momentum transfer, a detailed calculation incorporating a suitable momentum-transfer dependence of the parameters α seems called for.

With regard to the cross-section defect, we are able to make a much less definite statement because of the uncertainty in the proper choice of α_p and α_n. We observe, however, that for the case in which the real and imaginary parts of the two particle amplitudes are both equal to ± 1, the cross-section defect arises entirely from our correction term.

After submittal of this paper, a calculation was reported by Pumplin[17] in which the effect of the principal-value part of the double-scattering term was studied using relativistic kinematics in conjunction with the usual nonrelativistic Lippmann-Schwinger equations. Two points in this work are worth noticing. First, as a consequence of the use of relativistic kinematics, the principal-value part of the double-scattering term obtained by Pumplin differs from ours by an over-all factor of $[(k^2+m_N^2)/k^2]^{1/2}$—a factor which goes to unity in the limit of large k. This justifies to some extent our use of nonrelativistic kinematics in this region of extreme high-energy scattering—the error becoming less and less at higher and higher energies. It also shows that the use of relativistic kinematics is not likely to detract from the importance of the correction arising from the principal-value part of the double-scattering term. Second, Pumplin's numerical estimates of α_p and α_n differ from ours because in evaluating the principal-value integral he has neglected terms proportional to q^2 in the integrand while we have neglected terms proportional to Δ^2. The differential cross section for Pumplin's estimate of α_p and α_n lies lower than that obtained by Bennett *et al.* for $\alpha_p=-0.6$ and $\alpha_n=-1.2$. In order to reproduce the fit of Bennett *et al.* to the experimental data, particularly at the interference minimum, Pumplin would require larger magnitudes of α_p and α_n. This would tend to bring Pumplin's estimate of these parameters closer to ours.

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Theory of the Relativistic H Atom and Positronium

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It is shown that a relativistic infinite-component wave equation correctly describes the relativistic effects in the H atom including the motion of the nucleus. An exact mass formula for the singlet $l=n-1$ levels of positronium is derived.

IN 1939, Eddington[1] justly criticized the incorrect use of the principle of Lorentz invariance in problems such as the Dirac equation for the H atom, where the *relative* coordinate of two particles is treated as a four-vector when, in reality, it is not a four-vector. In reply to this criticism, Dirac, Peierls, and Pryce[2] started from a two-body equation and showed that the standard Dirac equation is correct and relativistic to a good approximation. Although the antogonists agreed on the incorrectness of the usual "textbook presentation" of the Dirac equation for H atom but on its approximate correctness in practice, the defense by Dirac, Peierls, and Pryce completely failed to satisfy Eddington[3] as a matter of principle. For the problem is not that the effect of the proton can be neglected in practice; the problem is precisely how to take this effect into account in a relativistically invariant way. It is possible that Eddington was looking for an equation that describes relativistically the atom as a whole. We know now that indeed the relativistic treatment of a two-body system, such as positronium, is very complicated in terms of the coordinates of the constituents. Even for the H atom there are a number of correction terms beyond those given by the Dirac equation which have to be taken into account.[4] At present this is done in the framework of (noncovariant) perturbation theory.

It is therefore of interest to point out that a completely relativistic equation exists for the H atom (or positronium) *as a whole* which quantitatively contains the relativistic spectrum (including recoil corrections) and which also describes the transitions of the atom

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under external interactions in a simple way. The problem of relativistic invariance for composite systems posed by Eddington is thus solved.

The underlying framework is the algebraic dynamical group theory, which sought to formulate the relativistic quantum theory in terms of the *global* quantum numbers (both external and internal) of the system as a whole and which took the H atom as a prototype right from the beginning.[5] There has been rapid development in this area because of the applicability of the theory to fundamental particles; the results have recently been reviewed.[6-8] Concerning the H atom, most of the interest has been in the algebraic description of the nonrelativistic H atom.[9] We are interested here in the quantitative aspects of the relativistic atom.

The relativistic equation in momentum space which describes the whole atom in an arbitrary frame has the following form:

$$[j^\mu P_\mu+\beta S+\gamma]\tilde{u}(p)=0, \qquad j_\mu=\alpha_1\Gamma_\mu+\alpha_2 P_\mu+\alpha_3 P_\mu S+U_\mu. \tag{1}$$

Here $\tilde{u}(p)$ is the wave function of the whole atom, P_μ is the total momentum of the atom with eigenvalues p_μ; β and γ are constants. A current j^μ has been introduced to indicate that the coupling of the atom to the external field is of the form $j^\mu A_\mu$, and *not* a minimal coupling obtained by replacing P_μ by $P_\mu-eA_\mu$. These two procedures would be equivalent if j^μ itself did not contain terms containing P_μ. The wave function $\tilde{u}(p)$ has two suppressed indices $\tilde{u}_{\sigma s}$, one a four-component Dirac index (σ), the other an infinite-component set of indices (s), short for the quantum numbers *nlm*, both given by the irreducible representations of the dynamical group $O(4,2)$. The spectrum provided by these irreducible representations exactly coincides with the spectrum of the *relativistic* atom with correct spin multiplicities. Note that neither the relative coordinates nor the Coulomb potential occurs in the equation. These have been eliminated and replaced by the infinite-component index s. The operators $\Gamma_\mu=(L_{56},L_{i6})$ and $S=L_{46}$ are among the generators L_{ab}; $a, b=1, 2, \cdots, 6$, of the group $O(4,2)$ and act on the index s. Finally, the quantity U_μ is a spin-orbit interaction term which, for simplicity, we shall neglect here.

Equation (1) was first used in connection with a hadron model,[10] and the analogy of the spectrum with the H atom has been noted. The nonrelativistic limit of the equation in a special case gives exactly the nonrelativistic H-atom spectrum.[11] Formally, the equation also applies to electrons and muons.[12] We shall now discuss the quantitative applicability of the relativistic equation to the H atom and positronium.

To solve Eq. (1), we first go to the rest frame of the atom by putting

$$\tilde{u}(p)=e^{i\xi\cdot M}\tilde{u}(0), \tag{2}$$

where $M_i=L_{i5}$ are the generators of pure Lorentz transformations and the four-momentum of the atom is specified by

$$P_\mu\equiv(P_0,\mathbf{P})=M(\cosh\xi,\hat{\xi}\sinh\xi), \qquad P_\mu^2=P_0^2-P_i^2=M^2. \tag{3}$$

We then obtain

$$[(\alpha_1\Gamma_0+\alpha_2 M+\alpha_3 MS)M+\beta S+\gamma]\tilde{u}(0)=0. \tag{4}$$

In order to diagonalize the equation we next put ("tilting" operation)

$$\tilde{u}(0)=e^{i\theta L_{45}}u(0) \tag{5}$$

and obtain, using commutation relations of the Lie algebra of $O(4,2)$,

$$\{[\alpha_1 M\cosh\theta-(\alpha_3 P_\mu^2+\beta)\sinh\theta]\Gamma_0 -[\alpha_1 M\sinh\theta-(\alpha_3 P_\mu^2+\beta)\cosh\theta]S +(\alpha_2 P_\mu^2+\gamma)\}u(0)=0. \tag{6}$$

If we choose θ in such a way that $\tanh\theta=(\alpha_3 P_\mu^2+\beta)/\alpha_1 M$, the coefficient of S in (6) vanishes and we are left with the eigenvalue equation

$$[\alpha_1 P_\mu^2-(\alpha_3 P_\mu^2+\beta)^2]^{1/2}L_{56}u(0)=-(\alpha_2 P_\mu^2+\gamma)u(0).$$

Now we take $u(0)$ to be the basis of the representation of $O(4,2)$ in which L_{56} is diagonal and has the discrete eigenvalues n, $n=1, 2, \ldots$; we then obtain the mass spectrum

$$n=(\alpha_2 P_\mu^2+\gamma)/[\alpha_1 P_\mu^2-(\alpha_3 P_\mu^2+\beta)^2]^{1/2}. \tag{7}$$

On the other hand, if we take θ such that $\tanh\theta=\alpha_1 M/(\alpha_3 P_\mu^2+\beta)$, then the coefficient of L_{56} vanishes in (6) and in a basis $u_\nu(0)$ of the representation of $O(4,2)$ in which the noncompact generator $S=L_{46}$ is diagonal with the continuous eigenvalues ν, we obtain the continuous spectrum corresponding to the scattering states

$$\nu^2=(\alpha_2 P_\mu^2+\gamma)^2/[\alpha_1 P_\mu^2-(\alpha_3 P_\mu^2+\beta)^2], \tag{8}$$

which is obtained from (7) by the substitution $n^2\rightarrow\nu^2$ or by $n\rightarrow\pm i\nu$.

If we use the special values

$$\alpha_1=1, \quad \alpha_2=0, \quad \alpha_3=(2m_p)^{-1}, \qquad \beta=(m_p^2-m_e^2)/2m_p, \quad \gamma=-m_e\alpha, \tag{9}$$

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we obtain from (7)

$$M_n^2=m_p^2+m_e^2\pm 2m_pm_e[1-(\alpha/n)^2]^{1/2}, \quad (10)$$

and from (8)

$$M_\nu^2=m_p^2+m_e^2\pm 2m_pm_e[1+(\alpha/\nu)^2]^{1/2}. \quad (10')$$

Now if we write $M_n=m_p+m_e+B_n$, Eq. (10) gives

$$1+B_n/\mu+B_n^2/2m_pm_e=\pm[1-(\alpha/n)^2]^{1/2}, \quad (11)$$

where μ is the reduced mass. In the limit $m_p\to\infty$, hence $\mu\to m_e$, we have

$$m_e+B_n=\pm m_e[1-(\alpha/n)^2]^{1/2}, \quad (12)$$

which coincides exactly with the Dirac formula[13] for energy levels for which $|k|=n$. As noted, we have neglected the spin-orbit term U_μ in (1) which will be reported separately. Even then it is remarkable that one gets the Dirac spectrum and not the Klein-Gordon spectrum. Thus, Eq. (10) or (11) is more accurate than the Dirac equation in that it contains automatically the corrections due to the motion of the nucleus. Indeed this correction is seen, from (11), to be very close to the accepted perturbation-theoretical value[13]

$$-\tfrac{1}{8}(m_e/m_p)(\alpha/n)^4.$$

Because our equation is relativistic and contains the masses of the constituents explicitly, it is applicable to the positronium as well. In this case we obtain

$$M_n^2=2m_e^2[1+(1-\alpha^2/n^2)^{1/2}], \quad (13)$$

and think that this is an exact formula for the singlet $l=n-1$ levels of positronium. Considering the great difficulty of obtaining the positronium energy levels in perturbation theory,[14] this simple formula should be of interest. Equation (13) gives for the binding energy

$$E_n=-m_e\left(\frac{\alpha^2}{4n^2}+\frac{5}{64}\frac{\alpha^4}{n^4}\cdots\right), \quad (14)$$

to be compared to the sum of five correction terms used in perturbation theory[14] for the $s=0$, $l=n-1$ levels:

$$E_n=-m_e\left(\frac{\alpha^2}{4n^2}+\frac{10n+11}{64(2n-1)}\frac{\alpha^4}{n^4}+\cdots\right),$$

which for large n coincides with ours.

It is important to discuss the normalization of the physical states $N_n^{-1}\tilde{u}_n(p)$. The normalization condition is

$$(\tilde{u}(p),j_0\tilde{u}(p))=1,$$

and gives for $j_0=C(\alpha_1\Gamma_0+\alpha_2P_0+\alpha_3P_0S)$,

$$|N_n|^2=\mp CP_0(n^2/m_p\alpha)(1-\alpha^2/n^2)^{1/2}.$$

For the continuous scattering states, we obtain

$$|N_\nu|^2=\mp 2CP_0\left(\frac{\nu^2}{m_p\alpha}\right)\left(1+\frac{\alpha^2}{\nu^2}\right)^{1/2}e^{-\pi\nu}\sinh\pi\nu$$

due to the orthonormality condition of the continuous states.[15] A choice of $C=-m_p\alpha/(m_p+m_e)$ is convenient, for then in the nonrelativistic limit we get $|N_n|^2\to n^2$ and $|N_\nu|^2\to\nu^2$, in the so-called "energy-scale" normalization of the scattering states.[13]

We see from the mass spectrum (10) or (11) that the plus sign is the usual spectrum of the atom and they are normalized to $+1$. The solutions with the minus sign in (10) cannot be normalized to $+1$ but to -1. These new solutions disappear in the nonrelativistic limit and are a new manifestation of the relativistic theory. These and the spacelike solutions of Eq. (1), both of negative norm, have recently been given a physical interpretation as corresponding to exchange effects or so-called cross-channel contribution[16] which is a purely relativistic phenomenon and has no nonrelativistic counterpart.

Finally, we discuss the case of the particles of equal charge. In this case the sign of γ in Eq. (1) changes, and we see from (7) that there are no discrete solutions but only continuous scattering solutions.

The second important feature of Eq. (1) besides the relativistic mass formula is the simple exact, and relativistic calculation of the electromagnetic transitions.[9,10] These are proportional to the matrix elements of j_μ given in (1) between the various levels.

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[16] A. O. Barut, Lettera al Nuovo Cimento **1**, 601 (1969). The essential ideas in the proposed new interpretation of the extra solutions of Eq. (1) are the following: The solutions with negative norm cannot be interpreted as asymptotic states of momentum P_μ. We therefore divide the solutions into two groups, solutions of type I with positive norm and solutions of type II with negative norm. The S matrix is defined between states of type I. States of type II occur as intermediate states. For solutions II, $P_\mu P^\mu$ is interpreted as the cross-channel variable u (or t). The poles of Green's functions of Eq. (1) then give kinematically correct singularities of the amplitudes. For example, in the $e^+e^-\to e^+e^-$, the bound states and scattering states in the s channel are the solutions of type I, and those in the t channel are of type II. There is an exact correspondence between the solutions of Eq. (1) and the singularities (right- and left-hand poles and cuts) in s of the amplitude. It has been proposed, therefore, to use the solutions of type II to evaluate the exchange diagrams. It is remarkable that, with this interpretation, the crossing properties follow from a wave equation.