We shall discuss implications of this Gaussian form for the current in Sec. VI.

[21] G. Cocconi, Nuovo Cimento **57**, 837 (1968).

IV. QUASIELASTIC PART—CONTRIBUTION TO UNITARITY

In our model for the quasielastic part, which we described above, the pions are produced in a coherent state characterized by the c-number source current β with Eqs. (3.1b) and (3.3) satisfied. To determine the contribution to unitarity, we take advantage of this structure and saturate the intermediate states with coherent states of pions plus two nucleons. The form of the saturation is obtained by considering the nucleon space to be contained in the vacuum of the pion part so that Eq. (A20), the resolution of the identity for coherent states of definite energy and momentum, can be used directly.

A. Simplification of Unitarity Integral

The quasielastic contribution to unitarity is represented by the diagram in Fig. 2, where the various symbols that are to be used are identified. There, P_1 and P_2 refer to the incident c.m. four-momenta of the protons, P_{1n}, P_{2n} their intermediate-state c.m. momenta, and P_1', P_2' their final-state c.m. momenta; the angles between the c.m. momenta of the initial-final, initial-intermediate, and intermediate-final states are denoted by θ_0, θ, and θ', respectively.

In order to obtain the general form for the quasielastic contribution to unitarity, we begin with the usual T operator form of unitarity

$$2\,\text{Im}T=T^\dagger T.$$

Taking matrix elements between initial and final proton states and restricting the sum over intermediate states to the quasielastic (Q.E.) states, we obtain

$$2\,\text{Im}\langle P_1'P_2'|T|P_1P_2\rangle|_{\text{Q.E.}} =\sum_{n=\text{Q.E.}}\langle P_1'P_2'|T^\dagger|n\rangle\langle n|T|P_1P_2\rangle. \qquad (4.1)$$

Within our model the sum over the quasielastic states corresponds to the sum over all possible meson states that

[22] See, e.g., A. D. Krisch, *Lectures in Theoretical Physics* (Gordon and Breach, Science Publishers, Inc., New York, 1967), Vol. IXB, p. 1.

can be produced. In the entire space of states modulo the two-particle states where this sum occurs, we assume that the unitarity sum can be saturated by the coherent states; i.e., we replace the sum by the unit operator in this space,

$$\sum_{n=\text{Q.E.}} |n\rangle\langle n| = I_{\text{Q.E.}} = \frac{1}{(2\pi)^6}\int [d\alpha] d^4q d^4k \times d^3P_{1n} d^3P_{2n} |\alpha(q)P_{1n}P_{2n}\rangle\langle\alpha(k)P_{1n}P_{2n}|\,, \quad (4.2)$$

where the measure $[d\alpha]$ is defined in the appendix.

Using Eq. (4.2) in (4.1), we find

$$2\,\text{Im}\langle P_1'P_2'|T|P_1P_2\rangle = (2\pi)^2\int d^4qd^4k[d\alpha] \times d^3P_{1n}d^3P_{2n}t^*_{1'2',1n2n}t_{1n2n,12}\langle\beta'|\alpha(q)\rangle\langle\alpha(k)|\beta\rangle \times\delta^4(k+P_n-P_i)\delta^4(P_f-q-P_n)\,, \quad (4.3)$$

where we have used the notation of Eqs. (3.1)–(3.3) and $\beta'\equiv\beta(\theta')$, $\beta\equiv\beta(\theta)$, the angle argument of the external current β distinguishing between the two vertices of Fig. 2; we also have put $P_n=P_{1n}+P_{2n}$, $P_i=P_1+P_2$, and $P_f=P_1'+P_2'$. Note that $\langle\beta'|$ and $|\beta\rangle$ are coherent states that depend on the c-number current source β while $|\alpha(q)\rangle$ is a coherent state of mesons with definite energy-momentum as defined in the appendix. We can simplify the form of Eq. (4.3) considerably by using the properties of coherent states discussed in the appendix. Consider the part

$$Y\equiv\int d^4qd^4k[d\alpha]\langle\beta'|\alpha(q)\rangle\langle\alpha(k)|\beta\rangle \times\delta^4(k+P_n-P_i)\delta^4(P_f-P_n-q)\,. \quad (4.4\text{a})$$

This reduces to

$$Y=\int d^4qd^4k[d\alpha]\langle\beta'(q)|\alpha\rangle\langle\alpha|\beta(k)\rangle \times\delta^4(k+P_n-P_i)\delta^4(P_f-P_n-q)\,.$$

Now since $\int[d\alpha]|\alpha\rangle\langle\alpha|=1$, we have, with some further simplification,

$$Y=\delta^4(P_f-P_i)\int d^4q\langle\beta'|\beta(q)\rangle\delta^4(P_i-P_n-q)\,. \quad (4.4\text{b})$$

This result when put into the unitarity integral (4.3) with

$$t_{ab,cd}=\frac{M^2}{(E_aE_bE_cE_d)^{1/2}}F(s,\theta_{ac})$$

gives

$$2\,\text{Im}F(s,\theta_0)|_{\text{Q.E.}}=\frac{M^2}{(2\pi)^2}\int d^4q\frac{d^3P_{1n}}{E_{1n}}\frac{d^3P_{2n}}{E_{2n}} \times\delta^4(P_i-P_n-q)F^*(s,\theta')F(s,\theta)\langle\beta'|\beta(q)\rangle\,, \quad (4.5)$$

where $s=(P_1+P_2)^2=W^2$, M is the nucleon mass, and $\cos\theta'=\cos\theta_0\cos\theta+\sin\theta_0\sin\theta\cos\phi$, with ϕ the azimuthal angle of the intermediate-state momentum vector.

The integrations over the intermediate-state momenta yield

$$2\,\text{Im}F(s,\theta_0)|_{\text{Q.E.}}=\frac{M^2}{(2\pi)^2}\int d^4q\,P^2d\Omega(\theta,\phi)F^*(s,\theta') \times\frac{F(s,\theta)\langle\beta'|\beta(q)\rangle}{(W-q_0)P+(E_{n1}/P)\mathbf{q}\cdot\mathbf{P}}\,, \quad (4.6)$$

where $\mathbf{P}$ is the c.m. three-momentum of the proton in the intermediate state with magnitude P. Since the recoil to the protons is negligible in our model, P is assumed to deviate negligibly from the magnitude of the initial c.m. momentum—small inelasticity. This allows one to treat P as a constant in Eq. (4.6) and thereby reduce the complexity of the q integrations considerably. Taking advantage of this fact, (4.6) becomes

$$2\,\text{Im}F(s,\theta_0)|_{\text{Q.E.}}=\frac{P^2M^2}{(2\pi)^2}\int d\Omega(\theta,\phi)F^*(s,\theta') \times(F(s,\theta)Z(s,\theta,\theta')\,, \quad (4.7\text{a})$$

where one finds after some simplification that

$$Z(s,\theta,\theta')=\frac{1}{2iP}\int_{-\infty}^{\infty}dx_0\epsilon(x_0)e^{iWx_0}\langle\beta'|(\alpha(x_0))\beta\rangle\,, \quad (4.7\text{b})$$

with $|(\alpha(x_0))\beta\rangle$ the coherent state translated by the four-vector $\alpha(x_0)$ and defined in Eq. (A18). The four-vector $\alpha(x_0)$ is given by

$$\alpha(x_0)\equiv(x_0,x_0(W/2P^2)\mathbf{P}) \quad (4.7\text{c})$$

and

$$\epsilon(x_0)=+1 \quad \text{for} \quad x_0>0$$
$$=-1 \quad \text{for} \quad x_0<0\,.$$

The form of the inner product of coherent states that appears in (4.7b), using the results in the Appendix and the relation of the current to the pion multiplicity of (3.4) and (3.5), has the form

$$\langle\beta'|[\alpha(x_0)]\beta\rangle=e^{-\frac{1}{2}\bar{n}(P,\theta)}e^{-\frac{1}{2}\bar{n}(P,\theta')}\exp\left[\int\frac{d^3k}{(2\pi)^32\omega} \times\beta^*(k,P,\theta')\beta(k,P,\theta)e^{ix_0(\omega-W\mathbf{P}\cdot\mathbf{k}/2P^2)}\right]. \quad (4.8)$$

We now take up the question of reducing $Z(s,\theta,\theta')$ to an analytic form in the high-energy limit and then derive the form of the quasielastic contribution in the impact-parameter representation.

B. $Z(s,\theta,\theta')$ in the High-Energy Limit

The kernel of (4.7a), i.e., $Z(s,\theta',\theta)$, can be simplified in the high-energy limit by standard asymptotic approximation methods. For purposes of discussion, we rewrite the kernel in the form

$$Z(s,\theta,\theta')=(1/P)e^{-\frac{1}{2}[\bar{n}(P,\theta)+\bar{n}(P,\theta')]}J(s,\theta,\theta'), \quad (4.9a)$$

where

$$J(s,\theta,\theta')\equiv\frac{1}{2i}\int_{-\infty}^{\infty}dx_0\epsilon(x_0)e^{iWx_0}e^{\Sigma(x_0)}, \quad (4.9b)$$

with

$$\Sigma(x_0)=\Sigma_R(x_0)+i\Sigma_I(x_0),$$

$$\begin{pmatrix}\Sigma_R(x_0)\\ \Sigma_I(x_0)\end{pmatrix}=\int\frac{d^3k}{(2\pi)^3 2\omega}\left\{\mathrm{Re}[M(k,\theta,\theta')]\begin{pmatrix}\cos(x_0 b\cdot k)\\ \sin(x_0 b\cdot k)\end{pmatrix}\mp\mathrm{Im}[M(k,\theta,\theta')]\begin{pmatrix}\sin(x_0 b\cdot k)\\ \cos(x_0 b\cdot k)\end{pmatrix}\right\}. \quad (4.9c)$$

For later use, we define

$$P\Delta_R(x_0)\equiv\Sigma_R(x_0), \quad P\Delta_I(x_0)\equiv\Sigma_I(x_0). \quad (4.9d)$$

In this final equation, we have defined,

$$b\cdot k\equiv\omega-\frac{W}{2P^2}\mathbf{P}\cdot\mathbf{k},$$
$$M(k,\theta,\theta')\equiv\beta^*(k,\theta')\beta(k,\theta), \quad (4.10a)$$

where the energy dependence in β and M has been suppressed.

We have already introduced our general model for the current $\beta(k)$ in the form given by Eq. (3.6). In terms of it, M has the form

$$M(k,\theta,\theta')\equiv P(\sin\theta\sin\theta')^{1/2}|\hat{\beta}(k)|^2e^{i[g(\theta)-g(\theta')]}. \quad (4.10b)$$

It is clear from the definition of M that for $\theta_0=0$, we have $\mathrm{Im}M=0$, since $\theta=\theta'$ at that point. Furthermore, at this point $\mathrm{Im}J(s,\theta,\theta')=0$ as well. However, $J(s,\theta,\theta')$ develops an imaginary part as $\mathrm{Im}M$ deviates from zero when θ_0 moves away from the forward direction. In our considerations we shall assume that the energy is sufficiently high and θ_0 small enough so that the deviation of the absorptive part of the amplitude from $\mathrm{Im}F$ is negligible. Thus, in the high-energy, small-angle region we have

$$J(s,\theta,\theta')\simeq\mathrm{Re}J(s,\theta,\theta')$$
$$=\frac{1}{2}\int_0^{\infty}dx_0\{e^{P\Delta_R(x_0)}\sin[Wx_0+P\Delta_I(x_0)]+e^{P\Delta_R(-x_0)}\sin[Wx_0-P\Delta_I(-x_0)]\} \quad (4.11)$$

using Eq. (4.9d).

In the high-energy limit, the integrals (4.11) can be approximated by their values near the maxima of $\Delta_R(x_0)$ and $\Delta_R(-x_0)$, provided that $\Delta_R(x_0)$ is sufficiently well-behaved.[23]

For the simple case that $\theta_0=0$, where $\mathrm{Im}M=0$, we see that

$$\Delta_R(x_0)=(\sin\theta\sin\theta')^{1/2}\int\frac{d^3k}{(2\pi)^3 2\omega}|\hat{\beta}(k)|^2\cos(x_0 k\cdot b).$$

With $\hat{\beta}(k)$ given by a Gaussian such as Eq. (3.7), the maximum in this case occurs at $x_0=0$. Furthermore, this maximum dominates, since damping due to oscillations of the cosine factor for $x_0\neq 0$ reduce the value of the integral. The effect of the small imaginary part of M in Δ_R, Eqs. (4.9c) and (4.9d), produces a small shift of this maximum away from $x_0=0$ to some point $x^0=a$ for the first term of (4.11) and a symmetrical point $x_0=-a$ for the second. The conditions for these dominant maxima are

$$\Delta_R'(a)\equiv\frac{d}{dx_0}\Delta_R(x_0)\bigg|_{x_0=a}=-\frac{d}{dx_0}\Delta_R(-x_0)\bigg|_{x_0=-a}=0,$$
$$\Delta_R''(a)\equiv\frac{d^2}{dx_0^2}\Delta_R(x_0)\bigg|_{x_0=a}=\frac{d^2}{dx_0^2}\Delta_R(-x_0)\bigg|_{x_0=-a}<0. \quad (4.11')$$

We evaluate $J(s,\theta,\theta')$ of (4.11) by noting that as W, $P\rightarrow\infty$, only a small region δ about the maxima at $x_0=\pm a$ determines the value of the integral.[23] This permits us to expand Δ_R and Δ_I in the first term of (4.11) about $x_0=a$, and about $x_0=-a$ in the second term. Taking δ here finite with $\delta>0$, we have after changing variables

$$J(s,\theta,\theta')=\int_0^{\delta}dy\, e^{P(\Delta_R(a)+\frac{1}{2}y^2\Delta_R''(a))}\times\sin[yW+yP\Delta_I'(a)]\cos[aW+P\Delta_I(a)]$$
$$+\int_0^{a}dy\, e^{P(\Delta_R(a)+\frac{1}{2}y^2\Delta_R''(a))}\times\sin[aW+P\Delta_I(a)-yW-yP\Delta_I'(a)]. \quad (4.12)$$

Turning next to explicitly introducting the limit $W\rightarrow\infty$, we see that (4.12) can only be defined if aW is some finite quantity. In fact, for the simple model of the current with $\hat{\beta}(k,P)$ a Gaussian, Eq. (3.7), one can show that $a\sim\tan[g(\theta)-g(\theta')]$, and this gives, for sufficiently small θ_0, $a\sim\theta_0$. Thus, for small angles and $W\rightarrow\infty$, $aW\sim\sqrt{(-t_0)}$. [We will limit our analysis to $-t_0\leq 1.0$ $(\mathrm{GeV}/c)^2$.] It may be possible to construct other models of $\beta(k,P)$ that give $a=O(W^{-n})$, where

[23] For a discussion of the method of approximation applied to various problems see D. V. Widder, *The Laplace Transform* (Princeton University Press, Princeton, N. J., 1946), pp. 227 and 296.

$n>1$. However, for n sufficiently large, the resulting form for $J(s,\theta,\theta')$ and thus the quasielastic contribution cannot produce the characteristic diffraction peak of pp scattering when combined with the N^* contribution.[24]

In addition to those properties of $\Delta_I(x_0)$ and $\Delta_R(x_0)$ already discussed, the evaluation of $J(s,\theta,\theta')$ in (4.12) also depends quite critically on the energy dependence of $\Delta_R(a)$ and $\Delta_R''(a)$. For $\Delta_R(a)$ we note from (4.9) and (4.10)

$$\Delta_R(a)=(\sin\theta\,\sin\theta')^{1/2}\{\cos[g(\theta')-g(\theta)]\sigma_R(a) -\sin[g(\theta')-g(\theta)]\,\sigma_I(a)\}\,, \quad (4.13a)$$

$$\begin{pmatrix}\sigma_R(a)\\ \sigma_I(a)\end{pmatrix}=\int\frac{d^3k}{(2\pi)^3 2\omega}|\hat{\beta}(k,P)|^2\begin{pmatrix}\cos(ab\cdot k)\\ \sin(ab\cdot k)\end{pmatrix}. \quad (4.13b)$$

With our assumption that $g(\theta')-g(\theta)\sim\theta_0$ which is small and $a\sim\theta_0$ as well, we have

$$\Delta_R(a)\simeq(\sin\theta\,\sin\theta')^{1/2}\sigma_R(0)\,.$$

In terms of our general form for the pion current and its relation to the pion multiplicity, i.e., Eqs. (3.4)–(3.6), we see that for α in (3.5) which is a constant from 10 to 30 GeV/c in the lab,[11] we have

$$\alpha=\tfrac{1}{2}\sigma_R(0)\,. \quad (4.13c)$$

Thus, $\Delta_R(a)\simeq\Delta_R(0)$ is either a constant or very slowly varying with energy.

In the case of $\Delta_R''(a)$, we consider the general form

$$\Delta_R''(a)=-W^nY_n(a),\quad \text{(no sum)} \quad (4.14)$$

where n is an integer, $n=0, 1, \cdots$, and $Y_n(a)$ is the appropriate coefficient corresponding to a particular assumed power of W. With this form for $\Delta_R''(a)$, $J(s,\theta,\theta')$ of (4.12) becomes with change of variables

$$J_n(s,\theta,\theta')\equiv\int_0^{\delta W^{(n+1)/2}}\frac{dy}{W^{(n+1)/2}}e^{P\Delta_R(a)}e^{-y^2Y_n(a)/4} \times\sin[yW^{(1-n)/2}(1+\tfrac{1}{2}\Delta_I'(a))] \times\cos[aW+\tfrac{1}{2}W\Delta_I(a)]+\int_0^{aW^{(n+1)/2}}\frac{dy}{W^{(n+1)/2}} \times e^{P\Delta_R(a)}e^{-y^2Y_n(a)/4}\sin[aW+\tfrac{1}{2}W\Delta_I(a) -yW^{(1-n)/2}(1+\tfrac{1}{2}\Delta_I'(a))]\,, \quad (4.15)$$

where the subscript n on J denotes the energy dependence of $\Delta_R''(a)$. Three different forms for J_n arise in the limit $W\to\infty$ and aW independent of energy corresponding to $n=0$, $n=1$ and $n=m$, where $m>1$.

$$J_0(s,\theta,\theta')=e^{P\Delta_R(a)}\frac{2[1+\frac{1}{2}\Delta_I'(a)]}{Y_0(a)} \times\cos[aW+\tfrac{1}{2}W\Delta_I(a)] \times M\left(1,\tfrac{3}{2},-\frac{W[1+\frac{1}{2}\Delta_I'(a)]^2}{Y_0(a)}\right), \quad (4.16a)$$

$$J_1(s,\theta,\theta')=e^{P\Delta_R(a)}\frac{2[1+\frac{1}{2}\Delta_I'(a)]}{WY_1(a)} \times\cos[aW+\tfrac{1}{2}W\Delta_I(a)] \times M\left(1,\tfrac{3}{2},-\frac{[1+\frac{1}{2}\Delta_I'(a)]^2}{Y_1(a)}\right) +\int_0^{aW}\frac{dy}{W}e^{P\Delta_R(a)}e^{-y^2Y_1(a)/4} \times\sin[aW+\tfrac{1}{2}W\Delta_I(a) -y(1+\tfrac{1}{2}\Delta_I'(a))]\,, \quad (4.16b)$$

$$J_m(s,\theta,\theta')\equiv e^{P\Delta_R(a)}\left(\frac{\pi}{Y_m(a)W^{m+1}}\right)^{1/2} \times\sin[aW+\tfrac{1}{2}W\Delta_I(a)] \times\exp\{-[1+\tfrac{1}{2}\Delta_I'(a)]^2/ W^{m-1}Y_m(a)\}\,. \quad (4.16c)$$

Here $M(a,b,c)$ is a confluent hypergeometric function.[18]

The energy dependence of $\Delta_R''(a)$ corresponding to J_m with $m>2$ cannot be obtained by any reasonable form for the current $\beta(k)$ such as the Gaussian form of (3.7) without introducing an energy dependence in $\Delta_R(a)$ of (4.13a). Further, the resultant general form for $J_m(s,\theta,\theta')$ of (4.16c), with $m>1$, has the undesirable feature of vanishing at the forward direction $\theta_0=0$, since $aW+\frac{1}{2}W\Delta_I(a)=0$ at this point, and at $\theta'=0$ and $\theta=0$ as well, since $Y_m(a)\sim(\sin\theta\,\sin\theta')^{1/2}$. This leads to the unphysical result that the quasielastic contribution has zero contribution to the total pp cross section.

When one considers the full integral equation for the elastic scattering amplitude alternatively with the kernel given by J_0 and J_1, only J_0 leads to the experimental falloff of the elastic differential cross section $d\sigma/dt$ in t. This will be discussed more fully in Sec. IV C, and in Sec. V.

It is interesting to note that if one constructs a Gaussian pion current $\hat{\beta}(k,P)$ which has at least the $\omega=(\mathbf{k}^2+\mu^2)^{1/2}$ dependence of δ-function source currents of classical models[9] for small pion momenta k, then one is led to

$$\beta(k,P)=g\frac{\exp[-\frac{1}{2}A^2(\gamma^{-2}k_{\parallel}^2+k_{\perp}^2)]}{\omega}, \quad (4.17a)$$

[24] See the discussion of the results in Sec. V for further details and comments.

where g is a constant. For this, one finds that

$$\Delta_R(0)\sim O(\ln W)\,,\quad \Delta_R''(0)\sim O(\ln W)\,,\qquad \Delta_I'(0)\sim O(W^0)\,. \tag{4.17b}$$

J_0 would be preferred on this basis alone.[25] It is clear that $\Delta_R(0)$ and its derivatives would be a constant if the Lorentz contraction of the meson cloud could not continue indefinitely, i.e., if the Gaussian in (4.17a) had instead the form

$$\exp[-\tfrac{1}{2}A^2(\gamma^{-2}k_{\parallel}^2+k_{\perp}^2)-\delta k_{\parallel}^{2n}] \tag{4.17c}$$

with $\delta>0$ and n an integer ≥ 1.

The forms of J_0 and J_1 given by (4.16a) are still quite complicated. However, the projection of only the small-angle contributions at high energies, i.e., small θ, θ', and θ_0, that occurs in the impact-parameter representation of the unitarity condition simplifies (4.16) considerably as we shall see in Sec. IV C. Briefly, this simplification occurs by noting that $\Delta_R(a)$, $\Delta_I(a)$, $\Delta_I'(a)$, and $Y_n(a)$ are all proportional to $f(\theta,\theta')\equiv(\sin\theta\,\sin\theta')^{1/2}$ and that this as well as a are negligibly small and can be set to zero, while Wa and $Wf(\theta,\theta')$ are finite and at least of order unity.

C. Quasielastic Contribution in the Impact-Parameter Representation at High Energy

We obtain the Fourier-Bessel transform of our unitarity condition for the quasielastic contribution Eq. (4.7a) using the notation of Eq. (1.1) and the various relevant relations among Bessel functions given in Ref. 15. Suppressing energy dependence in the partial-wave amplitude $\eta(b)$ we have

$$2\,\mathrm{Im}\eta(b)|_{\mathrm{Q.E.}}=\frac{M^2}{2\pi^2}P\int_0^{\infty}b_1db_1\int_0^{\infty}b_2db_2\eta^*(b_1)\eta(b_2)\times\int_0^{2P}r_0^2dr_0^2\int_0^{2P}r^2dr^2\int_0^{2\pi}d\phi J_0(br_0^2)J_0(b_1r^2)\times J_0(b_2r'^2)e^{-\alpha(r^2+r'^2)}J_n(P,r,r')\,, \tag{4.18}$$

where

$$r_0^2=2P\sin\tfrac{1}{2}\theta_0\,,\quad r^2\equiv 2P\sin\tfrac{1}{2}\theta\,,\quad r'^2\equiv 2P\sin\tfrac{1}{2}\theta'\,,\qquad r'^4=r^4+r_0^4-2r_0^2r^2\cos\phi\,, \tag{4.19a}$$

and where $J_n(P,r,r')$, for $n=0$ or 1, is given by Eq. (4.16). We have used (4.9a) for $Z(P,\theta,\theta')$, and Eq. (3.5) for $\bar{n}(P,\theta)$ and $\bar{n}(P,\theta')$.

Following Blankenbecler and Goldberger,[15] we make use of the fact that the Bessel functions in (4.18) oscillate rapidly for large arguments to simplify our unitarity contribution. Because of this property the major contributions to the integrals in the limit that $P\to\infty$ occur when r_0^2, r^2, and r'^2 are finite, i.e., for small angles θ, θ', and θ_0. Applying the small-angle and small-a limit to J_0 and J_1 of (4.16), we note that with

$$P\Delta_R(a)\simeq P\Delta_R(0)=2rr'\alpha\,,\qquad Wa+\tfrac{1}{2}W\Delta_I(a)\simeq Wa\,, \tag{4.19b}$$

and since[18]

$$\lim_{|z|\to\infty}M(a,b,z)=\frac{\Gamma(b)}{\Gamma(b-a)}(-z)^{-a}\,,$$

for $\mathrm{Re}z<0$, J_0 and J_1 become

$$J_0(s,\theta,\theta')\simeq e^{2rr'\alpha}(\cos aW)/W\,, \tag{4.20a}$$

$$J_1(s,\theta,\theta')\simeq e^{2rr'\alpha}/W\,. \tag{4.20b}$$

The derivative $\Delta_I'(a)$ is a constant in energy for both cases covered by J_0 and J_1 in any reasonable model including (4.17a). Since the form J_1 is a special case of J_0 that occurs when $aW\equiv 0$, hereafter, we take $J\equiv J_0$ and consider the comparison of theory to experiment for various limiting values of aW.

In order to take into account an averaged effect of the contribution of nonzero θ and θ' in J we introduce the multiplicative constant A which will be fixed by the normalization of the solution of our final integral equation. Thus, we have for J

$$J(s,\theta,\theta')=A(e^{+2rr'\alpha}/W)\cos aW\,. \tag{4.21}$$

The value of A should be close to unity if our arguments provide a reasonable description of pp scattering. Our final form for the quasielastic contribution is then obtained by substituting (4.21) into (4.18). Rewriting, with

$$\mathrm{Im}\eta\equiv\eta_I$$

and neglecting $(\mathrm{Re}\eta^2)$ compared to $(\mathrm{Im}\eta)^2$, a reasonable assumption above 20 GeV/c laboratory momentum, we have

$$\eta_I(b)|_{\mathrm{Q.E.}}=A\Omega\cdot\eta_I(b)\,, \tag{4.22a}$$

where the integral operator Ω is defined by

$$\Omega\cdot\eta_I(b)\equiv\frac{M^2}{(2\pi)^2}\frac{P}{W}\int b_1db_1\int b_2db_2\eta_I(b_1)\eta_I(b_2)\times\int r_0^2dr_0^2\int r^2dr^2\int_0^{2\pi}d\phi\,J_0(br_0^2)J_0(b_1r^2)\times J_0(b_2r'^2)e^{-\alpha(r-r')^2}\cos aW\,. \tag{4.22b}$$

V. FULL INTEGRAL EQUATION AND ITS SOLUTION

When we combine our results for the N^* [Eq. (2.6)] and quasielastic [Eq. (4.22)] contributions to unitarity, we obtain our full nonlinear inhomogeneous integral equation

$$\eta_I(b)=A\Omega\cdot\eta_I(b)+C(s)K_0^2(\mu b)\,. \tag{5.1}$$

[25] The introduction of a Gaussian form for $\beta(k,P)$ with a Lorentz contracted factor $\exp[-\frac{1}{2}A^2(\beta^2/\gamma^2)k_0^2]$ in addition to that given in Eq. (4.17a), as is done by Krisch in Ref. 22, has no net effect on the energy dependence of $\Delta_R(x_0)$ and its derivatives.

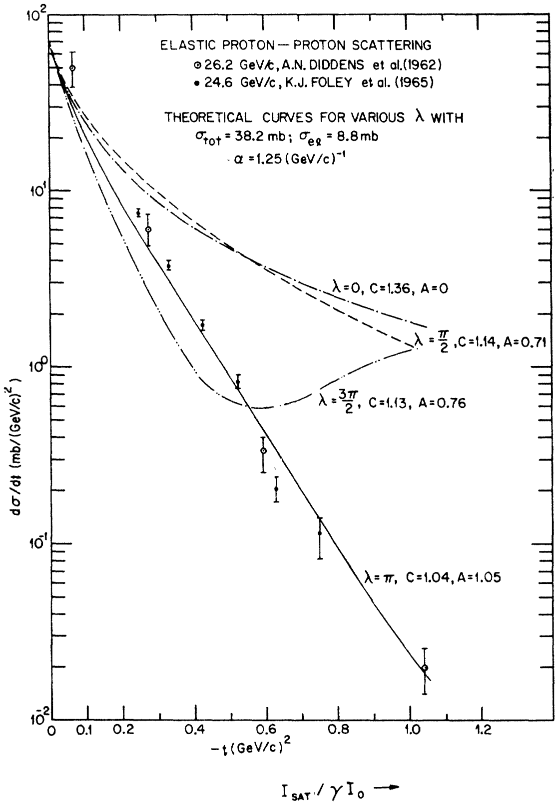

FIG. 3. Plot of $d\sigma/dt$ in mb/(GeV/c)^2 versus $-t$ in (GeV/c)^2 showing solutions to the inhomogeneous integral equation (5.1) for $\lambda=0$, $\frac{1}{2}\pi$, π, and $\frac{3}{2}\pi$, with $\sigma_{tot}=38.2$ mb, $\sigma_{el}=8.8$ mb, and $\alpha=1.25$ (GeV/c)$^{-1}$. Experimental points are from Ref. 29.

A. Normalization Condition and Parametrization of Solutions

The constants A and $C(s)$ of (5.1) are completely determined in terms of the total cross section σ_{tot} and the elastic cross section σ_{el} by the requirement that the solutions be normalized to σ_{tot} in the forward direction via the optical theorem.

An immediate condition on A and $C(s)$ occurs when we reexpress (5.1) in terms of scattering amplitudes by the Fourier-Bessel transform. Equation (5.1) then becomes

$$2\,\mathrm{Im}F(s,\theta_0)=A\frac{M^2}{(2\pi)^2}\frac{P}{W}\int d\Omega(\theta,\phi)F^*(s,\theta')F(s,\theta)\times e^{-\alpha(r-r')^2}\cos aW+C(s)(2P)^2\times\int_0^\infty bdb\,K_0^2(\mu b)J_0(br_0^2),\quad(5.2)$$

where r, r_0, and r' are given by (4.19a). At the forward direction, $\theta_0=0$, the optical theorem gives

$$2\,\mathrm{Im}F(s,\theta_0=0)=(WP/M^2)\sigma_{tot}.\quad(5.3a)$$

Furthermore, the right-hand side of (5.2) simplifies considerably at this point; e.g.,

$$r_0=0,\quad r=r',\quad a=0,$$

which enables one to relate the first integral to the elastic cross section. The final form for the right-hand side of (5.2) at $\theta_0=0$ is

$$A(WP/M^2)\sigma_{el}+2P^2C(s)/\mu^2.\quad(5.3b)$$

Combining this with the optical theorem (5.3a), we express A in terms of $C(s)$, σ_{tot}, and σ_{el} by

$$A=(\sigma_{tot}-CM^2/\mu^2)/\sigma_{el}.\quad(5.4)$$

(We suppress the possible energy dependence here and in the following.) Replacing A by (5.4) in the integral equation (5.1), the solution appears to depend yet on C. When, in fact, we solve the equation by iteration we find that the nonlinearity of the operator Ω and the optical theorem are sufficient to fix C. Thus both A and C are fixed in terms of σ_{tot} and σ_{el} which we have taken from experiment. The asymptotic value of σ_{tot} as determined from the data of Foley *et al.*[26] is

$$\sigma_{tot}=38.2\text{ mb},\quad(5.5a)$$

while the best value[27] of σ_{el} at the highest available energy of 24.5 (GeV/c) is

$$\sigma_{el}=8.8\pm0.8\text{ mb}.\quad(5.5b)$$

Since the normalization condition may not fix C uniquely, we note some additional restrictions. First of all, we only obtain a trivial solution to (5.1) if $C=(\mu^2/M^2)\sigma_{tot}$ since this leads to the vanishing of A by (5.4). Second, those values of C which give negative A, again by (5.4), are unphysical since this requires the quasielastic contribution to be negative in the forward direction, $\theta_0=0$.

The solution to our integral equation depends on only one free parameter. That is the one associated with the quantity aW that appears as the argument of the cosine in the kernal in Eq. (4.22b). During our discussion in (4.3) about the significance of a as a measure of the phase of the product of currents $\beta^*(k,\theta')\beta(k,\theta)$ we suggested that a be proportional to θ_0 for small θ_0. Following this point of view, we take

$$aW=\lambda r_0^2=\lambda W\sin\tfrac{1}{2}\theta_0,\quad(5.6)$$

where λ is a free parameter to be determined by fitting the elastic differential cross section obtained from (5.1), given σ_{tot} and σ_{el}, to the experimental forward elastic diffraction peak. The special case of $\lambda=0$ corre-

[26] K. J. Foley, R. S. Jones, S. J. Lindenbaum, W. A. Love, S. Ozaki, E. D. Platner, C. A. Quarles, and E. H. Willen, Phys. Rev. Letters **19**, 857 (1967). A fit to the total cross section σ_{tot} yields $\sigma_{tot}=(38.151+14.16/P^{0.92})$ mb (private communication from S. J. Lindenbaum).

[27] Summary of experimental data presented by Y. Sumi and T. Yoshida, Progr. Theoret. Phys. (Kyoto) Suppls. **41** & **42** (extra Nos.) 53 (1967).

sponds to the energy dependence of $\Delta_R(x_0)$ that gives J_1 in Eqs. (4.16b) and (4.20b).

We can relate λ above to parameters in a model for the current. Suppose we take the Gaussian form (4.17a) which is consistent with J_0, i.e., $\Delta_R''(0)$ independent of, or slowly varying in energy W and with the general form (3.6), (4.10a), and (4.10b). Then using the condition (4.11a) of a maximum in $\Delta_R(x_0)$ at $x_0=a$, one finds that with

$$g(\theta)-g(\theta')=\kappa\theta_0, \tag{5.7}$$

where κ is a finite constant, λ is determined by the product κA. Here A is the width of the pion cloud surrounding the nucleon. The result, assuming small a, is

$$\lambda=-\pi^{3/2}\kappa A/[\beta^2\ln(2\gamma^2+1)-1], \tag{5.8}$$

where β is the transformation velocity from the c.m. to the lab system, and $\gamma^2=(1-\beta^2)^{-1}$. If one uses a current of the form of (4.17c), then λ is independent of energy as $W\rightarrow\infty$.

B. Detail Characteristics of Solution

A solution to (5.1) for each value of λ was generated by iteration with the restrictions (5.3a) and (5.4) and is represented by the series

$$\eta_I(b)=CK_0^2(\mu b)+A\Omega\cdot[CK_0^2(\mu b)] +A\Omega\cdot\{A\Omega\cdot[CK_0^2(\mu b)]+\cdots\}. \tag{5.9}$$

This series was found to converge quite rapidly for the experimental values of σ_{tot} and σ_{el} given in (5.5) and for appropirate values of C. The multiple integrations that were required in each iteration of the integral equation were performed numerically on an IBM 360 Model 75 computer with appropriate precautions to ensure accuracy and reliability of our solution.[28]

We present the elastic differential cross section $d\sigma/dt$ as a function of t for $0\leqslant(-t)\leqslant1.0$ $(\text{GeV}/c)^2$ determined by the convergent iterative solutions of (5.1) for various values of the free parameter λ in Fig. 3 with σ_{el} taken to be 8.8 mb and $\alpha=1.25$ $(\text{GeV}/c)^{-1}$ as given in Ref. 11. The values of A and C fixed by the normalization are given for each curve; unphysical solutions, $A<0$, have been ignored. The general trend of the experimental points[29] is adequately provided by the curve for $\lambda=\pi$.[30] We note from the curves that values of $\lambda>\pi$ tend to produce shrinkage of the diffraction peak while $\lambda<\pi$ leads to antishrinkage. The

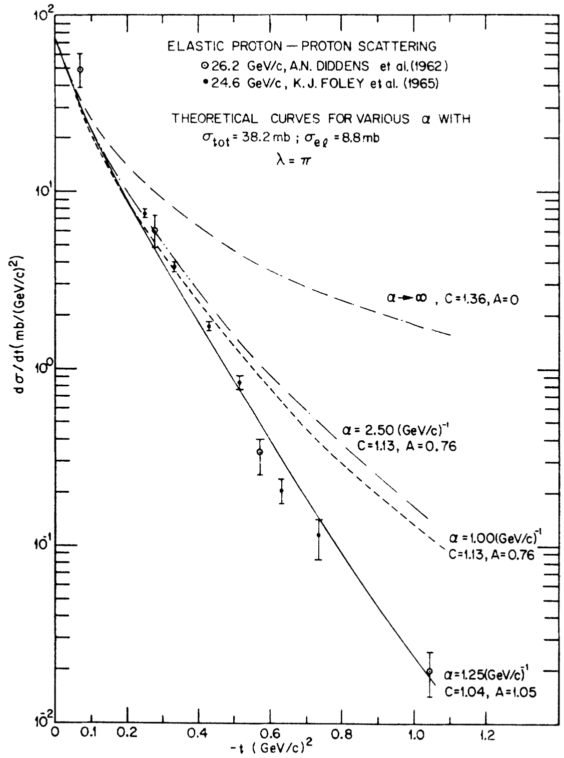

FIG. 4. Plot of $d\sigma/dt$ versus $-t$, showing solutions of the integral equation (5.1) for $\alpha=1.00$, 1.25, 2.50 $(\text{GeV}/c)^{-1}$, and $\alpha\rightarrow\infty$. Here $\lambda=\pi$, $\sigma_{tot}=38.2$ mb, and $\sigma_{el}=8.8$ mb. Experimental points are from Ref. 29.

importance of including the phase contribution in $\beta^*(k,\theta')\beta(k,\theta)$ is seen by the fact that the integral equation only has a trivial solution, $A=0$, for the case $\lambda=0$ (no phase contribution). This is the pure N^* contribution which clearly does not fit the experimental diffraction peak.

Recalling that the $\lambda=0$ solution also corresponds to the kernel function J_1 of Eq. (4.20b), its failure also excludes J_1 and the corresponding energy dependence of $\Delta_R''(x_0)$, i.e., $\Delta_R''(a)=O(W)$. Thus, the data and our model are consistent with J_0, i.e., $\Delta_R''(a)$ slowly varying in energy and a Gaussian form for the pion current (4.17a) or (4.17c).

In Fig. 4, we show the effect of varying the inelastic parameter α for $\lambda=\pi$ and σ_{tot}, σ_{el} set to the values used in Fig. 3. Again, the value of A and C for each solution is indicated. A significant variation of α away from the center value of 1.25 $(\text{GeV}/c)^{-1}$ disturbs the fit to experiment as shown by the change in the solutions for $\alpha=1.00$ and 2.50 $(\text{GeV}/c)^{-1}$. Note that antishrinkage of the diffraction peak for the normalized solutions occurs for both cases. When $\alpha\rightarrow\infty$, the quasielastic contribution vanishes and one is left with only the normalized N^* contribution, the same solution as that for $\lambda=0$ in Fig. 3. On the other hand, when $\alpha=0$, corresponding to zero average pion multiplicity, then

[28] We have not investigated the question of the existence of noniterative solutions. The existence of at least the iterative solution is sufficient for our purposes.

[29] Data at 26.2 GeV/c by A. N. Diddens, E. Lillethum, G. Manning, A. E. Taylor, T. G. Walker, and A. M. Wetherell, Phys. Rev. Letters **9**, 111 (1962). Data at 24.63 GeV/c by K. J. Foley, R. S. Gilmore, S. J. Lindenbaum, W. A. Lover, S. Ozaki, E. H. Willen, R. Yamada, and L. C. L. Yuan, *ibid.* **15**, 45 (1965).

[30] If we use Eq. (5.8) relating λ to κA we find that for $P_{lab}=25$ GeV/c, $\lambda=\pi$ gives $\kappa A=0.25$ F. In the event that $\kappa\simeq1$, then this gives a reasonable value for the size of the interaction region.

in (4.7a)

$$Z(s,\theta,\theta')=1/2P^2$$

and the quasielastic contribution corresponds to elastic unitarity. This leads to an algebraic equation for $\eta(b)$:

$$\eta_I(b)=(M^2/4\pi)[\eta_I(b)]^2+CK_0^2(\mu b)\,, \qquad (5.10)$$

which unfortunately has complex and thus unphysical solutions for $\eta_I(b)$ in this model.

The normalized solutions to the integral equation vary with σ_{el} as well. One finds that as σ_{el} decreases from its center value of 8.8 mb, the resultant diffraction peak shrinks, with the most change occurring for large t values. On the other hand, one has antishrinkage with increasing σ_{el} as expected. If another value of σ_{el} within the experimental errors were chosen, an appropriate change in λ would be made to reproduce the form of the experimental diffraction peak in Figs. 3 and 4.

Finally, an effort was made to solve the homogeneous form of the full integral equation (5.1), i.e., the case $C=0$. For a trial solution corresponding to

$$d\sigma/dt=e^{a+bt}\,,$$

where a and b were fitted to the elastic diffraction peak,[28] the resulting solution under successive iterations converged to zero.

VI. SUMMARY AND CONCLUSIONS

In the preceding sections we have presented a spinless model for the elastic diffraction peak for pp scattering and thus the Pomeranchukon as a sum over direct channel contributions via unitarity. The latter consist of (i) the sum over peripheral, two-body production dominated by the longest-range force—pion exchange—near the forward direction, and (ii) phase-space production of mesons represented by a phenomenological c-number current that allows saturation by coherent states and is directly related to the average pion multiplicity determined from experiment. In terms of the model, the unitarity condition is transformed into an inhomogeneous, nonlinear integral equation for the elastic amplitude. We found that this integral equation simplifies in the impact-parameter representation in the high-energy, small-angle limit. The solution was found to depend on σ_{tot}, σ_{el}, and a free parameter λ that characterizes the phase of the c-number current as described in Secs. IV and V.

Some of the important features of the model are as follows.

(1) The solution of the complete equation (5.1) normalized to the optical point gives the elastic diffraction peak, in terms of a free parameter λ, reasonable values for σ_{tot} and σ_{el} and inelastic pion production, which, as is seen from Fig. 4, can be fitted to agree with experimental falloff for $0\leqslant(-t)\leqslant 1.0$ (GeV/c)^2.

(2) The model does not introduce a detailed form for the c-number meson current except for the factorization of the proton momentum dependence as shown in Eq. (3.6). However, the energy dependence of the current and explicitly in terms of the integrals $\Delta_R(a)$ and $\Delta_R''(a)$ defined by (4.9c), (4.9d), and (4.10a) is quite important. One finds that $\Delta_R(a)$ and $\Delta_R''(a)$ must be constants or slowly varying with respect to energy W in order that a fit to experiment be possible.

(3) The phase of the meson current is quite crucial since it determines the maximum of $\Delta_R(x_0)$ and is eventually the source of the interference of the quasielastic and N^* contributions. Neglect of the phase effect, i.e., $\lambda=0$, in Fig. 3, makes the model disagree violently with experiment.

(4) Both the quasielastic and two-body N^* contributions are necessary in order that the falloff of the elastic diffraction peak be produced up to $-t=1.0$ (GeV/c)^2. These two contributions add at $t=0$ but interfere for larger values of $(-t)$. The pure N^* contribution as characterized by the $\alpha=\infty$ is seen to fail in Fig. 4. Further, the solution of the homogeneous equation, i.e., $C=0$, cannot produce anything resembling the elastic diffraction peak. Thus, $C(s)$ [see Sec. II or Eq. (5.1)] must be a slowly varying function of energy.

For a model of the current such as that in Eq. (4.17c), where λ and α are constants with respect to energy, the change in slope of the diffraction peak with energy would depend entirely on σ_{el}. For constant σ_{el} the diffraction peak in our model would remain unchanged and the net result would be equivalent to a fixed pole in the complex angular momentum plane.

In the case that λ and α are slowly varying with energy, i.e., such as the current model of Eq. (4.17a) then decreasing of σ_{el} with energy does not necessarily imply shrinkage or antishrinkage. Unfortunately, our model does not give very much information about the exact nature of the singularity in the complex angular momentum plane that generates the elastic scattering. In the event that σ_{el} is given, however, then the details of the model for the meson source current determine the nature of the singularity.

The novel application of coherent states of mesons in our simple model for high-energy pp scattering has permitted a simplification of the unitarity integral and led to some interesting results. In this model we have established a clear connection between the elastic diffraction peak and the driving force generated by the inelastic states. Perhaps an extension of these techniques to different processes as well as more detailed and realistic models will provide a deeper insight into the properties of various scattering processes.

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APPENDIX: SOME IMPORTANT PROPERTIES OF COHERENT STATES

Here we will discuss properties of coherent states that are necessary for the various developments in the body of the paper. In Part A, we will briefly discuss the coherent states of a single-mode harmonic oscillator where the properties are the simplest.[31] Then in Part B, we consider the form of the coherent states corresponding to a system with an infinite number of modes—the field case. Finally, in Part C, we introduce coherent states which are eigenstates of the energy-momentum operator.

A. Single-Mode Case

For a harmonic oscillator of one degree of freedom the states in Fock space are defined by the creation and annihilation operators a and $a^\dagger$, which satisfy

$$[a,a^\dagger]=1. \tag{A1}$$

One usually considers the eigenstates $|n\rangle$ of the number operator $N\equiv a^\dagger a$ in the Fock space with the properties

$$N|n\rangle=n|n\rangle;\quad a|n\rangle=n^{1/2}|n-1\rangle;$$
$$a^\dagger|n\rangle=(n+1)^{1/2}|n+1\rangle \tag{A2}$$

and

$$|n\rangle=(a^\dagger)^n/(n!)^{1/2}|0\rangle, \tag{A3}$$

where $|0\rangle$ is the vacuum state. Since the number states are complete and orthonormal, one can construct the unit operator in the Fock space in the form

$$1=\sum_n |n\rangle\langle n|. \tag{A4}$$

On the other hand, one can also construct a set of states in the Fock space that are eigenstates of the annihilation operator a which are complete, normalized, have a resolution of the unit operator, but are not orthogonal. These are the coherent states $|\alpha_0\rangle$, which we denote by the complex number α_0 since a is not Hermitian. Since they satisfy

$$a|\alpha_0\rangle=\alpha_0|\alpha_0\rangle, \tag{A5a}$$

then, using (A4), we see that the states $|\alpha_0\rangle$ are related to the number states by

$$|\alpha_0\rangle=\langle 0|\alpha_0\rangle\sum_n \frac{\alpha_0{}^n}{(n!)^{1/2}}|n\rangle, \tag{A5b}$$

where $\langle 0|\alpha_0\rangle=e^{-\frac{1}{2}|\alpha_0|^2}$ when we normalize the states by $\langle\alpha_0|\alpha_0\rangle=1$. One also finds that one can generate the coherent states from the vacuum by the displacement operator $D(\alpha_0)$, i.e.,

$$|\alpha_0\rangle=D(\alpha_0)|0\rangle, \tag{A6a}$$

where

$$D(\alpha_0)=\exp(\alpha_0 a^\dagger-\alpha_0{}^*a)$$
$$=\exp(-\tfrac{1}{2}|\alpha_0|^2)\exp(\alpha_0 a^\dagger)\exp(-\alpha_0{}^*a). \tag{A6b}$$

The resolution of the unit operator is furthermore expressible as an integral over the complex plane,

$$1=\frac{1}{\pi}\int d^2\alpha_0|\alpha_0\rangle\langle\alpha_0|, \tag{A7}$$

where $d^2\alpha_0\equiv dxdy$ when $\alpha_0=x+iy$. Finally, an inner product for the coherent states has the form

$$\langle\alpha_0|\beta_0\rangle=\exp(\alpha_0{}^*\beta_0-\tfrac{1}{2}|\alpha_0|^2-\tfrac{1}{2}|\beta_0|^2). \tag{A8}$$

B. Infinite-Mode Case

The various properties of coherent states discussed above can be generalized to the infinite-mode case.[32] Consider the inner-product space $\mathcal{H}$ in which annihilation and creation operators $a(k)$ and $a^\dagger(k)$ for a scalar-meson field are defined. They satisfy the usual commutation relations

$$[a(k),a^\dagger(k')]=(2\pi)^3 2\omega\delta^3(\mathbf{k}-\mathbf{k}'), \tag{A9}$$

where $\omega^2=\mathbf{k}^2+\mu^2$, with μ the meson mass. We further define in $\mathcal{H}$ the complex functions $\alpha(k)$ with bounded norm which we identified with the c-number meson source current in Sec. III. It is convenient to define the functional

$$(\alpha^*,\beta)\equiv\int\frac{d^3k}{(2\pi)^3 2\omega}\alpha^*(k)\beta(k); \tag{A10}$$

the norm of $\alpha(k)$ is (α^*,α) in this notation. From (A9) and (A10), the functionals $(a^\dagger,\alpha)$ and (α^*,a) satisfy

$$[(\alpha^*,a),(a^\dagger,\beta)]=(\alpha^*,\beta), \tag{A11}$$

and give a generalized form of (A1), the commutation relation of the single-mode case.

Now, we define the generalized coherent state $|\alpha\rangle$ as an eigenstate of the functional annihilation operator (β^*,a) as follows:

$$(\beta^*,a)|\alpha\rangle=(\beta^*,\alpha)|\alpha\rangle. \tag{A12}$$

One can then show that for normalized states $\langle\alpha|\alpha\rangle=1$, the unitary operator $U(\alpha)$ that generates $|\alpha\rangle$ from the vacuum, i.e., $|\alpha\rangle=U(\alpha)|0\rangle$, is

$$U(\alpha)=\exp[(a^\dagger,\alpha)-(\alpha^*,a)]$$
$$=\exp(a^\dagger,\alpha)\exp[-(\alpha^*,a)]\exp[-\tfrac{1}{2}(\alpha^*,\alpha)]; \tag{A13}$$

[31] We refer the reader to Ref. 12 for details concerning the derivation of the properties of coherent states discussed here and to Ref. 13 for numerous applications and further discussions.

[32] T. W. B. Kibble, J. Math. Phys. **9**, 315 (1968); Phys. Rev. **173**, 1527 (1968); **174**, 1882 (1968); **175**, 1624 (1968); J. K. Storrow, Nuovo Cimento **54A**, 15 (1968).

this is the generalized version of (A6b). The inner product of coherent states then has the form

$$\langle\beta|\alpha\rangle=\exp[(\beta^*,\alpha)-\tfrac{1}{2}(\alpha^*,\alpha)-\tfrac{1}{2}(\beta^*,\beta)], \quad (A14)$$

corresponding to (A8).

The resolution of the unit operator in the field case becomes

$$1=\int[d\alpha]|\alpha\rangle\langle\alpha|\,, \quad (A15)$$

where the measure $[d\alpha]$ can be defined in terms of an orthonormal set of functions $\{f_i\}$ defined in $\mathcal{H}$. With

$$a_i\equiv(f_i^*,a)\,,\quad \alpha_i\equiv(f_i^*,\alpha)\,, \quad (A16a)$$

then

$$[d\alpha]=\prod_i^\infty\left(\frac{d\alpha_i}{\pi}\right). \quad (A16b)$$

Furthermore, we can relate the α_i above to distinct modes, i.e.,

$$U(\alpha)=\prod_{i=1}^\infty U(\alpha_i)\,, \quad (A17a)$$

where

$$U(\alpha_i)=\exp(\alpha_i a_i^\dagger-\alpha_i^* a_i)\,. \quad (A17b)$$

The individual α_i and a_i have all of the independent properties of the single mode of Part A.

C. Coherent States with Definite Energy Momentum

The coherent states discussed above do not have a well-defined energy and momentum. They correspond to a combination of states with any number of particles that have any energy and momentum. Since we wish to use these states in an S-matrix formulation it is important to obtain a representation of them that is translation-invariant. Such a representation has been discussed by Kibble and by Storrow.[32]

To do this we note that the transformation of $a(k)$ and $a^\dagger(k)$ above under a translation $x=(x_0,\mathbf{x})$ is

$$e^{-iP\cdot x}a(k)e^{iP\cdot x}=e^{ik\cdot x}a(k)\,,$$
$$e^{-iP\cdot x}a^\dagger(k)e^{iP\cdot x}=e^{-ik\cdot x}a^\dagger(k)\,,$$

where P is the four momentum translation operator. Then a translation of the coherent state of Part B is given by

$$|(x)\alpha\rangle\equiv e^{-iP\cdot x}|\alpha\rangle=e^{-iP\cdot x}U(\alpha)e^{iP\cdot x}|0\rangle$$
$$=\exp[-\tfrac{1}{2}(\alpha^*,\alpha)]\exp\left[\int\frac{d^3k}{(2\pi)^3 2\omega}\alpha(k)a^\dagger(k)e^{-ik\cdot x}\right]$$
$$\times\exp\left[-\int\frac{d^3k}{(2\pi)^3 2\omega}\alpha^*(k)a(k)e^{ik\cdot x}\right]. \quad (A18)$$

In terms of the above, the state of definite energy and momentum q is obtained by the Fourier transform of $|(x)\alpha\rangle$.

$$|\alpha(q)\rangle=\frac{1}{(2\pi)^4}\int d^4x\, e^{iq\cdot x}|(x)\alpha\rangle$$
$$\equiv\delta^4(q-P)|\alpha\rangle\,, \quad (A19)$$

where we now have

$$P|\alpha(q)\rangle=q|\alpha(q)\rangle\,.$$

Finally, we can write the resolution of the identity (A15) in terms of the states $|\alpha(q)\rangle$. We find that

$$1=\int[d\alpha]|\alpha\rangle\langle\alpha|$$
$$=\int[d\alpha]e^{iP\cdot x}|(x)\alpha\rangle\langle(y)\alpha|e^{-iP\cdot y}, \quad (A20)$$
$$1=\int[d\alpha]d^4q d^4k|\alpha(q)\rangle\langle\alpha(k)|\,.$$