structed by Baacke, Jacob, and Pokorsky,[11] which is symmetric under $t \leftrightarrow u$ and therefore decouples the odd-spin states, gives Eq. (4c).[12]

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Similar considerations for other processes all giving results in agreement with (14) as well as results concerning strange and unnatural-parity trajectories will be published elsewhere.

We gratefully acknowledge very illuminating discussions with H. Harari, M. Kugler, and H. J. Lipkin.

PHYSICAL REVIEW VOLUME 184, NUMBER 5 25 AUGUST 1969

Nonrelativistic Model of Hadrons as Bound States of N Fermions

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(Received 4 February 1969)

A nonrelativistic model of hadrons as N-fermion bound states is proposed. It is concerned with the relative motion of the subparticles and the intrinsic spins of the fermions; the dependences on center-of-mass and internal variables (isospin, hypercharge, etc.) are separated out. Eigenvalue equations for the total angular momentum, parity, and mass are written down, with the requirement of invariance under the Born reciprocity principle applied to the relative coordinates and momenta. Ladder operators connecting different states are derived and are used to construct all solutions from a ground state. The solutions for three-fermion bound states are applied to baryonic resonances, yielding mass formulas for N^*, Y_0^*, and Y_1^* resonances.

I. INTRODUCTION

THE study of bound states of an arbitrary number of fermions is an imporant one since nuclei, and possibly also mesons and baryons,[1–3] are composite structures of more elementary fermions. The problem is a difficult one, since the binding mechanisms are either very complicated or completely unknown, and the many-body problem, even with known interactions, is hard to handle.

Our purpose is to investigate a model in which the dynamics is dictated by the Born reciprocity principle.[4–7] Several authors have treated reciprocity-invariant wave equations. Yukawa[8] has introduced reciprocity into nonlocal fields, while Takabayasi's quadrilocal model[9] is reciprocity-invariant, because the invariant binding potential in his model is the (3+1)-dimensional harmonic oscillator with $U(3,1)$ unitary symmetry. The difficulty arising within the relativistic treatment is that the states are either not normalizable or that they possess infinite degeneracy.[10] Yukawa and Takabayasi introduce a subsidiary condition to remove this degeneracy, while Shin,[10] in an attempt to establish a connection between a reciprocal wave equation and the theories of Nambu[11] and Kurşunoğlu,[12] reduces the problem to one in a one-dimensional mathematical space.

The same problem does not arise in a nonrelativistic treatment, since in such a framework reciprocity invariance leads to compact unitary symmetry. One has to assume, however, that the Born reciprocity transformations[4] for the spatial coordinates and momenta do not change in the nonrelativistic limit. This is not obvious, since reciprocity transformations introduce fundamental lengths, which could make reciprocity in the nonrelativistic limit meaningless. If this is the case, "nonrelativistic reciprocity" is to be understood to mean a harmonic-oscillator type of binding mechanism. The application of a nonrelativistic model to nuclear or elementary-particle physics is limited and requires justification. The reader is referred to the literature for a discussion on the feasibility of nonrelativistic approaches[13] and for pertinent models on baryons[14] and mesons.[15]

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We intend to construct and examine the consequences of a nonrelativistic model of N fermions, where the internal parameters obey the assumed nonrelativistic limit of the Born reciprocity principle. In Sec. II, center-of-mass motion will be separated out; eigenvalue equations for the relative variables will be studied in Sec. III. In Sec. IV, we will construct ladder operators connecting possible states. The solutions to the eigenvalue equations will be obtained in Sec. V, and applicability of the model to elementary particles will be examined in Sec. VI.

II. KINEMATICS

For a system of N subparticles, there are N independent momenta and N canonically conjugate position vectors, obeying

$$[p_i^m, x_j^n] = -i\delta_{ij}\delta^{mn}, \tag{1}$$

where $m, n = 1, 2, \cdots, N$, and $i, j = 1, 2, 3$.

Alternate descriptions are possible through linear transformations

$$k_i^\alpha = L^{\alpha m} p_i^m, \quad y_j^\beta = L'^{\beta n} x_j^n, \tag{2}$$

where $\alpha, \beta = 0, 1, 2, \cdots, N-1$, and summation over repeated indices is implied. The transformations have to preserve the canonical commutation relations

$$[k_i^\alpha, y_j^\beta] = -i\delta_{ij}\delta^{\alpha\beta}. \tag{3}$$

Then, we have

$$L^{\alpha m} L'^{\beta n} [p_i^m, x_j^n] = -i\delta_{ij}\delta^{mn} L^{\alpha m} L'^{\beta n}$$

and

$$L^{\alpha m} L'^{\beta n} \delta^{mn} = L^{\alpha m} L'^{\beta m} = \delta^{\alpha\beta}.$$

Thus, Eq. (3) is satisfied provided that

$$L' = (L^{-1})^T. \tag{4}$$

To separate out the center-of-mass motion, we impose

$$L^{0m} = c \quad \text{and} \quad L'^{0n} = \mu_n / c, \tag{5}$$

where c ($\neq 0$) is a constant, and $\mu_n = m_n / \sum_n m_n$ is the fractional mass of the nth subparticle. Now, we have

$$k_i^0 = c \sum_m p_i^m = cP_i, \quad y_j^0 = \frac{1}{c} \sum_n \mu_n x_j^n = \frac{1}{c} X_j, \tag{6}$$

where P and X are the center-of-mass momentum and coordinate, respectively.

The remaining $3(N-1)$ pairs of variables k_i^r, y_j^s, with $r, s = 1, 2, \cdots, N-1$, are then relative variables, obeying

$$[k_i^r, y_j^s] = -i\delta_{ij}\delta^{rs}. \tag{7}$$

The choice of the relative variables is not unique. Since Eq. (5) imposes $2N$ conditions with the introduction of one parameter, the transformation L has $N^2 - 2N + 1 = (N-1)^2$ undetermined constants. These are the parameters of the $(N-1)$-dimensional linear transformations existing among the possible sets of relative variables:

$$k'^s_i = \Lambda^{sr} k_i^r, \quad y'^s_i = \Lambda'^{sr} y_i^r, \tag{8}$$

where $\Lambda' = (\Lambda^{-1})^T$ to preserve the canonical commutation relations.

We will call a function of these relative variables reciprocity-invariant,[7] if upon the simultaneous replacements, $d_r k_i^r \to y_i^r / d_r$ and $y_i^r / d_r \to -d_r k_i^r$ (no summation over r), the function is not altered. The d_r are constants, associated with the rth degree of freedom, needed to make the reciprocity-invariant quantities dimensionally uniform.

The requirement of reciprocity invariance in a given representation of the relative variables ensures reciprocity invariance in another representation only if the transformations connecting the two representations are themselves reciprocally invariant. The group of these transformations is a subgroup of the $(N-1)$-dimensional linear group. Consequently, the requirement of reciprocity invariance eliminates a number of otherwise possible representations.

III. EIGENVALUE EQUATIONS

In the study of the bound states, we would like to select a representation in which the operators, corresponding to the basic physical observables—total angular momentum, parity, and mass—are diagonal.

These operators have to satisfy the following requirements:

(a) They must be scalars, to yield scalar eigenvalues.

(b) In order to be simultaneously diagonalizable, they have to commute mutually with each other.

(c) Reciprocity invariance is required as a basic postulate.

(d) Invariance under rotations for both integral and half-integral spins, i.e., invariance under $SU(2)$, is required.

(e) They can depend on internal quantum numbers, such as isotopic spin or hypercharge, but this dependence has to be separable from dependence on spatial variables.

To arrive at the total angular momentum, spin-orbit coupling will be used; that is, we take the total, relative orbital angular momentum to be

$$L_{ij} = \sum_r L_{ij}^r = \sum_r (y_i^r k_j^r - y_j^r k_i^r); \tag{9}$$

next, we couple subparticle spins according to

$$S_{ij} = \sum_m S_{ij}^m, \tag{10}$$

where the S_{ij}^m are components of the spin of the mth subparticle, and finally, we add total orbital and spin angular momenta:

$$J_{ij}=L_{ij}+S_{ij}. \tag{11}$$

The eigenvalue equation for angular momentum is, then,

$$J^2|\psi\rangle=\tfrac{1}{2}J_{ij}J_{ij}|\psi\rangle=(L^2+2\mathbf{L}\cdot\mathbf{S}+S^2)|\psi\rangle =j(j+1)|\psi\rangle. \tag{12}$$

We also require that

$$P|\psi\rangle=p|\psi\rangle, \tag{13}$$

where P and p $(=\pm1)$ are the parity operator and its eigenvalue, respectively. Both Eq. (12) and Eq. (13) satisfy conditions (a)–(e).

Now, we look for an equation of the form

$$K|\psi\rangle=f(M)|\psi\rangle, \tag{14}$$

where K is an operator satisfying the requirements (a)–(e), and $f(M)$ is a real function of the mass of the compound state.

The simplest reciprocity-invariant quantities in the relative variables[4–7] are

$$\Gamma_{ij}{}^{r}=\tfrac{1}{2}[k_i{}^{(r)}k_j{}^{(r)}+y_i{}^{(r)}y_j{}^{(r)}/d_{(r)}{}^4] \tag{15}$$

and the total angular momenta

$$L_{ij}=(y_i{}^rk_j{}^r-y_j{}^rk_i{}^r).$$

The operator K in Eq. (14) will have to be constructed from these quantities coupled with operators in spinor space; for simplicity we will only consider operators that are not higher than bilinear in y and k. Such operators, corresponding to our coupling scheme, are

$$\Gamma=\sum_r \Gamma_{ii}{}^r, \tag{16}$$

$$\Delta=L_{ij}S_{ij}=2\mathbf{L}\cdot\mathbf{S}, \tag{17}$$

and

$$S^2=\tfrac{1}{2}S_{ij}S_{ij}. \tag{18}$$

If we take a linear combination of these, Eq. (14) becomes

$$(A+B\Gamma+C\Delta+DS^2)|\psi\rangle=f(M)|\psi\rangle, \tag{19}$$

where A, B, C, and D, according to condition (e), can depend only on internal quantum numbers.

In Eq. (16) we took Γ to be the sum of the diagonal operators $\Gamma_{ii}{}^r$. This choice corresponds to normal modes of oscillations for the $(N-1)$ harmonic oscillators. There is only one representation of the relative variables in which Γ will be of this form; a reciprocity-invariant linear canonical transformation on these relative variables will introduce cross terms in Γ of the form $\Gamma_{ii}{}^{rr'}$.

The operators $\Omega_1=\Gamma$, $\Omega_2=L^2$, $\Omega_3=\Delta$, and $\Omega_4=S^2$ mutually commute; therefore, they can be simultaneously diagonalized:

$$\Gamma|\psi\rangle=\gamma|\psi\rangle, \tag{20}$$

$$L^2|\psi\rangle=l(l+1)|\psi\rangle, \tag{21}$$

$$\Delta|\psi\rangle=\delta|\psi\rangle, \tag{22}$$

$$S^2|\psi\rangle=s(s+1)|\psi\rangle. \tag{23}$$

Equations (20)–(23), together with Eqs. (12) and (19), yield the formulas

$$j(j+1)=l(l+1)+\delta+s(s+1) \tag{24}$$

and

$$f(M)=A'+B'\gamma+C'\delta+D's(s+1), \tag{25}$$

where A', B', C', and D' are the eigenvalues of A, B, C, and D.

IV. LADDER OPERATORS

If each member of the set of mutually commuting operators Ω_k $(k=1, 2, 3, 4)$ satisfies the relation

$$[\Omega_k,U]=U\omega_k, \tag{26}$$

where U is an operator and ω_k is a function of the Ω_k's, the operator U will be a ladder operator between two different states, for

$$(\Omega_kU-U\Omega_k)|\psi_\alpha\rangle=U\omega_k|\psi_\alpha\rangle,$$
$$\Omega_kU|\psi_\alpha\rangle=(\Omega_{k\alpha}'+\omega_{k\alpha}')U|\psi_\alpha\rangle,$$

where Ω_k' and ω_k' are the αth eigenvalues of Ω_k and ω_k, respectively. Thus $U|\psi_\alpha\rangle$ is again an eigenstate of Ω_k, with eigenvalue $\Omega_{k\alpha}'+\omega_{k\alpha}'$, unless $U|\psi_\alpha\rangle=0$.

We solve Eq. (26) for all possible U by means of commutator algebra. Since one of the operators in Eq. (19) is that of $N-1$ three-dimensional harmonic oscillators, we investigate first the effect of the raising and lowering operators for a harmonic oscillator. Let us define

$$a_i{}^{r\pm}=\frac{d_{(r)}}{\sqrt{2}}\left(k_i{}^{(r)}\pm i\frac{y_i{}^{(r)}}{d_{(r)}{}^2}\right). \tag{27}$$

The commutation relations of these operators are found from Eq. (7):

$$[a_i{}^{r-},a_j{}^{s+}]=\delta_{ij}\delta^{rs}$$
and
$$[a_i{}^{r-},a_j{}^{s-}]=[a_i{}^{r+},a_j{}^{s+}]=0. \tag{28}$$

In terms of these operators, we have

$$L_{ij}{}^r=i(a_i{}^{r-}a_j{}^{r+}-a_j{}^{r-}a_i{}^{r+}), \tag{29}$$

and

$$\Gamma_{ii}{}^r=\frac{a_i{}^{r-}a_i{}^{r+}-\frac{1}{2}}{d_r{}^2}=\frac{a_i{}^{r+}a_i{}^{r-}+\frac{1}{2}}{d_r{}^2}, \tag{30}$$

with no summation over the indices r and i. Then,

$$\Omega_1=\Gamma=\sum_r \Gamma_{ii}{}^r=\sum_r \frac{a_i{}^{r+}a_i{}^{r-}+\frac{3}{2}}{d_r{}^2}=\sum_r \frac{N^r+\frac{3}{2}}{d_r{}^2}, \tag{31}$$

$$\Omega_2=L^2=\tfrac{1}{2}L_{ij}L_{ij}, \tag{32}$$

$$\Omega_3=\Delta=L_{ij}S_{ij}, \quad (33)$$

$$\Omega_4=S^2=\tfrac{1}{2}S_{ij}S_{ij}. \quad (34)$$

Now, we have

$$[\Omega_1,a_k{}^{r\pm}]=\pm a_k{}^{r\pm}/d_r{}^2,$$
$$[\Omega_2,a_k{}^{r\pm}]=2iL_{kj}a_j{}^{r\pm}-2a_k{}^{r\pm}\equiv 2ic_k{}^{r\pm}-2a_k{}^{r\pm},$$
$$[\Omega_3,a_k{}^{r\pm}]=2iS_{kj}a_j{}^{r\pm}\equiv 2ie_k{}^{r\pm},$$
$$[\Omega_4,a_k{}^{r\pm}]=0.$$

The $a_k{}^{r\pm}$ alone are not solutions of Eq. (26), because their commutators with Ω_2 and Ω_3 contain new operators. If, however, there exists a set of operators, which, together with the Ω_k, forms a closed commutator algebra, then it is possible that one or more linear combinations of these operators will satisfy Eq. (26). To generate such a set, we find the commutators of each new member with the Ω_k:

$$[\Omega_1,c_k{}^{r\pm}]=\pm c_k{}^{r\pm}/d_r{}^2,$$
$$[\Omega_2,c_k{}^{r\pm}]=2iL_{kj}L_{jl}a_l{}^{r\pm}-2c_k{}^{r\pm}\equiv 2id_k{}^{r\pm}-2c_k{}^{r\pm},$$
$$[\Omega_3,c_k{}^{r\pm}]=2iS_{kj}L_{jl}a_l{}^{r\pm}\equiv 2ih_k{}^{r\pm},$$
$$[\Omega_4,c_k{}^{r\pm}]=0,$$
$$[\Omega_1,e_k{}^{r\pm}]=\pm e_k{}^{r\pm}/d_r{}^2,$$
$$[\Omega_2,e_k{}^{r\pm}]=2ih_k{}^{r\pm}-2e_k{}^{r\pm},$$
$$[\Omega_3,e_k{}^{r\pm}]=2iJ_{kj}S_{jl}a_l{}^{r\pm}-2ih_k{}^{r\pm}\equiv 2if_k{}^{r\pm}-2ih_k{}^{r\pm},$$
$$[\Omega_4,e_k{}^{r\pm}]=0,$$
$$[\Omega_1,d_k{}^{r\pm}]=\pm d_k{}^{r\pm}/d_r{}^2,$$
$$[\Omega_2,d_k{}^{r\pm}]=2iL_{kj}L_{jl}L_{lm}a_m{}^{r\pm}-2d_k{}^{r\pm}.$$

The operator $L_{kj}L_{jl}L_{lm}a_m{}^{r\pm}$ is expressible in terms of lower-order operators. If we define $L_i=\frac{1}{2}\epsilon_{ijk}L_{jk}$, or $L_{ij}=\epsilon_{ijk}L_k$, then

$$L_{ij}L_{jk}=-\delta_{ik}L^2+L_kL_i,$$

and

$$L_{ij}L_{jk}L_{kl}=-\epsilon_{ilk}L_kL_l-2iL_lL_i+\epsilon_{ilk}L_k+i\delta_{il}L^2$$
$$=-i\delta_{il}L^2+(1-L^2)L_{il}-2iL_{ij}L_{jl}.$$

Thus,

$$[\Omega_2,d_k{}^{r\pm}]=2L^2a_k{}^{r\pm}+2i(1-L^2)c_k{}^{r\pm}+4d_k{}^{r\pm}-2d_k{}^{r\pm}$$
$$=2a_k{}^{r\pm}(L^2-2)+2ic_k{}^{r\pm}(5-L^2)+6d_k{}^{r\pm},$$
$$[\Omega_3,d_k{}^{r\pm}]=2iS_{kj}L_{jl}L_{lm}a_m{}^{r\pm}\equiv 2iv_k{}^{r\pm},$$
$$[\Omega_4,d_k{}^{r\pm}]=0,$$
$$[\Omega_1,h_k{}^{r\pm}]=\pm h_k{}^{r\pm}/d_r{}^2,$$
$$[\Omega_2,h_k{}^{r\pm}]=2iv_k{}^{r\pm}-2h_k{}^{r\pm},$$
$$[\Omega_3,h_k{}^{r\pm}]=2iJ_{kj}S_{jl}L_{lm}a_m{}^{r\pm}-2iv_k{}^{r\pm}\equiv 2iq_k{}^{r\pm}-2iv_k{}^{r\pm},$$
$$[\Omega_4,h_k{}^{r\pm}]=0,$$
$$[\Omega_1,f_k{}^{r\pm}]=\pm f_k{}^{r\pm}/d_r{}^2,$$
$$[\Omega_2,f_k{}^{r\pm}]=2iq_k{}^{r\pm}-2f_k{}^{r\pm},$$
$$[\Omega_3,f_k{}^{r\pm}]=2iJ_{kj}J_{jl}S_{lm}a_m{}^{r\pm}-2iq_k{}^{r\pm}\equiv 2ig_k{}^{r\pm}-2iq_k{}^{r\pm},$$
$$[\Omega_4,f_k{}^{r\pm}]=0,$$
$$[\Omega_1,v_k{}^{r\pm}]=\pm v_k{}^{r\pm}/d_r{}^2,$$
$$[\Omega_2,v_k{}^{r\pm}]=2e_k{}^{r\pm}(L^2-2)+2ih_k{}^{r\pm}(5-L^2)+6v_k{}^{r\pm},$$
$$[\Omega_3,v_k{}^{r\pm}]=2iJ_{kj}S_{jl}L_{lm}L_{mn}a_n{}^{r\pm}+2e_k{}^{r\pm}(2-L^2)$$
$$-2ih_k{}^{r\pm}(5-L^2)-8v_k{}^{r\pm}\equiv 2iw_k{}^{r\pm}$$
$$+2e_k{}^{r\pm}(2-L^2)-2ih_k{}^{r\pm}(5-L^2)-8v_k{}^{r\pm},$$
$$[\Omega_4,v_k{}^{r\pm}]=0,$$
$$[\Omega_1,q_k{}^{r\pm}]=\pm q_k{}^{r\pm}/d_r{}^2,$$
$$[\Omega_2,q_k{}^{r\pm}]=2iw_k{}^{r\pm}-2q_k{}^{r\pm},$$
$$[\Omega_3,q_k{}^{r\pm}]=2iJ_{kj}J_{jl}S_{lm}L_{mn}a_n{}^{r\pm}-2iw_k{}^{r\pm}$$
$$\equiv 2it_k{}^{r\pm}-2iw_k{}^{r\pm},$$
$$[\Omega_4,q_k{}^{r\pm}]=0,$$
$$[\Omega_1,g_k{}^{r\pm}]=\pm g_k{}^{r\pm}/d_r{}^2,$$
$$[\Omega_2,g_k{}^{r\pm}]=2it_k{}^{r\pm}-2g_k{}^{r\pm},$$
$$[\Omega_3,g_k{}^{r\pm}]=-2e_k{}^{r\pm}(2-J^2)+2if_k{}^{r\pm}(5-J^2)$$
$$+8g_k{}^{r\pm}-2it_k{}^{r\pm},$$
$$[\Omega_4,g_k{}^{r\pm}]=0,$$
$$[\Omega_1,w_k{}^{r\pm}]=\pm w_k{}^{r\pm}/d_r{}^2,$$
$$[\Omega_2,w_k{}^{r\pm}]=2f_k{}^{r\pm}(L^2-2)+2iq_k{}^{r\pm}(5-L^2)+6w_k{}^{r\pm},$$
$$[\Omega_3,w_k{}^{r\pm}]=2iJ_{kj}J_{jl}S_{lm}L_{mn}L_{np}a_p{}^{r\pm}+2f_k{}^{r\pm}(2-L^2)$$
$$-2iq_k{}^{r\pm}(5-L^2)-8w_k{}^{r\pm}\equiv 2iz_k{}^{r\pm}$$
$$+2f_k{}^{r\pm}(2-L^2)-2iq_k{}^{r\pm}(5-L^2)-8w_k{}^{r\pm},$$
$$[\Omega_4,w_k{}^{r\pm}]=0,$$
$$[\Omega_1,t_k{}^{r\pm}]=\pm t_k{}^{r\pm}/d_r{}^2,$$
$$[\Omega_2,t_k{}^{r\pm}]=2iz_k{}^{r\pm}-2t_k{}^{r\pm},$$
$$[\Omega_3,t_k{}^{r\pm}]=-2h_k{}^{r\pm}(2-J^2)+2iq_k{}^{r\pm}(5-J^2)+8t_k{}^{r\pm}$$
$$-2iz_k{}^{r\pm},$$
$$[\Omega_4,t_k{}^{r\pm}]=0,$$
$$[\Omega_1,z_k{}^{r\pm}]=\pm z_k{}^{r\pm}/d_r{}^2,$$
$$[\Omega_2,z_k{}^{r\pm}]=2g_k{}^{r\pm}(L^2-2)+2it_k{}^{r\pm}(5-L^2)+6z_k{}^{r\pm},$$
$$[\Omega_3,z_k{}^{r\pm}]=-2v_k{}^{r\pm}(2-J^2)+2iw_k{}^{r\pm}(5-J^2)$$
$$-2g_k{}^{r\pm}(2-L^2)-2it_k{}^{r\pm}(5-L^2),$$
$$[\Omega_4,z_k{}^{r\pm}]=0.$$

The set of 12 operators found, together with the Ω_k, forms a closed algebra. If we exclude the operators $a_k{}^{r\pm}$, $c_k{}^{r\pm}$, and $d_k{}^{r\pm}$, the resulting subalgebra is also closed. The linear combination of the remaining operators,

$$U_k{}^{r\pm}=e_k{}^{r\pm}\lambda_1+f_k{}^{r\pm}\lambda_2+g_k{}^{r\pm}\lambda_3+h_k{}^{r\pm}\mu_1+q_k{}^{r\pm}\mu_2+t_k{}^{r\pm}\mu_3$$
$$+v_k{}^{r\pm}\nu_1+w_k{}^{r\pm}\nu_2+z_k{}^{r\pm}\nu_3,$$

then satisfies

$$[\Omega_1,U_k{}^{r\pm}]=\pm U_k{}^{r\pm}/d_r{}^2=U_k{}^{r\pm}\omega_1;\ \text{thus}\ \omega_1=\pm 1/d_r{}^2, \quad (35)$$

and

$$[\Omega_2,U_k{}^{r\pm}]=e_k{}^{r\pm}[-2\lambda_1+2(L^2-2)\nu_1]+f_k{}^{r\pm}[-2\lambda_2+2(L^2-2)\nu_2]+g_k{}^{r\pm}[-2\lambda_3+2(L^2-2)\nu_3]$$
$$+h_k{}^{r\pm}[2i\lambda_1-2\mu_1+2i(5-L^2)\nu_1]+q_k{}^{r\pm}[2i\lambda_2-2\mu_2+2i(5-L^2)\nu_2]+t_k{}^{r\pm}[2i\lambda_3-2\mu_3+2i(5-L^2)\nu_3]$$
$$+v_k{}^{r\pm}[2i\mu_1+6\nu_1]+w_k{}^{r\pm}[2i\mu_2+6\nu_2]+z_k{}^{r\pm}[2i\mu_3+6\nu_3]=U_k{}^{r\pm}\omega_2,$$

provided that

$$-2\lambda_p+2(L^2-2)\nu_p=\lambda_p\omega_2,$$
$$2i\lambda_p-2\mu_p+2i(5-L^2)\nu_p=\mu_p\omega_2, \tag{36}$$
$$2i\mu_p+6\nu_p=\nu_p\omega_2,\ p=1,2,3.$$

These nine equations have three solutions:

$$\begin{aligned}&(1)\ \omega_2=0,\quad \lambda_p=(L^2-2)\nu_p,\quad \mu_p=3i\nu_p;\\&(2)\ \omega_2=1+(1+4L^2)^{1/2}=L_+,\quad \lambda_p=\tfrac{1}{2}(L_+-4)\nu_p,\quad \mu_p=\tfrac{1}{2}i(6-L_+)\nu_p;\\&(3)\ \omega_2=1-(1+4L^2)^{1/2}=L_-,\quad \lambda_p=\tfrac{1}{2}(L_--4)\nu_p,\quad \mu_p=\tfrac{1}{2}i(6-L_-)\nu_p.\end{aligned}\tag{37}$$

Furthermore,

$$[\Omega_3,U_k{}^{r\pm}]=e_k{}^{r\pm}[-2(2-J^2)\lambda_3+2(2-L^2)\nu_1]+f_k{}^{r\pm}[2i\lambda_1+2i(5-J^2)\lambda_3+2(2-L^2)\nu_2]+g_k{}^{r\pm}[2i\lambda_2+8\lambda_3+2(2-L^2)\nu_3]$$
$$+h_k{}^{r\pm}[-2i\lambda_1-2(2-J^2)\mu_3-2i(5-L^2)\nu_1]+q_k{}^{r\pm}[-2i\lambda_2+2i\mu_1+2i(5-J^2)\mu_3-2i(5-L^2)\nu_2]$$
$$+t_k{}^{r\pm}[-2i\lambda_3+2i\mu_2+8\mu_3-2i(5-L^2)\nu_3]+v_k{}^{r\pm}[-2i\mu_1-8\nu_1-2(2-J^2)\nu_3]$$
$$+w_k{}^{r\pm}[-2i\mu_2+2i\nu_1-8\nu_2+2i(5-J^2)\nu_3]+z_k{}^{r\pm}[-2i\mu_3+2i\nu_2]=U_k{}^{r\pm}\omega_3.$$

Equating coefficients results in nine equations:

$$\begin{aligned}&-2(2-J^2)\lambda_3+2(2-L^2)\nu_1=\lambda_1\omega_3,\\&2i\lambda_1+2i(5-J^2)\lambda_3+2(2-L^2)\nu_2=\lambda_2\omega_3,\\&2i\lambda_2+8\lambda_3+2(2-L^2)\nu_3=\lambda_3\omega_3,\\&-2i\lambda_1-2(2-J^2)\mu_3-2i(5-L^2)\nu_1=\mu_1\omega_3,\\&-2i\lambda_2+2i\mu_1+2i(5-J^2)\mu_3-2i(5-L^2)\nu_2=\mu_2\omega_3,\\&-2i\lambda_3+2i\mu_2+8\mu_3-2i(5-L^2)\nu_3=\mu_3\omega_3,\\&-2i\mu_1-8\nu_1-2(2-J^2)\nu_3=\nu_1\omega_3,\\&-2i\mu_2+2i\nu_1-8\nu_2+2i(5-J^2)\nu_3=\nu_2\omega_3,\\&-2i\mu_3+2i\nu_2=\nu_3\omega_3.\end{aligned}\tag{38}$$

These equations have to be satisfied simultaneously with the conditions imposed in Eqs. (37). With the substitution of λ_p and μ_p in terms of ν_p into Eqs. (38), one arrives at the following solutions:

(1) $\omega_2=0$

(a) U_1: $\omega_3=0,\quad \nu_1=(J^2-2)\nu_3,\quad \nu_2=3i\nu_3;$

(b) U_2: $\omega_3=1+(1+4J^2)^{1/2}=J_+,$

$$\nu_1=\frac{2(J^2-2)}{2+J_+}\nu_3,\quad \nu_2=\tfrac{1}{2}i(6-J_+)\nu_3;$$

(c) U_3: $\omega_3=1-(1+4J^2)^{1/2}=J_-,$

$$\nu_1=\frac{2(J^2-2)}{2+J_-}\nu_3\quad (J_-\neq-2)$$
$$=-3\nu_3\qquad (J_-=-2),$$
$$\nu_2=\tfrac{1}{2}i(6-J_-)\nu_3.$$

(2) $\omega_2=L_+$

(a) U_4: $\omega_3=-L_+,\quad \nu_1=(J^2-2)\nu_3,\quad \nu_2=3i\nu_3;$

(b) U_5: $\omega_3=J_+-L_+,\quad \nu_1=\dfrac{2(J^2-2)}{2+J_+}\nu_3,$

$$\nu_2=-\tfrac{1}{2}i(J_+-6)\nu_3;$$

(c) U_6: $\omega_3=L_+-J_-,$

$$\nu_1=\frac{2(J^2-2)}{2+J_-}\nu_3\quad (J_-\neq-2)$$
$$=-3\nu_3\qquad (J_-=-2),$$
$$\nu_2=\tfrac{1}{2}i(6-J_-)\nu_3.$$

(3) $\omega_2=L_-$

(a) U_7: $\omega_3=L_-,\quad \nu_1=(J^2-2)\nu_3,\quad \nu_2=3i\nu_3;$

(b) U_8: $\omega_3=J_+-L_-,\quad \nu_1=\dfrac{2(J^2-2)}{2+J_+}\nu_3,$

$$\nu_2=\tfrac{1}{2}i(6-J_+)\nu_3;$$

(c) U_9: $\omega_3=J_--L_-,$

$$\nu_1=\frac{2(J^2-2)}{2+J_-}\nu_3\quad (J_-\neq-2)$$
$$=-3\nu_3\qquad (J_-=-2),$$
$$\nu_2=\tfrac{1}{2}i(6-J_-)\nu_3.$$

Finally, we have

$$[\Omega_4,U_k{}^{r\pm}]=0,\quad \omega_4=0. \tag{39}$$

TABLE I. Tabulation of the operators U_1–U_9.

	$(U_1)_i^{r\pm}$	$(U_2)_i^{r\pm}$	$(U_3)_i^{r\pm}$ $j\neq 1$	$(U_3)_i^{r\pm}$ $j=1$	$(U_4)_i^{r\pm}$	$(U_5)_i^{r\pm}$
λ_1	$(J^2-2)(L^2-2)\nu_3$	$\frac{2(J^2-2)(L^2-2)\nu_3}{(2+J_+)}$	$\frac{2(J^2-2)(L^2-2)\nu_3}{(2+J_-)}$	$-3(L^2-2)\nu_3$	$\frac{1}{2}(L_+-4)(J^2-2)\nu_3$	$\frac{(L_+-4)(J^2-2)\nu_3}{(2+J_+)}$
λ_2	$3i(L^2-2)\nu_3$	$\frac{1}{2}i(6-J_+)(L^2-2)\nu_3$	$\frac{1}{2}i(6-J_-)(L^2-2)\nu_3$	$4i(L^2-2)\nu_3$	$\frac{3}{2}i(L_+-4)\nu_3$	$\frac{1}{4}i(L_+-4)(6-J_+)\nu_3$
λ_3	$(L^2-2)\nu_3$	$(L^2-2)\nu_3$	$(L^2-2)\nu_3$	$(L^2-2)\nu_3$	$\frac{1}{2}(L_+-4)\nu_3$	$\frac{1}{2}(L_+-4)\nu_3$
μ_1	$3i(J^2-2)\nu_3$	$\frac{6i(J^2-2)\nu_3}{(2+J_+)}$	$\frac{6i(J^2-2)\nu_3}{(2+J_-)}$	$-9i\nu_3$	$\frac{1}{2}i(J^2-2)(6-L_+)\nu_3$	$\frac{i(J^2-2)(6-L_+)\nu_3}{(2+J_+)}$
μ_2	$-9\nu_3$	$-\frac{3}{2}(6-J_+)\nu_3$	$-\frac{3}{2}(6-J_-)\nu_3$	$-12\nu_3$	$-\frac{3}{2}(6-L_+)\nu_3$	$-\frac{1}{4}(6-J_+)(6-L_+)\nu_3$
μ_3	$3i\nu_3$	$3i\nu_3$	$3i\nu_3$	$3i\nu_3$	$\frac{1}{2}i(6-L_+)\nu_3$	$\frac{1}{2}i(6-L_+)\nu_3$
ν_1	$(J^2-2)\nu_3$	$\frac{2(J^2-2)\nu_3}{(2+J_+)}$	$\frac{2(J^2-2)\nu_3}{(2+J_-)}$	$-3\nu_3$	$(J^2-2)\nu_3$	$\frac{2(J^2-2)\nu_3}{(2+J_+)}$
ν_2	$3i\nu_3$	$\frac{1}{2}i(6-J_+)\nu_3$	$\frac{1}{2}i(6-J_-)\nu_3$	$4i\nu_3$	$3i\nu_3$	$\frac{1}{2}i(6-J_+)\nu_3$

	$(U_6)_i^{r\pm}$	$(U_7)_i^{r\pm}$	$(U_8)_i^{r\pm}$	$(U_9)_i^{r\pm}$ $j\neq 1$	$(U_9)_i^{r\pm}$ $j=1$
λ_1	$\frac{(L_+-4)(J^2-2)\nu_3}{(2+J_-)}$	$\frac{1}{2}(L_--4)(J^2-2)\nu_3$	$\frac{(L_--4)(J^2-2)\nu_3}{(2+J_+)}$	$\frac{(L_--4)(J^2-2)\nu_3}{(2+J_-)}$	$-\frac{3}{2}(L_--4)\nu_3$
λ_2	$\frac{1}{4}i(L_+-4)(6-J_-)\nu_3$	$\frac{3}{2}i(L_--4)\nu_3$	$\frac{1}{4}i(L_--4)(6-J_+)\nu_3$	$\frac{1}{4}i(L_--4)(6-J_-)\nu_3$	$2i(L_--4)\nu_3$
λ_3	$\frac{1}{2}(L_+-4)\nu_3$	$\frac{1}{2}(L_--4)\nu_3$	$\frac{1}{2}(L_--4)\nu_3$	$\frac{1}{2}(L_--4)\nu_3$	$\frac{1}{2}(L_--4)\nu_3$
μ_1	$\frac{i(J^2-2)(6-L_+)\nu_3}{(2+J_-)}$	$\frac{1}{2}i(J^2-2)(6-L_-)\nu_3$	$\frac{i(J^2-2)(6-L_-)\nu_3}{(2+J_+)}$	$\frac{i(J^2-2)(6-L_-)\nu_3}{(2+J_-)}$	$-\frac{3}{2}i(6-L_-)\nu_3$
μ_2	$-\frac{1}{4}(6-J_-)(6-L_+)\nu_3$	$-\frac{3}{2}(6-L_-)\nu_3$	$-\frac{1}{4}(6-J_+)(6-L_-)\nu_3$	$-\frac{1}{4}(6-J_-)(6-L_-)\nu_3$	$-2(6-L_-)\nu_3$
μ_3	$\frac{1}{2}i(6-L_+)\nu_3$	$\frac{1}{2}i(6-L_-)\nu_3$	$\frac{1}{2}i(6-L_-)\nu_3$	$\frac{1}{2}i(6-L_-)\nu_3$	$\frac{1}{2}i(6-L_-)\nu_3$
ν_1	$\frac{2(J^2-2)\nu_3}{(2+J_-)}$	$(J^2-2)\nu_3$	$\frac{2(J^2-2)\nu_3}{(2+J_+)}$	$\frac{2(J^2-2)\nu_3}{(2+J_-)}$	$-3\nu_3$
ν_2	$\frac{1}{2}i(6-J_-)\nu_3$	$3i\nu_3$	$\frac{1}{2}i(6-J_+)\nu_3$	$\frac{1}{2}i(6-J_-)\nu_3$	$4i\nu_3$

The coefficients of the corresponding nine solutions U_1–U_9 are listed in Table I; the effect of these operators on eigenstates is shown in Table III.

Constructing ladder operators from the members of the whole algebra, i.e., with the inclusion of the three operators $a_k^{r\pm}$, $c_k^{r\pm}$, and $d_k^{r\pm}$, will not produce new solutions, since the coefficients of these terms in the ladder operators will have to be identically zero to satisfy Eqs. (26).

The ladder operators constructed so far are the only linearly independent vector operators that can be generated from Euclidean three-vectors of the $6(N-1)$-dimensional phase space. One can easily construct tensor operators by taking the product of two or more of the operators U_1–U_9.

All of the operators U_1–U_9 commute with Ω_4; therefore, they do not produce spin flip of the subparticles. However, one can generate another closed commutator algebra by starting from the basic elements of spinor space, the S_{ij}^m.

For this purpose let us define

$$S_i^m=\tfrac{1}{2}\epsilon_{ijk}S_{jk}^m;$$

then

$$S_i=\textstyle\sum_m S_i^m=\tfrac{1}{2}\epsilon_{ijk}S_{jk}.$$

Now,

$$[\Omega_1,S_i^m]=0$$

$$[\Omega_2,S_i^m]=0,$$

$$[\Omega_3,S_i^m]=-2i\epsilon_{ijk}L_jS_k^m\equiv-2iE_i^m,$$

$$[\Omega_4,S_i^m]=-2i\epsilon_{ijk}S_jS_k^m-2S_i^m\equiv-2iF_i^m-2S_i^m,$$

$$[\Omega_1,E_i^m]=0,$$

$$[\Omega_2,E_i^m]=0,$$

$$\begin{aligned}[\Omega_3,E_i^m]&=2iS_i\mathbf{L}\cdot\mathbf{S}^m-2iL_i\mathbf{S}\cdot\mathbf{S}^m-2iL_i\mathbf{L}\cdot\mathbf{S}^m\\&\quad+2iS_i^mL^2+2E_i^m\\&=2iQ_i^m-2iG_1^m-2iN_i^m+2iS_i^mL^2+2E_i^m,\end{aligned}$$

where $Q_i^m\equiv S_i\mathbf{L}\cdot\mathbf{S}^m$, $G_i^m\equiv L_i\mathbf{S}\cdot\mathbf{S}^m$, and $N_i^m\equiv L_i\mathbf{L}\cdot\mathbf{S}^m$.

$$[\Omega_4,E_i^m]=-2iQ_i^m+2iS_i^m\mathbf{L}\cdot\mathbf{S},$$

$$[\Omega_1,F_i^m]=0,$$

$$[\Omega_2,F_i^m]=0,$$

$$[\Omega_3,F_i^m]=-2iQ_i^m+2iS_i^m\mathbf{L}\cdot\mathbf{S}+2E_i^m,$$

$$\begin{aligned}[\Omega_4,F_i^m]&=-2iS_i\mathbf{S}\cdot\mathbf{S}^m+2iS_i^m(S^2-2)+4F_i^m\\&\equiv-2iH_i^m+2iS_i^m(S^2-2)+4F_i^m,\end{aligned}$$

$$[\Omega_1,G_i^m]=0,$$

$$[\Omega_2,G_i^m]=0,$$

$$[\Omega_3,G_i^m]=2i\epsilon_{ijk}L_jS_k\mathbf{S}\cdot\mathbf{S}^m\equiv 2iK_i^m,$$
$$[\Omega_4,G_i^m]=0,$$
$$[\Omega_1,H_i^m]=0,$$
$$[\Omega_2,H_i^m]=0,$$
$$[\Omega_3,H_i^m]=-2iK_i^m,$$
$$[\Omega_4,H_i^m]=0,$$
$$[\Omega_1,K_i^m]=0,$$
$$[\Omega_2,K_i^m]=0,$$
$$[\Omega_3,K_i^m]=2iH_i^m(\mathbf{L}\cdot\mathbf{S}+L^2)-2iG_i^m(\mathbf{L}\cdot\mathbf{S}+S^2)+2K_i^m,$$
$$[\Omega_4,K_i^m]=0,$$
$$[\Omega_1,N_i^m]=0,$$
$$[\Omega_2,N_i^m]=0,$$
$$[\Omega_3,N_i^m]=2i\epsilon_{ijk}L_jS_k\mathbf{L}\cdot\mathbf{S}^m+2i\epsilon_{ljk}L_iL_lS_jS_k^m+2N_i^m =2iR_i^m+2iT_i^m+2N_i^m,$$

where $R_i^m\equiv\epsilon_{ijk}L_jS_k\mathbf{L}\cdot\mathbf{S}^m$ and $T_i^m\equiv\epsilon_{ljk}L_iL_lS_jS_k^m$.

$$[\Omega_4,N_i^m]=-2iT_i^m-2N_i^m,$$
$$[\Omega_1,Q_i^m]=0,$$
$$[\Omega_2,Q_i^m]=0,$$
$$[\Omega_3,Q_i^m]=-2i\epsilon_{ijk}L_jS_k\mathbf{L}\cdot\mathbf{S}^m+2i\epsilon_{ljk}S_iL_lS_jS_k^m+2Q_i^m \equiv -2iR_i^m+2iV_i^m+2Q_i^m,$$
$$[\Omega_4,Q_i^m]=-2iV_i^m-2Q_i^m,$$
$$[\Omega_1,R_i^m]=0,$$
$$[\Omega_2,R_i^m]=0,$$
$$[\Omega_3,R_i^m]=2iQ_i^m(L^2+\mathbf{L}\cdot\mathbf{S}+1)+2iN_i^m(1-\mathbf{L}\cdot\mathbf{S}-S^2) -2V_i^m-2T_i^m+2i\epsilon_{ijk}\epsilon_{lnp}L_jS_kL_lS_nS_p^m +4R_i^m\equiv 2iQ_i^m(L^2+\mathbf{L}\cdot\mathbf{S}+1) +2iN_i^m(1-\mathbf{L}\cdot\mathbf{S}-S^2)-2V_i^m-2T_i^m +2iW_i^m+4R_i^m,$$
$$[\Omega_4,R_i^m]=-2iW_i^m-2R_i^m,$$
$$[\Omega_1,T_i^m]=0,$$
$$[\Omega_2,T_i^m]=0,$$
$$[\Omega_3,T_i^m]=2iG_i^m\mathbf{L}\cdot\mathbf{S}-2iN_i^m(S^2-2)-4T_i^m+2iW_i^m,$$
$$[\Omega_4,T_i^m]=-2iG_i^m\mathbf{L}\cdot\mathbf{S}+2iN_i^m(S^2-2)+4T_i^m,$$
$$[\Omega_1,V_i^m]=0,$$
$$[\Omega_2,V_i^m]=0,$$
$$[\Omega_3,V_i^m]=2iH_i^m-2iQ_i^m(S^2-2)-4V_i^m-2iW_i^m,$$
$$[\Omega_4,V_i^m]=-2iH_i^m\mathbf{L}\cdot\mathbf{S}+2iQ_i^m(S^2-2)+4V_i^m,$$
$$[\Omega_1,W_i^m]=0,$$
$$[\Omega_2,W_i^m]=0,$$
$$[\Omega_3,W_i^m]=2iV_i^m(\mathbf{L}\cdot\mathbf{S}+L^2-2)+2Q_i^m(S^2-2) -2H_i^m\mathbf{L}\cdot\mathbf{S}-2iT_i^m(S^2+\mathbf{L}\cdot\mathbf{S}+2) -2G_i^m\mathbf{L}\cdot\mathbf{S}-2iR_i^m(S^2-2)+2iK_i^m\mathbf{L}\cdot\mathbf{S} +2N_i^m(S^2-2)-2W_i^m,$$
$$[\Omega_4,W_i^m]=-2iK_i^m\mathbf{L}\cdot\mathbf{S}+2iR_i^m(S^2-2)+4W_i^m.$$

The operators S_i^m, E_i^m, F_i^m, G_i^m, H_i^m, K_i^m, N_i^m, Q_i^m, R_i^m, T_i^m, V_i^m, and W_i^m, together with the Ω_k, form a closed algebra. Two kinds of subalgebras exist: The smaller is composed of the elements G_i^m, H_i^m, K_i^m, and Ω_k, while the larger is made up of the operators G_i^m, H_i^m, K_i^m, N_i^m, Q_i^m, R_i^m, T_i^m, V_i^m, W_i^m, and Ω_k. Since the larger type of subalgebra contains the corresponding smaller one for given i and m, we can form a linear combination of the nine operators (excepting the Ω_k) of the larger subalgebra and examine the effect of the exclusion of S_i^m, E_i^m, and F_i^m later. The linear combinations

$$U_i^m=G_i^m\rho_1+H_i^m\rho_2+K_i^m\rho_3+N_i\sigma_1+Q_i^m\sigma_2+R_i^m\sigma_3 +T_i^m\tau_1+V_i^m\tau_2+W_i^m\tau_3$$

satisfy

$$[\Omega_1,U_i^m]=U_i^m\omega_1=0 \quad\text{with}\quad \omega_1=0, \tag{40}$$
$$[\Omega_2,U_i^m]=U_i^m\omega_2=0 \quad\text{with}\quad \omega_2=0, \tag{41}$$

$$[\Omega_3,U_i^m]=U_i^m\omega_3=G_i^m[-2i(\mathbf{L}\cdot\mathbf{S}+S^2)\rho_3+2i\mathbf{L}\cdot\mathbf{S}\tau_1-2\mathbf{L}\cdot\mathbf{S}\tau_3]+H_i^m[2i\tau_2+2i(\mathbf{L}\cdot\mathbf{S}+L^2)\rho_3-2\mathbf{L}\cdot\mathbf{S}\tau_3] +K_i^m[2i\rho_1-2i\rho_2+2\rho_3+2i\mathbf{L}\cdot\mathbf{S}\tau_3]+N_i^m[2\sigma_1+2i(1-\mathbf{L}\cdot\mathbf{S}-S^2)\sigma_3-2i(S^2-2)\tau_1+2(S^2-2)\tau_3] +Q_i^m[2\sigma_2+2i(L^2+\mathbf{L}\cdot\mathbf{S}+1)\sigma_3-2i(S^2-2)\tau_2+2(S^2-2)\tau_3]+R_i^m[-2i\sigma_2+2i\sigma_1+4\sigma_3-2i(S^2-2)\tau_3] +T_i^m[2i\sigma_1-2\sigma_3-4\tau_1-2i(S^2+\mathbf{L}\cdot\mathbf{S}+2)\tau_3]+V_i^m[2i\sigma_2-2\sigma_3-4\tau_2+2i(\mathbf{L}\cdot\mathbf{S}+L^2-2)\tau_3] +W_i^m[2i\sigma_3-2i\tau_2+2i\tau_1-2\tau_3],$$

with

$$\begin{aligned} &-2i(\mathbf{L}\cdot\mathbf{S}+S^2)\rho_3+2i\mathbf{L}\cdot\mathbf{S}\tau_1-2\mathbf{L}\cdot\mathbf{S}\tau_3=\omega_3\rho_1,\\ &2i\tau_2+2i(\mathbf{L}\cdot\mathbf{S}+L^2)\rho_3-2i\mathbf{L}\cdot\mathbf{S}\tau_3=\omega_3\rho_2,\\ &2i\rho_1-2i\rho_2+2\rho_3+2i\mathbf{L}\cdot\mathbf{S}\tau_3=\omega_3\rho_3,\\ &2\sigma_1+2i(1-\mathbf{L}\cdot\mathbf{S}-S^2)\sigma_3-2i(S^2-2)\tau_1+2(S^2-2)\tau_3=\omega_3\sigma_1,\\ &2\sigma_2+2i(L^2+\mathbf{L}\cdot\mathbf{S}+1)\sigma_3-2i(S^2-2)\tau_2+2(S^2-2)\tau_3=\omega_3\sigma_2,\\ &-2i\sigma_2+2i\sigma_1+4\sigma_3-2i(S^2-2)\tau_3=\omega_3\sigma_3,\\ &2i\sigma_1-2\sigma_3-4\tau_1-2i(S^2+\mathbf{L}\cdot\mathbf{S}+2)\tau_3=\omega_3\tau_1,\\ &2i\sigma_2-2\sigma_3-4\tau_2+2i(\mathbf{L}\cdot\mathbf{S}+L^2-2)\tau_3=\omega_3\tau_2,\\ &2i\sigma_3-2i\tau_2+2i\tau_1-2\tau_3=\omega_3\tau_3, \end{aligned} \tag{42}$$

and

$$[\Omega_4,U_i^m]=U_i^m\omega_4=G_i^m(-2i\mathbf{L}\cdot\mathbf{S}\tau_1)+H_i^m(-2i\mathbf{L}\cdot\mathbf{S}\tau_2) +K_i^m(-2i\mathbf{L}\cdot\mathbf{S}\tau_3)+N_i^m[-2\sigma_1+2i(S^2-2)\tau_1] +Q_i^m[-2\sigma_2+2i(S^2-2)\tau_2]+R_i^m[-2\sigma_3 +2i(S^2-2)\tau_3]+T_i^m(-2i\sigma_1+4\tau_1)+V_i^m(-2i\sigma_2+4\tau_2) +W_i^m(-2i\sigma_3+4\tau_3),$$

with

$$\left.\begin{aligned} &-2i\mathbf{L}\cdot\mathbf{S}\tau_q=\omega_4\rho_q,\\ &-2\sigma_q+2i(S^2-2)\tau_q=\omega_4\sigma_q,\\ &-2i\sigma_q+4\tau_q=\omega_4\tau_q, \end{aligned}\right\}\quad q=1,2,3. \tag{43}$$

TABLE II. Tabulation of the operators U_{10}–U_{17}.

	$(U_{10})_i^m$	$(U_{11})_i^m$ $j\neq 0$	$(U_{12})_i^m$	$(U_{13})_i^m$
ρ_1	$-i(\Delta+2S^2)\rho_3/J_+$	$-i(\Delta+2S^2)\rho_3/J_-$	$-i\Delta\tau_1/S_+$	$-(\Delta/S_+)(S_++2S^2+\Delta)\tau_3/J_+$
ρ_2	$i(\Delta+2L^2)\rho_3/J_+$	$i(\Delta+2L^2)\rho_3/J_-$	$-i\Delta\tau_1/S_+$	$(\Delta/S_+)(\Delta+2L^2-S_+)\tau_3/J_+$
ρ_3	ρ_3	ρ_3	0	$-i(\Delta/S_+)\tau_3$
σ_1	0	0	$-\frac{1}{2}i(4-S_+)\tau_1$	$-\frac{1}{2}(4-S_+)(S_++2S^2+\Delta)\tau_3/J_+$
σ_2	0	0	$-\frac{1}{2}i(4-S_+)\tau_1$	$-\frac{1}{2}(4-S_+)(S_+-\Delta-2L^2)\tau_3/J_+$
σ_3	0	0	0	$-\frac{1}{2}i(4-S_+)\tau_3$
τ_1	0	0	τ_1	$-i(S_++2S^2+\Delta)\tau_3/J_+$
τ_2	0	0	τ_1	$i(\Delta+2L^2-S_+)\tau_3/J_+$
τ_3	0	0	0	τ_3

	$(U_{14})_i^m$ $j\neq 0$	$U(_{15})_i^m$ $s\neq 0$	$(U_{16})_i^m$ $s\neq 0$	$(U_{17})_i^m$ $j\neq 0,\ s\neq 0$
ρ_1	$-(\Delta/S_+)(S_++2S^2+\Delta)\tau_3/J_-$	$-i\Delta\tau_1/S_-$	$-(\Delta/S_-)(S_-+2S^2+\Delta)\tau_3/J_+$	$-(\Delta/S_-)(S_-+2S^2+\Delta)\tau_3/J_-$
ρ_2	$(\Delta/S_+)(\Delta+2L^2-S_+)\tau_3/J_-$	$-i\Delta\tau_1/S_-$	$(\Delta/S_-)(\Delta+2L^2-S_-)\tau_3/J_+$	$(\Delta/S_-)(\Delta+2L^2-S_-)\tau_3/J_-$
ρ_3	$-i(\Delta/S_+)\tau_3$	0	$-i(\Delta/S_-)\tau_3$	$-i(\Delta/S_-)\tau_3$
σ_1	$-\frac{1}{2}(4-S_+)(S_++2S^2+\Delta)\tau_3/J_-$	$-\frac{1}{2}i(4-S_-)\tau_1$	$-\frac{1}{2}(4-S_-)(S_-+2S^2+\Delta)\tau_3/J_+$	$-\frac{1}{2}(4-S_-)(S_-+2S^2+\Delta)\tau_3/J_-$
σ_2	$-\frac{1}{2}(4-S_+)(S_+-\Delta-2L^2)\tau_3/J_-$	$-\frac{1}{2}i(4-S_-)\tau_1$	$-\frac{1}{2}(4-S_-)(S_--\Delta-2L^2)\tau_3/J_+$	$-\frac{1}{2}(4-S_-)(S_--\Delta-2L^2)\tau_3/J_-$
σ_3	$-\frac{1}{2}i(4-S_+)\tau_3$	0	$-\frac{1}{2}i(4-S_-)\tau_3$	$-\frac{1}{2}i(4-S_-)\tau_3$
τ_1	$-i(S_++2S^2+\Delta)\tau_3/J_-$	τ_1	$-i(S_-+2S^2+\Delta)\tau_3/J_+$	$-i(S_-+2S^2+\Delta)\tau_3/J_-$
τ_2	$i(\Delta+2L^2-S_+)\tau_3/J_-$	τ_1	$i(\Delta+2L^2-S_-)\tau_3/J_+$	$i(\Delta+2L^2-S_-)\tau_3/J_-$
τ_3	τ_3	0	τ_3	τ_3

The solutions of Eqs. (43) are

(1) $\omega_4=0$, $\sigma_q=\tau_q=0$, and the ρ_q are arbitrary;

(2) $\omega_4=1+(1+4S^2)^{1/2}=S_+$, $\rho_q=-2i(\mathbf{L}\cdot\mathbf{S}/S_+)\tau_q$,
$$\sigma_q=\tfrac{1}{2}i(S_+-4)\tau_q;\quad (44)$$

(3) $\omega_4=1-(1+4S^2)^{1/2}=S_-$, $\rho_q=-2i(\mathbf{L}\cdot\mathbf{S}/S_-)\tau_q$ $(S_-\neq 0)$, $\sigma_q=\frac{1}{2}i(S_--4)\tau_q$.

Substituting into Eqs. (42) for the three cases, we obtain

(1) $\omega_4=0$

(a) $\omega_3=0$, $\rho_1=\rho_2$, $\rho_3=0$; this case is trivial, since the corresponding operator commutes with all four Ω_k.

(b) U_{10}: $\omega_3=J_+\neq 0$, $\rho_1=-2i(\mathbf{L}\cdot\mathbf{S}+S^2)/J_+\rho_3$,
$$\rho_2=2i(\mathbf{L}\cdot\mathbf{S}+L^2)/J_+\rho_3;$$

(c) U_{11}: $\omega_3=J_-\neq 0$, $\rho_1=-2i(\mathbf{L}\cdot\mathbf{S}+S^2)/J_-\rho_3$,
$$\rho_2=2i(\mathbf{L}\cdot\mathbf{S}+L^2)/J_-\rho_3.$$

(2) $\omega_4=S_+$

(a) U_{12}: $\omega_3=-S_+$, $\tau_1=\tau_2$, $\tau_3=0$;

(b) U_{13}: $\omega_3=J_+-S_-$, $\tau_1=-i\dfrac{2S^2+2\mathbf{L}\cdot\mathbf{S}+S_+}{J_+}\tau_3$,
$$\tau_2=i\frac{2\mathbf{L}\cdot\mathbf{S}+2L^2-S_+}{J_+}\tau_3;$$

(c) U_{14}: $\omega_3=J_--S_+$, $(J_-\neq 0)$
$$\tau_1=-i\frac{2S^2+2\mathbf{L}\cdot\mathbf{S}+S_+}{J_-}\tau_3,$$
$$\tau_2=i\frac{2\mathbf{L}\cdot\mathbf{S}+L^2-S_+}{J_-}\tau_3.$$

(3) $\omega_4=S_-\neq 0$

(a) U_{15}: $\omega_3=-S_-$, $\tau_1=\tau_2$, $\tau_3=0$;

(b) U_{16}: $\omega_3+J_+-S_-$, $\tau_1=-i\dfrac{2S^2+2\mathbf{L}\cdot\mathbf{S}+S_-}{J_+}\tau_3$,
$$\tau_2=i\frac{2\mathbf{L}\cdot\mathbf{S}+2L^2-S_-}{J_+}\tau_3;$$

(c) U_{17}: $\omega_3=J_--S_-$, $(J_-\neq 0)$
$$\tau_1=-i\frac{2S^2+2\mathbf{L}\cdot\mathbf{S}+S_-}{J_-}\tau_3,$$
$$\tau_2=i\frac{2\mathbf{L}\cdot\mathbf{S}+2L^2-S_-}{J_-}\tau_3.$$

The eight nontrivial solutions U_{10}–U_{17} are listed with their coefficients in Table II, and their effect on eigenstates is shown in Table III.

For ladder operators constructed from the members of the whole algebra, the coefficients of the operators S_i^m, E_i^m, and F_i^m have to be zero to satisfy Eqs. (26); thus they are identical with U_{10}–U_{17}.

V. SOLUTIONS OF EIGENVALUE EQUATIONS

The operators Ω_k ($k=1, 2, 3, 4$) defined in Sec. III are the only ones entering into the eigenvalue equations, but they do not comprise a complete set of mutually commuting operators. As a result, the states satisfying Eqs. (20)–(23) are degenerate. Let us consider, then, additional operators that commute with the Ω_k and among each other, in order to narrow down the repre-

TABLE III. Properties of the U operators.

	ω_1'	ω_2'	ω_3'	ω_4'	$\Delta[j(j+1)]$	$\Delta(M^2)$
$(U_1)_i^{r\pm}$	$\pm 1/d_r^2$	0	0	0	0	$\pm B'/d_r^2$
$(U_2)_i^{r\pm}$	$\pm 1/d_r^2$	0	$2(j+1)$	0	$2(j+1)$	$\pm B'/d_r^2+2C'(j+1)$
$(U_3)_i^{r\pm}$	$\pm 1/d_r^2$	0	$-2j$	0	$-2j$	$\pm B'/d_r^2-2C'j$
$(U_4)_i^{r\pm}$	$\pm 1/d_r^2$	$2(l+1)$	$-2(l+1)$	0	0	$\pm B'/d_r^2-2C'(l+1)$
$(U_5)_i^{r\pm}$	$\pm 1/d_r^2$	$2(l+1)$	$2(j-l)$	0	$2(j+1)$	$\pm B'/d_r^2+2C'(j-l)$
$(U_6)_i^{r\pm}$	$\pm 1/d_r^2$	$2(l+1)$	$-2(j+l+1)$	0	$-2j$	$\pm B'/d_r^2-2C'(j+l+1)$
$(U_7)_i^{r\pm}$	$\pm 1/d_r^2$	$-2l$	$2l$	0	0	$\pm B'/d_r^2+2C'l$
$(U_8)_i^{r\pm}$	$\pm 1/d_r^2$	$-2l$	$2(j+l+1)$	0	$2(j+1)$	$\pm B'/d_r^2+2C'(j+l+1)$
$(U_9)_i^{r\pm}$	$\pm 1/d_r^2$	$-2l$	$-2(j-l)$	0	$-2j$	$\pm B'/d_r^2-2C'(j-l)$
$(U_{10})_i^m$	0	0	$2(j+1)$	0	$2(j+1)$	$2C'(j+1)$
$(U_{11})_i^m$	0	0	$-2j$	0	$-2j$	$-2C'j$
$(U_{12})_i^m$	0	0	$-2(s+1)$	$2(s+1)$	0	$-2C'(s+1)+2D'(s+1)$
$(U_{13})_i^m$	0	0	$2(j-s)$	$2(s+1)$	$2(j+1)$	$2C'(j-s)+2D'(s+1)$
$(U_{14})_i^m$	0	0	$-2(j+s+1)$	$2(s+1)$	$-2j$	$-2C'(j+s+1)+2D'(s+1)$
$(U_{15})_i^m$	0	0	$2s$	$-2s$	0	$2C's-2D's$
$(U_{16})_i^m$	0	0	$2(j+s+1)$	$-2s$	$2(j+1)$	$2C'(j+s+1)-2D's$
$(U_{17})_i^m$	0	0	$-2(j-s)$	$-2s$	$-2j$	$-2C'(j-s)-2D's$

sentations and to examine the nature of the arising degeneracy.

Third component of total spin: $J_3=J_{12}$ [*Eq. (11)*]. The degeneracy is an expected one, since the eigenvalue j_3 of this operator merely specifies the spatial direction of the compound spin. Specifically, the operators $(U)_3^{r\pm}$ and $(U)_3^m$ (see Table III) leave j_3 unchanged, while the combinations $(U)_1^{r\pm}\pm i(U)_2^{r\pm}$ and $(U)_1^m\pm i(U)_2^m$ will raise (lower) j_3. We can also require, then, that

$$J_3|\psi\rangle=j_3|\psi\rangle. \tag{45}$$

N−1 occupation number operators: $N^r=a_i^{(r)+}a_i^{(r)-}$. Since the operator $\Omega_1=\Gamma$ is a linear combination of these with unequal coefficients, no degeneracy arises. From Eqs. (20) and (30) the eigenvalue is given in terms of the occupation numbers as

$$\gamma=\sum_r\frac{n^r+\frac{3}{2}}{d_r^2}. \tag{46}$$

N subparticle spins $(S^m)^2$. The subparticles are fermions; therefore, these operators have unique eigenvalues ($\frac{3}{4}$) resulting in no degeneracy.

N−2 asymmetric linear combinations of the $\mathbf{S}^m\cdot\mathbf{S}^n$. The corresponding degeneracy is related to the $N-2$ degrees of freedom of spin direction of the subparticles, consistent with Eq. (10), and it always arises whenever more than two angular momenta are added together. The superscript m of the operators U_{10}–U_{17} in Table III corresponds to this degeneracy.

N−1 orbital angular momenta: $(L^r)^2$. The eigenvalues of these operators are not unique, although they can take on only the values

$$l_r=n_r,\quad n_r-2,\quad n_r-4,\cdots,1\quad\text{or}\quad 0; \tag{47}$$

thus, for given n_r any interchange of two or more l_r consistent with Eq. (47) leads to a degenerate state.

N−3 asymmetric linear combination of the $\mathbf{L}^r\cdot\mathbf{L}^s$. The arising degeneracy is analogous to the one due to the $\mathbf{S}^m\cdot\mathbf{S}^n$ discussed above, since it arises from the addition of more than two orbital angular momenta and corresponds to the $N-3$ degrees of freedom of orientation of these angular momenta consistent with Eq. (9).

In order to find the solutions we will look at two distinct cases:

(1) N even. Taking $|\psi_0\rangle$ to be the ground state, we require that

$$\Gamma|\psi_0\rangle=\tfrac{3}{2}\sum_r(1/d_r^2)|\psi_0\rangle=\tfrac{3}{2}(1/d^2)|\psi_0\rangle,$$

where $(1/d^2)=\sum_r(1/d_r^2)$,

$$L^2|\psi_0\rangle=\Delta|\psi_0\rangle=S^2|\psi_0\rangle=J_3|\psi_0\rangle=0,$$

$$(S^m)^2|\psi_0\rangle=\tfrac{3}{4}|\psi_0\rangle,$$

and

$$P|\psi_0\rangle=|\psi_0\rangle.$$

That is, our ground state is of positive parity with compound spin $j_3=0$ [Eq. (24)] and [from Eq. (25)]

$$f(M)=A'+\tfrac{3}{2}B'/d^2.$$

$|\psi_0\rangle$ is of the form

$$|\psi_0\rangle=\pi^{-3N/4}(d_1d_2\cdots d_N)^{-3/2}\phi\exp(-y_i^ry_i^r/2d_r^2)|u_0\rangle,$$

where ϕ is a normalized eigenstate of the operators A, B, C, and D, yielding eigenvalues A', B', C', and D', and $|u_0\rangle$ is a normalized asymmetric combination of N-fold direct products of two-spinors, satisfying

$$S^2|u_0\rangle=S_3|u_0\rangle=0$$

and

$$(S^m)^2|u_0\rangle=\tfrac{3}{4}|u_0\rangle.$$

With the help of the ladder operators found in Sec. IV, all other solutions can be generated from this ground state.

Since the operators U_1–U_9 are of negative parity and change the total occupation number by one unit, while U_{10}–U_{17} are of positive parity and leave the occupation numbers unchanged, with the choice of positive parity for the ground state we have the condition

$$p=(-1)^n. \tag{48}$$

It should also be noted here that the operators U_{10}–U_{17} annihilate the ground state, although U_{13} is the ladder operator that, in general, is responsible for the allowed transition $\Delta s=\Delta j=+1$, $\Delta l=\Delta\gamma=0$. To achieve such a transition using the ladder operators of Sec. IV, one has to use the product of three of them, e.g., $(U_7)_3{}^{r-}(U_{13})_3{}^m(U_4)_3{}^{r+}|\psi_0\rangle$.

There exist, however, special ladder operators that are good only for single states. If we write Eq. (26) in a modified form, namely,

$$[\Omega_k,U]=U\omega_k+\theta_k \quad \text{with} \quad \theta_k|\psi'\rangle=0, \tag{49}$$

i.e., with θ_k an annihilation operator for a certain state, then any solution of Eq. (49) will be a ladder operator if acting on that state. The operator $(U_0)_i{}^m=\alpha(S_i{}^m+iF_i{}^m+H_i{}^m)$ (see Sec. IV) is such a solution for the ground state, since

$$[\Omega_1,(U_0)_i{}^m]=0,$$
$$[\Omega_2,(U_0)_i{}^m]=0,$$
$$\begin{aligned}[\Omega_3,(U_0)_i{}^m]&=\alpha(-2iE_i{}^m+2Q_i{}^m-2S_i{}^m\mathbf{L}\cdot\mathbf{S}\\&\qquad+2iE_i{}^m-2iK_i{}^m)\\&=\omega_3(U_0)_i{}^m+\theta_3,\end{aligned}$$

where $\theta_3|\psi_0\rangle=0$ and $\omega_3=0$,

$$\begin{aligned}[\Omega_4,(U_0)_i{}^m]&=\alpha(-2iF_i{}^m-2S_i{}^m+2H_i{}^m\\&\qquad-2S_i{}^m(S^2-2)+4iF_i{}^m)\\&=2\alpha(S_i{}^m+iF_i{}^m+H_i{}^m)-2\alpha S_i{}^mS^2\\&=2(U_0)_i{}^m-2\alpha S_i{}^mS^2=\omega_4(U_0)_i{}^m+\theta_4,\end{aligned}$$

where $\theta_4|\psi_0\rangle=-2\alpha S_i{}^mS^2|\psi_0\rangle=0$ and $\omega_4=2$.

Example: $N=2$.

$$|\psi_0\rangle=(\pi d)^{-3/2}\phi\exp(-y_iy_i/2d^2)|u_0\rangle,$$

with

$$|u_0\rangle=\frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}-\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right].$$

(2) N odd. Since the lowest possible spin now is $\frac{1}{2}$, there are two types of ground states $|\psi_0\rangle_\pm$, satisfying

$$\begin{aligned}\Gamma|\psi_0\rangle_\pm&=\tfrac{3}{2}(1/d^2)|\psi_0\rangle_\pm,\\L^2|\psi_0\rangle_\pm&=\Delta|\psi_0\rangle_\pm=0,\\S^2|\psi_0\rangle_\pm&=\tfrac{3}{4}|\psi_0\rangle_\pm,\\J_3|\psi_0\rangle_\pm&=\pm\tfrac{1}{2}|\psi_0\rangle_\pm,\\(S^m)^2|\psi_0\rangle_\pm&=\tfrac{3}{4}|\psi_0\rangle_\pm,\\P|\psi_0\rangle_\pm&=|\psi_0\rangle_\pm.\end{aligned}$$

The ground states are therefore of positive parity, with compound spin $\frac{1}{2}$, and with

$$f(M)=A'+\tfrac{3}{2}B'/d^2+\tfrac{3}{4}D',$$

having the form

$$|\psi_0\rangle_\pm=\pi^{-3N/4}(d_1d_2\cdots d_{N-1})^{-3/2}\phi\times\exp(-y_i{}^ry_i{}^r/2d_r)|u_0\rangle_\pm,$$

where ϕ is again a normalized eigenstate of A, B, C, and D, and $|u_0\rangle_\pm$ are normalized spinors, satisfying

$$S^2|u_0\rangle_\pm=\tfrac{3}{4}|u_0\rangle_\pm,$$
$$S_3|u_0\rangle_\pm=\pm\tfrac{1}{2}|u_0\rangle_\pm,$$

and

$$(S^m)^2|u_0\rangle_\pm=\tfrac{3}{4}|u_0\rangle_\pm.$$

Example: $N=3$. We can choose

$$|u_0\rangle_+=\frac{1}{\sqrt{6}}\left[2\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}-\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}-\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right],$$

$$|u_0\rangle_-=\frac{1}{\sqrt{6}}\left[2\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}-\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}-\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\right],$$

then,

$$|\psi_0\rangle_\pm=\pi^{-9/4}(d_1d_2)^{-3/2}\phi\exp(-y_i{}^ry_i{}^r/2d_r{}^2)|u_0\rangle_\pm.$$

VI. APPLICATION TO ELEMENTARY PARTICLES

In this section we will investigate the mass formula, Eq. (25), for groups of elementary particles and resonances. Under a "group" we will understand the collection of particles of the same internal quantum numbers: isospin and hypercharge. We will examine mass formulas for $f(M)=M$ and $f(M)=M^2$.

Since Eq. (25) has four undetermined constants A', B', C', and D', the results of the theory have to be partially empirical, for, in any group of elementary particles and resonances under consideration, these constants will have to be determined from experimental values of particle masses. The eigenvalues γ of the operator Γ also contain undetermined constants: the fundamental lengths d_r. On the assumption that the mesons are made up of two, and the baryons of three subparticles, i.e., if there are three degrees of freedom of relative motion for mesons, corresponding to one fundamental length d, while there are 2×3 degrees of freedom for baryons, corresponding to two fundamental lengths d_1 and d_2, one arrives at a total of four undetermined constants for mesons: A', B'/d^2, C', and D', and five for baryons: A', $B'/d_1{}^2$, $B'/d_2{}^2$, C', and D'. Consequently, one will have to use experimental mass

TABLE IV. Quantum-number assignments and resulting masses for nuclear resonances.

Symbol	j^p	l	s	δ	n_1	n_2	n	M^2(theoret) (BeV^2)	M^2(expt)[a] (BeV^2)
N	$\frac{1}{2}^+$	0	$\frac{1}{2}$	0	0	0	0	(0.88)	0.88[b]
N'(1470)	$\frac{1}{2}^+$	0	$\frac{1}{2}$	0	2	0	2	2.12	2.16
N(1518)	$\frac{3}{2}^-$	1	$\frac{3}{2}$	−2	1	0	1	(2.33)	2.33[b]
N(1550)	$\frac{1}{2}^-$	1	$\frac{1}{2}$	−2	0	1	1	(2.40)	2.40[b]
N(1680)	$\frac{5}{2}^-$	1	$\frac{3}{2}$	−7	1	0	1	2.78	2.82
N(1688)	$\frac{5}{2}^+$	2	$\frac{3}{2}$	−1	2	0	2	(2.86)	2.86[b]
N'(1710)	$\frac{1}{2}^-$	1	$\frac{1}{2}$	−2	3	0	3	2.92	2.92
N(2190)	$\frac{7}{2}^-$	2	$\frac{3}{2}$	6	4	1	5	4.81	4.84
N(2650)	$[\frac{11}{2}^-]$[c]	5	$\frac{3}{2}$	2	5	2	7	7.13	7.02
N(3030)	$[\frac{11}{2}^-]$[c]	6	$\frac{1}{2}$	−7	6	3	9	9.25	9.18
Δ(1236)	$\frac{3}{2}^+$	0	$\frac{3}{2}$	0	0	0	0	(1.53)	1.53[b]
Δ(1640)	$\frac{1}{2}^-$	1	$\frac{3}{2}$	−5	1	0	1	2.60	2.69
Δ(1920)	$\frac{7}{2}^+$	3	$\frac{1}{2}$	3	3	1	4	3.81	3.80
Δ(2420)	$\frac{11}{2}^+$	5	$\frac{3}{2}$	2	5	1	6	5.79	5.86
Δ(2850)	$[\frac{15}{2}^+]$[c]	8	$\frac{1}{2}$	−9	6	2	8	8.09	8.12
Δ(3230)	$[19/2^+]$[c]	10	$\frac{3}{2}$	−14	8	2	10	10.43	10.40

$M^2=[0.718+0.620n_1+1.340n_2-0.090\delta+0.217s(s+1)]$ (BeV^2)

[a] From A. H. Rosenfeld *et al.*, Rev. Mod. Phys. **40**, 77 (1968).
[b] The mass formula was based on these values.
[c] Quantities in square brackets are suggested values; they have not been established experimentally.

values of four mesons and five baryons in any group to arrive at the masses of the other members of the group.

The determination of these constants is not unique, since it depends on the assignment of the quantum numbers n_r, l, s, and δ for the particles from which these constants are determined.

These quantum numbers have to satisfy the following conditions:

(a) Eq. (48): $p=(-1)^n$.

(b) The total spin j of each particle satisfies

$$|l-s|\leq j\leq l+s$$

from the theory of addition of angular momenta.

(c) $s=0$, 1 for mesons,
$s=\frac{1}{2}$, $\frac{3}{2}$ for baryons,

since **S** is the vector sum of two or three $\frac{1}{2}$ spins, respectively.

(d) For a three-dimensional harmonic oscillator

$$l_r=n_r,\ n_r-2,\ n_r-4,\cdots,\ 1 \text{ or } 0;$$

therefore, for mesons

$$l=n,\ n-2,\ n-4,\cdots,\ 1 \text{ or } 0;$$

for baryons,

$$l_1=n_1,\ n_2-2,\ n_1-4,\cdots,\ 1 \text{ or } 0,$$
$$l_2=n_2,\ n_2-2,\ n_2-4,\cdots,\ 1 \text{ or } 0,$$

with

$$|l_1-l_2|\leq l\leq l_1+l_2,$$

since $\mathbf{L}=\mathbf{L}_1+\mathbf{L}_2$.

(e) The quantum number δ is given by [from Eq. (24)]

$$\delta=j(j+1)-l(l+1)-s(s+1).$$

The general procedure is, then, to select the necessary number of particles for the determination of the constants within a group, to find all possible combinations of quantum-number assignments satisfying the above conditions, to find the constants for each, and to check the resulting mass formula for the other members of the group.

In carrying out this procedure, we found no apparent agreement with the choice of $f(M)=M$. With $f(M)=M^2$, the mass formula is still inapplicable to mesons, but for groups of baryon resonances, there is more than

TABLE V. Quantum-number assignments and resulting masses for $Y_0{}^*$ resonances.

Symbol	j^p	l	s	δ	n_1	n_2	n	M^2(theoret) (BeV^2)	M^2(expt)[a] (BeV^2)
Λ	$\frac{1}{2}^+$	0	$\frac{1}{2}$	0	0	0	0	(1.24)	1.24[b]
Λ(1405)	$\frac{1}{2}^-$	1	$\frac{1}{2}$	−2	1	0	1	(1.97)	1.97[b]
Λ(1520)	$\frac{3}{2}^-$	1	$\frac{3}{2}$	−2	1	0	1	(2.31)	2.31[b]
Λ'(1670)	$\frac{1}{2}^-$	1	$\frac{3}{2}$	−5	0	1	1	(2.79)	2.79[b]
Λ'(1690)	$\frac{3}{2}^-$	1	$\frac{3}{2}$	−2	0	1	1	(2.86)	2.86[b]
Λ(1815)	$\frac{5}{2}^+$	2	$\frac{1}{2}$	2	1	1	2	3.39	3.30
Λ(1830)	$\frac{5}{2}^-$	3	$\frac{1}{2}$	−4	3	0	3	3.48	3.34
Λ(2100)	$\frac{7}{2}^-$	3	$\frac{3}{2}$	0	2	1	3	4.46	4.41
Λ(2350)	$[\frac{9}{2}^+]$[c]	4	$\frac{1}{2}$	4	2	2	4	5.54	5.52

$M^2=[1.155+0.777n_1+1.327n_2+0.023\delta+0.113s(s+1)]$ (BeV^2)

[a] From A. H. Rosenfeld *et al.*, Rev. Mod. Phys. **40**, 77 (1968).
[b] The mass formula was based on these values.
[c] Quantities in square brackets are suggested values; they have not been established experimentally.

one possible assignment scheme, each giving satisfactory results for the entire spectrum.

It appeared to be reasonable to select those assignment schemes that yielded, at least in the higher-lying resonances, approximate equipartition between the two independent harmonic oscillators. With this condition imposed, we found that the $N_{1/2}^*$ and $N_{3/2}^*$ resonances can be fitted with a single-mass formula. This is not surprising on the basis of quark theory, according to which, both of these groups of particles are made up of the same kind of subparticles. On this basis one would expect a single mass formula to hold for the Y_0^* and Y_1^* resonances as well. We found this not to be the case, although the corresponding constants in the mass formulas are not drastically different. In particular,

$$\frac{B}{d^2}=\frac{B}{d_1^2}+\frac{B}{d_2^2}=2.104 \text{ BeV}^2 \quad \text{for the} \quad Y_0^*$$
$$=2.033 \text{ BeV}^2 \quad \text{for the} \quad Y_1^*,$$

as compared with 1.96 BeV^2 for the N^* resonances. If B is of the order of unity, these values correspond to a fundamental length of the order of 10^{-15} meters.

The selected assignment schemes, the mass formulas, and the resulting masses for nuclear resonances are listed in Table IV, for Y_0^* resonances in Table V, and for the Y_1^* in Table VI.

For the Z_0^*, $\Xi_{1/2}^*$, and Ω_0^* groups, the number of known resonances is not sufficient for the determination of mass formulas.

In the schemes of all the groups of particles considered above there are holes, that is, possible states for which there are no established resonances. These states for nuclear resonances for $n_1=1$, $n_2=0$, and for $n_1=n_2=1$ are listed in Table VII. As indicated there, seven of these fifteen states can be identified with resonances reported by the CERN group (Donnachie *et al.*[16]) and not included in Table IV.

TABLE VI. Quantum-number assignments and resulting masses for Y_1^* resonances.

Symbol	j^p	l	s	δ	n_1	n_2	n	M^2(theoret) (BeV^2)	M^2(expt)[a] (BeV^2)
Σ	$\frac{1}{2}^+$	0	$\frac{1}{2}$	0	0	0	0	(1.42)	1.42[b]
Σ(1385)	$\frac{3}{2}^+$	0	$\frac{3}{2}$	0	0	0	0	(1.92)	1.92[b]
Σ(1660)	$\frac{3}{2}^-$	1	$\frac{3}{2}$	−2	1	0	1	(2.76)	2.76[b]
Σ(1690)	$[\frac{5}{2}^-]$[c]	1	$\frac{3}{2}$	3	1	0	1	2.84	2.89
Σ(1770)	$\frac{5}{2}^-$	1	$\frac{3}{2}$	3	0	1	1	(3.13)	3.13[b]
Σ(1910)	$\frac{5}{2}^+$	2	$\frac{3}{2}$	−1	2	0	2	(3.65)	3.65[b]
Σ(2030)	$\frac{7}{2}^+$	2	$\frac{3}{2}$	6	1	1	2	4.05	4.12
Σ(2250)	$[\frac{9}{2}^-]$[c]	3	$\frac{3}{2}$	9	2	1	3	4.98	5.06
Σ(2455)	$[\frac{9}{2}^+]$[c]	4	$\frac{3}{2}$	1	2	2	4	6.00	6.03
Σ(2595)	$[\frac{9}{2}^-]$[c]	5	$\frac{3}{2}$	−9	3	2	5	6.71	6.73
Σ(1616)[d]	$[\frac{3}{2}^-]$[c]	1	$\frac{1}{2}$	1	0	1	1	2.60	2.61

$$M^2=[1.295+0.873n_1+1.160n_2+0.017\delta+0.167s(s+1)] \text{ (BeV}^2\text{)}$$

[a] From A. H. Rosenfeld *et al.*, Rev. Mod. Phys. **40**, 77 (1968); with the exception of Σ(1616).
[b] The mass formula was based on these values.
[c] Quantities in square brackets are suggested values; they have not been established experimentally
[d] Reported by D. J. Crennell *et al.*, Phys. Rev. Letters **21**, 648 (1968).

VII. DISCUSSION

Although the motivation for this work has been some observed regularities in the squared masses of hadron resonances, we would like to stress that the application of our model to hadrons was not the primary objective of this paper. Whether or not the model has anything to do with hadrons, the problem of an arbitrary number of independent harmonic oscillators with broken $U(3)$ symmetry remains an interesting one.

It should also be pointed out that the material contained in Secs. I–IV is in no way dependent upon the value of the spin of the subparticles; therefore, these sections are applicable to bound states of bosons as well.

The application of the model to fermions has to be with the use of the Pauli exclusion principle. For our model the application of the Pauli principle is not straightforward. As was pointed out in Sec. III, in one particular representation of the relative variables the operators Γ_{ii}^r (and, therefore, the N^r) will be diagonal. This representation is obtained from the coordinates and momenta of the subparticles by a unique canonical transformation that separates out center-of-mass motion, i.e., a transformation that satisfies Eqs. (5) and (6). Since all elements of this transformation matrix are not given, it is not known in what way individual subparticles enter into a given harmonic oscillator, i.e., what the association is between subparticles and harmonic-oscillator quantum numbers n_r and l_r. It is our intention to study this problem in the future.

TABLE VII. Some low-mass resonance states of the model that do not correspond to established resonances.

j^p	l	s	δ	n_1	n_2	n	$M^{2\,a}$ (BeV^2)	$M^{2\,b}$ (BeV^2)
$\frac{3}{2}^-$	1	$\frac{1}{2}$	1	1	0	1	1.41	
$\frac{1}{2}^-$	1	$\frac{1}{2}$	−2	1	0	1	1.68	
$\frac{5}{2}$	2	$\frac{1}{2}$	2	1	1	2	2.66	
$\frac{3}{2}$	1	$\frac{1}{2}$	1	1	1	2	2.75	
$\frac{3}{2}$	0	$\frac{1}{2}$	0	1	1	2	2.84	2.85
$\frac{7}{2}$	2	$\frac{3}{2}$	6	1	1	2	2.95	
$\frac{1}{2}$	1	$\frac{1}{2}$	−2	1	1	2	3.02	3.07
$\frac{3}{2}$	2	$\frac{1}{2}$	−3	1	1	2	3.11	
$\frac{5}{2}$	1	$\frac{3}{2}$	3	1	1	2	3.32	
$\frac{3}{2}$	0	$\frac{3}{2}$	0	1	1	2	3.49	3.47
$\frac{5}{2}$	2	$\frac{3}{2}$	−1	1	1	2	3.58	3.66
$\frac{3}{2}$	1	$\frac{3}{2}$	−2	1	1	2	3.67	3.74
$\frac{1}{2}$	1	$\frac{3}{2}$	−5	1	1	2	3.94	3.93
$\frac{3}{2}$	2	$\frac{3}{2}$	−6	1	1	2	4.03	
$\frac{1}{2}$	2	$\frac{3}{2}$	−9	1	1	2	4.30	4.23

[a] Computed from the mass formula of Table IV.
[b] Resonances reported by A. Donnachie, R. G. Kirsopp, and C. Lovelace, Phys. Rev. Letters **26B**, 161 (1968).

[16] A. Donnachie, R. G. Kirsopp, and C. Lovelace, Phys. Letters **26B**, 161 (1968).

In applying our model to baryon resonances, we found good agreements with experimental values by assigning known resonances to possible solutions of the theory. As pointed out in Sec. VI, however, it would also have been possible to obtain fairly good agreements for different assignment schemes. We selected the ones shown in Tables IV–VI on the basis of simplicity and equipartition among the independent harmonic oscillators. These conditions seem reasonable, although there are a few exceptions to the latter even in our assignment schemes, and, therefore, it cannot be considered absolute. It would be necessary to know a much larger number of resonances in each group to be able to test a given assignment scheme or the model as a whole.

The number of missing resonances in the model, although not unreasonably large compared with the number of established ones, is of concern, because there are some states corresponding to fairly low masses, and one would think that such resonances, if they existed, could not have easily escaped attention. However, if one could apply the Pauli exclusion principle to the quarks within the model, some of the states would be forbidden; thus, a possible state in our model could either correspond to a resonance yet unknown, or could be excluded by the Pauli principle or other applicable conditions that have been ignored. It is interesting to note in this respect that in all of the Y^* resonances of Tables V and VI the quantum number l is maximal. Such a condition, if it existed, would eliminate a large number of states that do not correspond to known resonances.

The results of Sec. VI were obtained by taking $f(M)=M^2$ in Eq. (25). There is no *a priori* reasons for selecting either M or M^2 in Eq. (19). If an eigenvalue equation exists for M, one exists also for M^2. Since we are not familiar with the exact dynamics of the problem, the one we select to be the eigenvalue of a reciprocity-invariant operator has to be dictated by empirical considerations.

Mesonic resonances apparently cannot be fitted by our model. Since baryonic resonances decay via the emission of mesons, it has been suggested[17] that, perhaps, the more fundamental members of the meson family can be considered as the agents of the excitation and deexcitation of baryonic resonances. If so, these mesons would not be associated with bound-state solutions, but with the ladder operators of Sec. IV, which establish transitions between baryonic states. However, the question of which mesons are more "fundamental" than others remains to be answered.

ACKNOWLEDGMENTS

The author wishes to express his sincere gratitude to Professor Ernest E. H. Shin for suggesting the problem and for his continuous guidance throughout the course of the investigation. He also wishes to thank Professor James C. Nearing and Professor Joseph H. Wojtaszek for many helpful discussions. The work was partially supported by a National Science Foundation Institutional Grant.

[17] E. E. H. Shin (private communication).